Optimal Execution:
III. Game Theory & Predatory Trading

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Premises for Predatory Trading

▶ Large Trader facing a Forced Liquidation
▶ Especially if the need to liquidate is known by other traders
  ▶ hedge funds with (nearing) margin call
  ▶ traders who use portfolio insurance, stop loss orders, . . .
  ▶ some institutions / funds cannot hold on to downgraded instruments
  ▶ Index-replication funds (at re-balancing dates) e.g. Russell 3000

Forced liquidation can be very costly because of price impact

Business Week

...if lenders know that a hedge fund needs to sell something quickly, they will sell the same asset, driving the price down even faster. Goldman Sachs and other counter-parties to LTCM did exactly that in 1998.

When you smell blood in the water, you become a shark . . . . when you know that one of your number is in trouble . . . you try to figure out what he owns and you start shorting those stocks . . .

Cramer (2002)
Typical Predatory Trading Scenario

- Distressed trader needs to unload a large position
- Size will have impact on price
- Predator initially trades in the same direction as the prey
- Effect is to withdraw liquidity
- Market impact of the liquidation becomes greater
- Price fall is exaggerated (over-shooting)
- Predator reverses direction, profiting from the over-shoot
- Predator closes position for a profit.
Optimal Portfolio Liquidation: Multi-Player Case

- New issues when other market participants know that our client is selling:
  - The market impact of our client creates a drift in the market price
  - This drift can be exploited by the other market participants
  - Since we know about this danger, we will adjust our strategy
    - Brunnermeier and Pedersen (2005),
    - Carlin, Lobo, and Viswanathan (2005)
    - SChied, Schöneborn (2008)
Game Model

- One risk free asset and one risky asset
- Trading in continuous time, interest rate $r = 0$
- $n + 1$ strategic players and a number of noise traders
- $X_0(t), X_1(t), \ldots, X_n(t)$ risky asset positions of the strategic players
- Trades at time $t$ are executed at the price (Chriss-Almgren price impact model)

$$P(t) = \tilde{P}(t) + \gamma \sum_{i=0}^{n} [X_i(t) - X_i(0)] + \lambda \sum_{i=0}^{n} \dot{X}_i(t)$$

where $\tilde{P}(t)$ is a mean zero martingale (say a Wiener process).
Goal of the Mathematical Analysis

- Understand predation
- Illustrate benefits of
  - Stealth trading
  - Sunshine trading

**Modeling extreme markets**

- **Elastic** (truly illiquid) markets:
  - temporary impact $\lambda \gg$ permanent impact $\gamma$
- **Plastic** (nervous) markets:
  - permanent impact $\gamma \gg$ temporary impact $\lambda$
Assumptions of the One Period Game

- Each strategic player \( i \in \{0, 1, \ldots, n\} \) knows
  - all other strategic players initial asset positions \( X^i(0) \)
  - Their target \( X^i(T) \) at some fixed time point \( T > 0 \) in the future
- Objective (all players are risk neutral)
  - Players maximize their expected return by choosing an optimal trading strategy \( X^i(t) \) satisfying their constraints \( X^i(0) \) and \( X^i(T) \)

One distressed trader / prey (e.g seller), player 0

\[
X_0(0) = x_0 > 0, \quad X_0(T) = 0
\]

\( n \) predators players 1, 2, \ldots, \( n \)

\[
X_i(0) = X_i(T) = 0, \quad i = 1, \ldots, n
\]
Optimization Problem

A strategy \( X_i = (X_i(t))_{0 \leq t \leq T} \) is admissible (for player \( i \)) if it is an a
- adapted process
- with continuously differentiable sample paths

Given a set \( X = (X_0, X_1, \cdots, X_n) \) of admissible strategies
- Each player \( i \in \{0, 1, \cdots, n\} \) tries to maximize his expected return
  \[
  J^i(X) = \mathbb{E} \left[ \int_0^T (-\dot{X}_i(t))P(t)dt \right]
  \]
  under the constraint
  \[
  P(t) = \tilde{P}(t) + \gamma \sum_{i=0}^{n} [X_i(t) - X_i(0)] + \lambda \sum_{i=0}^{n} \dot{X}_i(t)
  \]
- Search for Nash Equilibrium
Deterministic Strategies

If we restrict the admissible strategies $X = (X_0, X_1, \cdots, X_n)$ to be DETERMINISTIC

$$J^i(X) = \mathbb{E}\left[\int_0^T (-\dot{X}_i(t))P(t)dt\right] = \mathbb{E}\left[\int_0^T (-\dot{X}_i(t))\overline{P}(t)dt\right]$$

where

$$\overline{P}(t) = P(0) + \gamma \sum_{i=0}^{n} [X_i(t) - X_i(0)] + \lambda \sum_{i=0}^{n} \dot{X}_i(t)$$

THE SOURCE OF RANDOMNESS IS GONE!

Solution in the Deterministic Case

Unique Optimal Strategies

\[ X_i(t) = ae^{-\frac{n}{n+2} \frac{\gamma}{X} t} + b_i e^{\frac{\gamma}{X} t} \]

where

\[
\begin{align*}
  a &= \frac{n}{n+2} \frac{\gamma}{\lambda} \left( 1 - e^{-\frac{n}{n+2} \frac{\gamma}{X} T} \right)^{-1} \frac{1}{n+1} \sum_{i=0}^{n} [X_i(T) - X_i(0)] \\
  b_i &= \frac{\gamma}{\lambda} \left( e^{\frac{\gamma}{X} T} - 1 \right)^{-1} \left( X_i(T) - X_i(0) - \frac{1}{n+1} \sum_{i=0}^{n} [X_i(T) - X_i(0)] \right)
\end{align*}
\]

Carlin, Lobo, and Viswanathan (2005)
$n = 1$ predator, $\gamma/\lambda = 0.3$
$n = 1$ predator, $\gamma = \lambda$
$n = 1$ predator, $\gamma = 15.5\lambda$
Holdings of the Distressed Trader & Predator
Fancy Plots of the Holdings of the Distressed Trader & Predator

Holdings of Distressed Trader (black) & Predator (red)
Impact of the Number of Predators: $\gamma = \lambda$

Holdings of Distressed Trader & 1 Predator

Holdings of Distressed Trader & 50 Predators
Impact of the Number of Predators: $\gamma = 15.5\lambda$
Expected Price: $\gamma = \lambda$
Expected Price: $\gamma = 15\lambda$
Impact of Nb of Predators on Expected Returns

Expected Returns of Distressed Trader $GOL=1 \& GOL=15$

Expected Returns of Predators $GOL=1 \& GOL=15$
Two Period Model

- Prey has to liquidate $X_0 > 0$ by time $T_1$, i.e. $X_0(T_1) = 0$
- Predators can stay in the game longer $X_i(0) = X_i(T_2) = 0$ for some $T_2 > T_1$ for $i = 1, \ldots, n$
- Prey does not trade in second period $[T_1, T_2]$, i.e. $X_0(t) = 0$ for $T_1 \leq t \leq T_2$.

Markovian Structure $\implies$

Solution determined by predators’ positions at time $T_1$
UNIQUE Nash Equilibrium

- **ALL** Predators have the same position at time $T_1$

$$X_i(T_1) = \frac{A_2 n^2 + A_1 n + A_0}{B_3 n^3 + B_2 n^2 + B_1 n + B_0} X_0, \quad i = 1, \ldots, n$$

- Coefficients depend upon $n$ but converge as $n \rightarrow \infty$
- Asymptotic formulas for expected returns
- Asymptotic comparison of **Stealth** versus Sunshine trading for some regimes of $\gamma/\lambda$

Schöneborn - Schied (2008)