Optimal Execution:
II. Trade Optimal Execution

René Carmona

Bendheim Center for Finance
Department of Operations Research & Financial Engineering
Princeton University

Purdue June 21, 2012
Optimal Execution Set-Up

Goal: sell $x_0 > 0$ shares by time $T > 0$

- $X = (X_t)_{0 \leq t \leq T}$ execution strategy
- $X_t$ position (nb of shares held) at time $t$. $X_0 = x_0$, $X_T = 0$
- Assume $X_t$ absolutely continuous (differentiable)

- $\tilde{P}_t$ mid-price (unaffected price), $P_t$ transaction price, $I_t$ price impact

\[
P_t = \tilde{P}_t + I_t
\]

e.g. Linear Impact A-C model:

\[
I_t = \gamma [X_t - X_0] + \lambda \dot{X}_t
\]

- **Objective:** Maximize *form of revenue* at time $T$

Revenue $\mathcal{R}(X)$ from the execution strategy $X$

\[
\mathcal{R}(X) = \int_0^T (-\dot{X}_t)P_t dt
\]
Specific Challenges

- **First generation**: Price impact models (e.g. Almgren - Chriss)
  - Risk Neutral framework (maximize $\mathbb{E} R(X)$) versus utility criteria
  - More complex portfolios (including options)
  - Robustness and performance constraints (e.g. slippage or tracking market VWAP)

- **Second generation**: Simplified LOB models
  - Simple liquidation problem
  - performance constraints (e.g. slippage or tracking market VWAP) and using both market and limit orders
Optimal Execution Problem in A-C Model

\[ \mathcal{R}(X) = \int_0^T (-\dot{X}_t) P_t dt \]

\[ = -\int_0^T \dot{X}_t \tilde{P}_t dt - \int_0^T X_t I_t dt \]

\[ = x_0 \tilde{P}_0 + \int_0^T X_t d\tilde{P}_t - C(X) \]

with \( C(X) = \int_0^T \dot{X}_t I_t dt \). Interpretation

- \( x_0 \tilde{P}_0 \) (initial) **face value** of the portfolio to liquidate
- \( \int_0^T X_t d\tilde{P}_t \) **volatility risk** for selling according to \( X \) instead of immediately!
- \( C(X) \) **execution costs** due to market impact
Special Case: the Linear A-C Model

\[ \mathcal{R}(X) = x_0 \tilde{P}_0 + \int_0^T X_t d\tilde{P}_t - \lambda \int_0^T \dot{X}_t^2 \, dt - \frac{\gamma}{2} x_0^2 \]

**Easy Case:** Maximizing \( \mathbb{E}[\mathcal{R}(X)] \)

\[ \mathbb{E}[\mathcal{R}(X)] = x_0 P_0 - \frac{\gamma}{2} x_0^2 - \lambda \mathbb{E} \int_0^T \dot{X}_t^2 \, dt \]

Jensen’s inequality & constraints \( X_0 = x_0 \) and \( X_T = 0 \) imply

\[ \dot{X}_t^* = -\frac{x_0}{T} \]

trade at a constant rate **indepdt of volatility** ! **Bertsimas - Lo (1998)**
Almgren - Chriss propose to maximize

\[ \mathbb{E}[\mathcal{R}(X)] - \alpha \text{var}[\mathcal{R}(X)] \]

(\(\alpha\) risk aversion parameter – late trades carry volatility risk)

For **DETERMINISTIC** trading strategies \(X\)

\[ \mathbb{E}[\mathcal{R}(X)] - \alpha \text{var}[\mathcal{R}(X)] = x_0 P_0 - \frac{\gamma}{2} x_0^2 - \int_0^T \left( \frac{\alpha \sigma^2}{2} X_t^2 + \lambda \dot{X}_t^2 \right) dt \]

maximized by (standard variational calculus with constraints)

\[ \dot{X}_t^* = x_0 \frac{\sinh \kappa (T - t)}{\sinh \kappa T} \quad \text{for} \quad \kappa = \sqrt{\frac{\alpha \sigma^2}{2\lambda}} \]

For **RANDOM** (adapted) trading strategies \(X\), more difficult as **Mean-Variances not amenable to dynamic programming**
Maximizing Expected Utility

Choose $U : \mathbb{R} \rightarrow \mathbb{R}$ increasing concave and

$$\text{maximize} \quad \mathbb{E}[U(R(X_T))]$$

Stochastic control formulation over a state process $(X_t, R_t)_{0 \leq t \leq T}$.

$$\nu(t, x, r) = \sup_{\xi \in \Xi(t, x)} \mathbb{E}[u(R_T)|X_t = x, R_T = r]$$

denoted as the value function, where $\Xi(t, x)$ is the set of admissible controls

$$\left\{ \xi = (\xi_s)_{t \leq s \leq T}; \text{progressively measurable, } \int_t^T \xi_s^2 ds < \infty, \int_t^T \xi_s ds = x \right\}$$

$$X_s = X_s^{\xi} = x - \int_t^s \xi_u du, \quad \dot{X}_s = -\xi_s, \quad X_t = x$$

and (choosing $\tilde{P}_t = \sigma W_t$)

$$R_s = R_s^{\xi} = R + \sigma \int_t^s X_u dW_u - \lambda \int_t^s \xi_u^2 du, \quad dR_s = \sigma X_s dW_s - \lambda \xi_s^2 ds, \quad R_t = r$$
Finite Fuel Problem

Non Standard Stochastic Control problem because of the constraints

\[ \int_0^T \xi_s ds = x_0. \]

Still, one expects

- For any admissible \( \xi \), \( [v(t, X_{t}^{\xi}, R_{t}^{\xi})]_{0 \leq t \leq T} \) is a super-martingale
- For some admissible \( \xi^* \), \( [v(t, X_{t}^{\xi^*}, R_{t}^{\xi^*})]_{0 \leq t \leq T} \) is a true martingale

If \( v \) is smooth, and we set \( V_t = v(t, X_{t}^{\xi}, R_{t}^{\xi}) \), Itô’s formula gives

\[
dV_t = \left( \partial_t v(t, X_t, R_t) + \frac{\sigma^2}{2} \partial_{rr}^2 v(t, X_t, R_t) 
- \lambda \xi_t^2 \partial_r v(t, X_t, R_t) - \xi_t \partial_x v(t, X_t, R_t) \right) dt \\
+ \sigma \partial_x v(t, X_t, R_t) dW_t
\]
Hamilton-Jabobi-Bellman Equation

One expects that $v$ solves the HJB equation (nonlinear PDE)

$$\partial_t v + \frac{\sigma^2}{2} \partial^2_{xx} v - \inf_{\xi \in \mathbb{R}} [\xi^2 \lambda \partial_r v + \xi \partial_x v] = 0$$

in some sense, with the (non-standard) terminal condition

$$v(T, x, r) = \begin{cases} 
U(r) & \text{if } x = X_0 \\
-\infty & \text{otherwise}
\end{cases}$$
Solution for CARA Exponential Utility

For \( u(x) = -e^{-\alpha x} \) and \( \kappa \) as before

\[
v(t, x, r) = e^{-\alpha r + x_0^2 \alpha \lambda \kappa \coth \kappa (T - t)}
\]

solves the HJB equation and the unique maximizer is given by the DETERMINISTIC

\[
\xi_t^* = x_0 \kappa \frac{\cosh \kappa (T - t)}{\sinh \kappa T}
\]

Schied-Schöneborn-Tehranchi (2010)

- Optimal solution same as in Mean - Variance case
- Schied-Schöneborn-Tehranchi’s trick shows that optimal trading strategy is generically deterministic for exponential utility
- Open problem for general utility function
- Partial results in infinite horizon versions
Shortcomings

- Optimal strategies
  - are **DETERMINISTIC**
  - do not react to **price changes**
  - are **time inconsistent**
  - are **counter-intuitive** in some cases

- Computations require
  - solving **nonlinear PDEs**
  - with **singular** terminal conditions
In the spirit of Almgren-Chriss mean-variance criterion, maximize

\[ \mathbb{E} \left[ \mathcal{R}(X) - \tilde{\lambda} \int_0^T X_t \tilde{P}_t dt \right] \]

The solution happens to be ROBUST

\( \tilde{P}_t \) can be a semi-martingale, optimal solution does not change
Recent Developments

Almgren - Li (2012), Hedging a large option position

- \( g(t, \tilde{P}_t) \) price at time \( t \) of the option (from Black-Scholes theory)
- Revenue

\[
\mathcal{R}(X) = g(T, \tilde{P}_T) + X_T \tilde{P}_T - \int_0^T \tilde{P}_t \dot{X}_t dt - \lambda \int_0^T \dot{X}_t^2 dt
\]

- Using Itô's formula and the fact that \( g \) solves a PDE,

\[
\mathcal{R}(X) = R_0 + \int_0^T [X_t + \partial_x g(t, \tilde{P}_t)] dt - \lambda \int_0^T \dot{X}_t^2 dt \quad R_0 = x_0 \tilde{P}_0 + g(0, \tilde{P}_0)
\]

- Introduce \( Y_t = X_t + \partial_x g(t, \tilde{P}_t) \) for hedging correction

\[
\begin{align*}
    d\tilde{P}_t &= \gamma \dot{X}_t dt + \sigma dW_t \\
    dY_t &= [1 + \gamma \partial_{xx} g(t, \tilde{P}_t)] dt + \sigma \partial_{xx} g(t, \tilde{P}_t) dW_t
\end{align*}
\]

- Minimize

\[
\mathbb{E} \left[ G(\tilde{P}_T, Y_T) + \int_0^T \left( \frac{\sigma^2}{2} Y_t^2 - \gamma \dot{X}_t Y_t + \lambda \dot{X}_t^2 \right) dt \right]
\]

Explicit solution in some cases (e.g. \( \partial_{xx} g(t, x) = c \), \( G \) quadratic)
Transient Price Impact

Flexible price impact model

- **Resilience function** $G : (0, \infty) \to (0, \infty)$ measurable bounded
- Admissible $X = (X_t)_{0 \leq t \leq T}$ cadlag, adapted, **bounded variation**
- Transaction price

$$ P_t = \tilde{P}_t + \int_0^t G(t - s) \, dX_s $$

- Expected cost of strategy $X$ given by

$$ -x_0 P_0 + \mathbb{E} [C(X)] $$

where

$$ C(X) = \int \int G(|t - s|) \, dX_s \, dX_t $$
Transient Price Impact: Some Results

- No **Price Manipulation** in the sense of *Huberman - Stanzl (2004)* if $G(| \cdot |)$ positive definite
- Optimal strategies (if any) are **deterministic**
- Existence of an optimal $X^*$ ⇔ solvability of a Fredholm equation
- Exponential Resilience $G(t) = e^{-\rho t}$

$$dX_t^* = -\frac{x_0}{\rho T + 2} \left( \delta_0(dt) + \rho dt + \delta_T(dt) \right)$$

- $X^*$ purely discrete measure on $[0, T]$ when $G(t) = (1 - \rho t)^+$ with $\rho > 0$
  - $dX_t^* = -\frac{x_0}{2} \left[ \delta_0(dt) + \delta_T(dt) \right]$ if $\rho < 1/T$
  - $dX_t^* = -\frac{x_0}{n+1} \sum_{i=0}^{n} \delta_{iT/n}(dt)$ if $\rho < n/T$ for some integer $n \geq 1$

*Obizhaeva - Wang (2005), Gatheral - Schied (2011)*
Optimal Execution in a LOB Model

- Unaffected price $\tilde{P}_t$ (e.g. $\tilde{P}_t = P_0 + \sigma W_t$)
- Trader places only market sell orders
  - Placing buy orders is not optimal
- Bid side of LOB given by a function $f : \mathbb{R} \rightarrow (0, \infty)$ s.t. $\int_0^\infty f(x)dx = \infty$. At any time $t$

$$\int_a^b f(x)dx = \text{bids available in the price range } [\tilde{P}_t + a, \tilde{P}_t + b]$$

- The shape function $f$ does not depend upon $t$ or $\tilde{P}_t$

Optimal Execution in a LOB Model (cont.)

- **Price Impact** process $D = (D_t)_{0 \leq t \leq T}$ adapted, cadlag.
  At time $t$ a market order of size $A$ moves the price from $\tilde{P}_t + D_{t-}$ to $\tilde{P}_t + D_t$ where
  $$\int_{D_{t-}}^{D_t} f(x) dx = A$$

- **Volume Impact** $Q_t = F(D_t)$ where $F(x) = \int_0^x f(x') dx'$.

- **LOB Resilience**: $Q_t$ and $D_t$ decrease between trades, e.g.
  $$dQ_t = -\rho Q_t dt, \quad \text{for some } \rho > 0$$

- At time $t$, a sell of size $A$ will bring
  $$\int_{D_{t-}}^{D_t} (\tilde{P}_t + x) f(x) dx = A\tilde{P}_t + \int_{D_{t-}}^{D_t} x dF(x)$$
  $$= A\tilde{P}_t + \int_{Q_{t-}}^{Q_t} \psi(x) dx = A\tilde{P}_t + \psi(Q_t) - \psi(Q_{t-})$$
  if $\psi = F^{-1}$ and $\psi(x) = \int_0^x \psi(x') dx'$. 
Holding trajectories / Trading strategies

\( \Xi(t, x) = \left\{ (\Xi_s)_{t \leq s \leq T} : \text{càdlàg, adapted, bounded variation, } \Xi_t = x \right\} \)

\( \Xi_{ac}(t, x) = \left\{ (\Xi_s)_{t \leq s \leq T} : \Xi_s = x + \int_t^s \xi_r dr \text{ for } (\xi_s)_{t \leq s \leq T} \text{ bounded adapted} \right\} \)

\[
\begin{align*}
  dX_t &= -d\Xi_t \\
  dQ_t &= -d\Xi_t - \rho Q_t dt \\
  dR_t &= -\rho Q_t \psi(Q_t) dt - \sigma \Xi_t dW_t
\end{align*}
\]
Value Function Approach

State space process $Z_t = (X_t, Q_t, R_t)$, value function

$$v(t, x, q, r) = v(t, z) = \sup_{\xi \in \Omega(t, x)} \mathbb{E}[U(R_T - \Psi(Q_T))]$$

First properties

- $U(r - \Psi(q + r)) \leq v(t, x, q, r) \leq U(r - \Psi(q))$
- $v(t, x, q, r) = U(r - \Psi(q + r))$ for $x = 0$ and $t = T$
- Functional approximation arguments imply

$$v(t, x, q, r) = \sup_{\xi \in \Omega(t, x)} \mathbb{E}[U(R_T - \Psi(Q_T))]$$

$$= \sup_{\xi \in \Omega_{ac}(t, x)} \mathbb{E}[U(R_T - \Psi(Q_T))]$$

$$= \sup_{\xi \in \Omega_d(t, x)} \mathbb{E}[U(R_T - \Psi(Q_T))]$$
QVI Formulation

As before

- Assume \( v \) smooth and apply Itô’s formula to \( v(t, X_t, Q_t, R_t) \)
- \( v(t, X_t, Q_t, R_t) \) is a super-martingale for a typical \( \xi \) implies

\[
\partial_t v + \frac{\sigma^2}{2} x^2 \partial_{rr} v - \rho q \psi(q) \partial_r v - \rho q \partial_q v \geq 0
\]

- \( \partial_x v - \partial_q v \geq 0 \)

**QVI (Quasi Variational Inequality)** instead of **HJB** nonlinear PDE

\[
\min[\partial_t v + \frac{\sigma^2}{2} x^2 \partial_{rr} v - \rho q \psi(q) \partial_r v - \rho q \partial_q v, \partial_x v - \partial_q v] = 0
\]

with terminal condition \( v(T, x, q, r) = U(r - \Psi(x + q)) \)

**Existence and Uniqueness of a viscosity solution**

Special Cases

Assuming a **flat** LOB \( f(x) = c \) and \( U(c) = x \)

\[
v(t, x, q, r) = r - \frac{q^2(1 - e^{-2\rho s})}{2c} - \frac{(x + qe^{-\rho s})^2}{c(2 + \rho(T - t - s))}
\]

with \( s = (T - t) \wedge \inf\{u \in [0, T]; (1 + \rho(T - t - u))qe^{-\rho u} \leq x\} \)

Still with \( f(x) = c \) but for a CARA utility \( U(x) = -e^{-\alpha x} \)

\[
v(t, x, q, r) = -\exp\left[-\alpha r - \frac{\alpha}{2c}\left(\alpha c\sigma^2 xx^2 + q^2(1 - e^{-2\rho s}) + \varphi(t+s)(x + qe^{-\rho s})^2\right)\right]
\]

where \( \varphi \) is the solution of the Riccati's equation

\[
\dot{\varphi}(t) = \frac{\rho^2}{2\rho + \alpha c\sigma^2} \varphi(t)^2 + \frac{2\rho\alpha c\sigma^2}{2\rho + \alpha c\sigma^2} \varphi(t) - \frac{2\rho\alpha c\sigma^2}{2\rho + \alpha c\sigma^2}, \quad \varphi(T) = 1
\]

and

\[
s = (T - t) \wedge \inf\{u \in [0, T]; (\alpha c\sigma^2 + \rho \varphi(t + u))x \geq \rho(2 - \varphi(t + u))qe^{-\rho u}\} \]