# The Agents

## Market Maker

- Nasdaq definition: agent that places competitive orders on both sides of the order book in exchange for privileges.
- In this lecture: **Liquidity provider**, someone who posts an order book (equivalently, a transaction cost curve).
- Strategy: adapt pricing and volumes by *reading client flows*.

## Clients

- In this lecture: **Liquidity takers**, agents who trade with the Market maker.
- Clients place market orders.
- Each client has his/her *own information* and acts accordingly.
Theoretical literature

- **Early approaches**: Hasbrouck (2007), Chakrborti - Toke - Patriarca - Abergel (2011)
- **Inventory models**: Garman (1976), Amihud - Mendelson (1980)
**Objective: Endogenous Order Book**

Propose a **stochastic, agent-based** model in which existence and *(tractable and realistic)* properties of the LOB appear as a result of the analysis *(not as hypotheses)*

<table>
<thead>
<tr>
<th>Client model</th>
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<tbody>
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<td>➤ Should capture the dependence between trades and price dynamics.</td>
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<td>➤ Assumes the clients are rational, and optimizes his/her order book choice</td>
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R.C. - K. Webster (2012)
Setup: Heterogeneous Beliefs

Mathematically

1. \((\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) with \(W\) a \(\mathbb{P}\)-BM that generates \(\mathbb{F}\).
2. \(\mathbb{F}^k \subset \mathbb{F}\) generated by a \(\mathbb{P}\)-BM \(W^k\).
3. \(\mathbb{P}^k\) s.t. \(\mathbb{P}^k|_{\mathcal{F}_t} \sim \mathbb{P}|_{\mathcal{F}_t}\).
4. \(P_t\) an Itô process adapted to all \((\mathbb{F}^k)_{k=0\ldots n}\).

NB

- Each agent has his /her own filtration & probability measure.
- The filtrations (information structures) are potentially different,
- The price process is adapted to all of them (i.e each client sees the price)
Anatomy of a Trade

- **Midprice** $P_t$ announced by the market at time $t$
- **Market maker** proposes an **order book** around $P_t$
- **Market maker** cannot differentiate clients **pre-trade**
- **Client** triggers a **trade** of volume $l_t$
- **Client** obtains volume $l_t$ and pays **cash flow** $P_t l_t + c_t(l_t)$
  ($l \leftrightarrow c_t(l)$ transaction cost function at time $t$)
- **Market maker** learns the **identity** of the client **post-trade**
  (assumption depends upon market, true for FX)
Setup: Transaction Costs

Agents behaviors
- Market maker controls transaction cost function \( \ell \mapsto c_t(\ell) \).
- Client \( i \) controls trading volumes/speeds \( l_i \).

Hypotheses

1. **Marginal costs** are defined: \( \ell \mapsto c_t(\ell) \) is differentiable in \( \ell \).
2. Clients may choose **not to trade**, \( c_t(0) = 0 \)
3. The **midprice** is well defined, \( c'_t(0) = 0 \).
4. **Marginal costs** increase with volume: \( c_t \) is convex.
5. \( c_t \) has ”compact domain” (\( \infty \) outside an interval)
Duality Relationship

Legendre transform

\[
\gamma_t(\alpha) := \sup_{l \in \text{supp}(c_t)} (\alpha l - c_t(l))
\]

Duality

\(c_t\) convex with compact domain \(\iff\ \gamma_t''\) is a positive finite measure.

- The distribution \(\gamma_t''\) represents the order book formed by the orders of the market maker.
- If \(\gamma_t''\) has a density \(f(x)\), it is the shape function we used earlier.
Disclaimer: *We are NOT* trying to implement an optimal trading strategy.

### Assumptions

- The client only tries to *predict*, not *cause* price movements.
- The client’s decision does not affect $c_t$. 
Client Optimization Problem

- **Exogeneous state variables**
  - $P_t$ non-negative Itô process
  - $c_t$ (random adapted) convex function in a fixed domain

- **Endogeneous state variables**

  \[
  \begin{cases} 
  dL^i_t = l^i_t dt \\
  dX^i_t = L^i_t dP_t - c_t(l^i_t) dt 
  \end{cases}
  \]

  - $l^i_t$ rate at which client trades (control variable).
  - $L^i_t$ volume or total position of the client
  - $X^i_t$ wealth, marked to the mid-price.

- **Objective function**

  \[ J^i = E_{\mathbb{P}^i} \left[ U^i(X^i_{\tau^i}, P_{\tau^i}) \right] \]

  - $U^i$ utility function
  - $\tau^i$ stopping time
Optimal Trading Strategy

Theorem

Under suitable integrability assumptions on $U^i$ and $\tau^i$, the optimal strategy is

$$\alpha_t^i := c_t'(l_t^i) = \mathbb{E}_{Q^i} \left[ P_{\tau^i} - P_t \mid \mathcal{F}_t^i \right]$$

with

$$\frac{dQ^i}{dP^i} = \frac{\partial_X U^i(X_{\tau^i}^i, P_{\tau^i})}{\mathbb{E}_{P^i} \left[ \partial_X U^i(X_{\tau^i}^i, P_{\tau^i}) \right]}.$$
Testing the Client Model

Hypotheses

- Under $Q^i$, $\tau^i \sim \exp(\beta^i)$ independent of $P_t$.
- $\sigma^i_t := \left| c_t' \left( \frac{r^i_t}{l^i_t} \right) - (p_{\tau^i} - P_t) \right| \leq \frac{\text{spread}}{2}$

This leads to a two parameter model linking trade to price dynamics: $(\beta^i, \sigma^i)$.

Testing the hypotheses on data

- Assume all clients have one of two time scales.
- Choose $(\beta_1, \beta_2)$ that minimizes error between implied and realized alpha.
Source

- Nasdaq ‘fullview’ data: all public quotes, all trades, nanosecond timestamps.
- Long parsing time: Data goes from 7:00-10:00am.
Two Time Scales

- $L^1$ regression used.
- Time scales: 9 ($\approx 0.5$ seconds) and 158 ticks.
- Mean error: 0.026.
- Mean half-spread: 0.063.
- Lower bound on error: 0.005.
Market Maker Optimization Problem

With **primal** variables

\[
\begin{align*}
\frac{dL_t}{dt} &= -\frac{1}{n} \sum_i l^i_t dt \\
\frac{dX_t}{dt} &= L_t dP_t + \frac{1}{n} \sum_i c_t(l^i_t) dt
\end{align*}
\]

Recall \( \alpha^i_t = c^i_t(l^i_t) \) so equivalently \( l^i_t = [c^i_t]^{-1}(\alpha^i_t) = \gamma^i_t(\alpha^i_t) \)

With **dual** variables

\[
\begin{align*}
\frac{dL_t}{dt} &= -\frac{1}{n} \sum_i \gamma^i_t(\alpha^i_t) dt \\
\frac{dX_t}{dt} &= L_t dP_t + \frac{1}{n} \sum_i [\alpha^i_t \gamma^i_t(\alpha^i_t) - \gamma_t(\alpha^i_t)] dt
\end{align*}
\]

We assume the market maker is **risk-neutral**
Model for the $\alpha^i_t$

- **Notation**
  We will denote by $\mu_t(\alpha)$ the client belief distribution, that is, the empirically observed distribution of the $(\alpha^i_t)$.

- **Microscopic model (SDE)**
  
  \[
  d\alpha^i_t = -\rho \alpha^i_t \, dt + \sigma dB^i_t + \nu dB_t
  \]

  Mean reversion corresponds to decay of information.

- **Macroscopic model (SPDE)**
  
  \[
  d\mu_t(\alpha) = \left[ \frac{1}{2} (\sigma^2 + \nu^2) \Delta \mu_t(\alpha) + \rho \nabla (\alpha \mu_t(\alpha)) \right] dt - \nu \nabla \mu_t(\alpha) dB_t
  \]
What does that tell us about $P_t$?

▶ **Intuition**
  - Do not want to make an explicit model for the price process.
  - Instead, would like to *infer* the price from client trades.

▶ **Implied alpha relationship**

\[
\alpha^i_t := c^i_t(l^i_t) = \mathbb{E}_Q^i \left[ \int_t^\infty e^{-\beta^i(t-s)} dP_s \bigg| \mathcal{F}^i_t \right]
\]

▶ **Price Proxy**

\[
dP^\lambda_t := \sum_{i=1}^n \lambda^i \left( \beta^i \alpha^i_t dt - d\alpha^i_t \right)
\]

for any set of weights $\lambda^i$ s.t. $\sum \lambda^i = 1.$
Estimation Result

Entropic feedback

There exists $\lambda$ s.t.

$$\mathbb{E} \left| P_t - P_t^\lambda \right|^2 \leq \epsilon^2 \frac{1}{n} \sum_i E(Q^i, P) \approx -\epsilon^2 \int_0^t \left\langle \log \left( \frac{\gamma_s''}{\mu_s} \right), \mu_s \right\rangle ds$$

with $E$ the relative entropy (Kullback - Leibler) and

$$\epsilon = \sqrt{\frac{n}{\sum_i (\sigma^i)^{-2}}} \leq \frac{1}{n} \sum_i \sigma^i$$
Approximate Control Problem

State variables

\[
\begin{aligned}
\begin{cases}
\ dL_t &= -\langle \gamma'_t, \mu_t \rangle \ dt \\
\ d\mu_t(\alpha) &= \left[ \frac{1}{2} \left( \sigma^2 + \nu^2 \right) \Delta \mu_t(\alpha) + \rho \nabla (\alpha \mu_t(\alpha)) \right] \ dt - \nu \nabla \mu_t(\alpha) dB_t 
\end{cases}
\end{aligned}
\]

Objective function

\[
J^\lambda = \int_0^\infty e^{-\beta t} \mathbb{E} \left[ L_t \langle \text{id}, (\beta \lambda)_t \rangle + \langle -L_t \beta \text{id} + (\text{id} - \bar{\alpha}_t) \gamma'_t - \gamma_t, \mu_t \rangle \right] \ dt
\]

under the constraint \( \int_0^\infty \left\langle e^{-\beta t} \log \left( \frac{\gamma''_t}{\mu_t} \right), \mu_t \right\rangle \ dt \leq C. \)
(Pontryagin) Stochastic Maximum Principle

**BSDE**

The solution to the Pontryagin BSDE gives rise to the market maker’s ‘shadow alpha’:

\[
\alpha^*_t = \left\langle id, \lambda_t + \frac{(\beta \lambda)_t - \beta \mu_t}{\beta + \rho} \right\rangle
\]

**Hamiltonian**

\[
\mathcal{H}(\gamma, \mu, \alpha^*) = \left\langle (id - \alpha^*)\gamma' - \gamma + \epsilon \log \gamma'', \mu \right\rangle
\]
Profitability of an order without feedback

Define

\[ m(\alpha) = (\alpha - \alpha^*) \cdot \underbrace{\int_0^\infty}_\text{spread} \underbrace{\mu}_{\text{filling probability}} \quad \text{if } \alpha \geq 0 \]

then we have:

\[ \mathcal{H}(\gamma, \mu, \alpha^*) = \langle \gamma'', m \rangle + \epsilon \langle \log \gamma'', \mu \rangle \]

Optimal Strategy with Feedback

\[ \gamma''(\alpha) \quad \mu(\alpha) = \frac{\epsilon}{C - m(\alpha)} \]

where \( C \) is a renormalization constant.
Simulation Example

Figure: Blue: Optimal order book $\gamma''$. Green: Client alpha distribution $\mu$. 