Optimal Execution Tracking a Benchmark

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Optimal Execution Market Set-Up

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Goal: sell $v > 0$ shares by time $T > 0$ (finite horizon)

- $P_t$ mid-price (unaffected price),
  \[ P_t = P_0 + \int_0^t \sigma(u) dW_u, \quad 0 \leq t \leq T, \]

- $V(t)$ volume traded in the market up to (and including) time $t$

- Market VWAP = $\frac{1}{V} \int_0^T P_t dV(t)$

- Fraction of shares still to be executed in the market
  \[ X(t) = \frac{V - V(t)}{V} = \frac{T - t}{T} \]

  (deterministic $V(t)$ used to change clock). Convenient simplification!
Broker Problem

\( v_t \) volume executed by the broker up to time \( t \)

\[ x_t = \frac{V - v_t}{V} \]

fraction of shares left to be executed by the broker at time \( t \)

\[ x_t = 1 - \ell_t - m_t \]

Where

- \( \ell_t \) cumulative volume executed through limit orders
- \( m_t \) cumulative volume executed through market orders
- Broker average liquidation price \( \text{vwap} = \frac{1}{V} \int_0^T \left( P_t - \frac{S}{2} \right) \, dt + \left( P_t + \frac{S}{2} \right) \, d\ell_t \)
- **Objective:** Minimize discrepancy between \( \text{vwap} \) and VWAP
Controls of the broker:

- \((m_t)_{0 \leq t \leq T}\) non-decreasing adapted process
- \((L_t)_{0 \leq t \leq T}\) predictable process

\[
\ell_t = \int_0^t \int_{[0,1]} y \wedge L_u \mu(du, dy) = \sum_{i=1}^{N_t} Y_i \wedge L_{\tau_i}
\]

where

\[\mu(du, dy)\]

point measure (Poisson) compensator \(\nu_t(du)\nu(t)dt\).

\[
x_t = 1 - \int_0^t \int_{[0,1]} y \wedge L_u \mu(du, dy) - m_t = 1 - \sum_{i=1}^{N_t} Y_i \wedge L_{\tau_i} - m_t
\]

So the dynamics of \(x_t\) are given by

\[
dx_t = - \int_{[0,1]} y \wedge L_t \mu(dt, dy) - dm_t,
\]

with initial condition \(x_{0^-} = 1\).
Optimization Problem

Goal of the broker

\[
\sup_{(L,m) \in \mathcal{A}} \mathbb{E}\left[U(\text{vwap} - \text{VWAP})\right],
\]

For the CARA exponential utility, approximately

\[
\inf_{(L,m) \in \mathcal{A}} \mathbb{E}\left[\exp\left(-\gamma\left(\frac{S}{2} + \int_0^T [x_u^{L,m} - X(u)]dP_u - Sdm_u\right)\right]\right],
\]

We will work with a **Mean - Variance** criterion

\[
\inf_{(L,m) \in \mathcal{A}} \mathbb{E}\left[\int_0^T \gamma \frac{\sigma(u)^2}{2} [x_u^{L,m} - X(u)]^2 du + S m_T\right],
\]

- **S spread**
- **X(u) = (T - u)/T** fraction of shares left to be executed in the market.
Stochastic Control Problem

Singular control problem of a pure jump process

Value function

\[ J(t, x) = \inf_{(L, m) \in \mathcal{A}(t, x)} J(t, x, L, m) \]

where

\[ J(t, x, L, m) = \mathbb{E} \left[ \int_t^T \gamma \frac{\sigma(u)^2}{2} [x_{u,L,m}^L - X(u)]^2 du + Sm_T \right] . \]

\( J(t, x) \) is non-decreasing in \( t \) for \( x \in [0, 1] \) fixed. (\( \mathcal{A}(t_2, x) \subset \mathcal{A}(t_1, x) \) whenever \( t_1 \leq t_2 \))
The set $A$ of admissible controls is not convex.

For any number $\ell \in (0, 1)$, the two controls $(L^1, m^1)$ and $(L^2, m^2)$ by:

\[
L^1_t = 1_{\{t \leq \tau_1\}} + \sum_{k=2}^{\infty} x_{\tau_{k-1}} 1_{\{\tau_{k-1} < t \leq \tau_k\}}, \quad \text{and} \quad m^1_t = x_T 1_{\{T \leq t\}},
\]

and:

\[
L^2_t = \frac{\ell}{2} 1_{\{t \leq \tau_1\}} + \sum_{k=2}^{\infty} x_{\tau_{k-1}} 1_{\{\tau_{k-1} < t \leq \tau_k\}}, \quad \text{and} \quad m^2_t = x_T 1_{\{T \leq t\}},
\]

are admissible, but the pair $(L, m)$ defined by

\[
L_t = \frac{1}{2} (L^1_t + L^2_t), \quad \text{and} \quad m_t = \frac{1}{2} (m^1_t + m^2_t),
\]

IS NOT
Closest Related Works

- Poisson random measure \( \mu(dt, dy) \) for claim sizes \( Y_t \)
- **insurer** pays \( Y_t \wedge \alpha_t \) up to a **retention level** \( \alpha_t \)
- **re-insurer** covers the excess \( (Y_t - \alpha_t)^+ \)

Wealth process of the Insurance Company

\[
X_t = x + \int_0^t p(\alpha_s) ds - \int_0^t y \wedge \alpha_s \mu(ds, dy) - \int_0^t dD_s
\]

- \( p(\alpha) \) insurer net premium (after paying the reinsurance company)
- \( D_t \) cumulative dividends paid up to (and including) time \( t \)

\[
\sup_{(\alpha_t), (D_t)} \mathbb{E} \left[ \int_0^\tau e^{-ru} dD_u \right]
\]

- time of bankruptcy \( \tau = \inf \{ t \geq 0; X_t \leq 0 \} \)

Similarities & Differences

**Similarities**
- $\alpha_t \leftrightarrow$ standing limit orders $L_t$
- $D_t \leftrightarrow$ cumulative market orders $m_t$

**Differences**
- We work in a **finite horizon** (PDEs instead of ODEs)
- We use a **Mean - Variance** criterion
- We exhibit a **classical** solution (as opposed to a viscosity solution)
- We derive a **system of ODEs** identifying
  - the value function
  - the optimal strategy
Technical Assumptions

\( \nu_t(dy)\nu(t)dt \) intensity of Poisson measure \( \mu(dt, dy) \) with \( \nu_t([0, 1]) = 1. \)

- \( \int_0^T \sigma(t)^2 dt < \infty \)
- \( \sup_{0 \leq t \leq T} \nu(t) < \infty \)
- \( t \mapsto \frac{\sigma(t)^2}{\nu(t)} (X(t) - x) \) is increasing for each \( x \in [0, 1] \)
- \( t \mapsto \frac{1}{\nu(t)} \nu_t(\cdot) \) is decreasing (in the sense of stochastic dominance)
Hamilton-Jabobi-Bellman Equation (QVI)

\[
\min \left[ [A\phi](t, x), \partial_t \phi(t, x) + [B\phi](t, x) \right] = 0.
\]

where

\[
[A\phi](t, x) = S - \partial_x \phi(t, x)
\]

and

\[
[B\phi](t, x) = \gamma \frac{\sigma(t)^2}{2} [X(t) - x]^2 + \nu(t) \inf_{0 \leq L \leq x} \int_{[0, 1]} \left[ \phi(t, x - y \land L) - \phi(t, x) \right] \nu_t(dy)
\]

with terminal condition

\[
\phi(T, x) = Sx, \quad \text{(notice that } \phi(T, x) = 0)\]

and boundary condition:

\[
\phi(t, 0) = \int_t^T \frac{\gamma \sigma(u)^2}{2} X(u) du.
\]
Theorem

The value function is the unique solution of

\[-\dot{J}(t, x) = \min \left\{ \inf_{0 \leq y \leq x} -\dot{J}(t, x), \right.\]

\[
\gamma \frac{\sigma(t)^2}{2} [X(t) - x]^2 + \nu(t) \int_{[0,1]} [J(t, (x - y) \lor \tilde{L}(t, y)) - J(t, x)] \nu_t(dy) \left. \right\}
\]

with

\[J(t, 0) = \gamma \int_0^t \frac{\sigma(u)^2}{2} X(u)^2 du, \quad \text{and} \quad J(T, x) = Sx\]

where

\[\tilde{L}(t, x) = \arg \min_{0 \leq y \leq x} J(t, y)\]

- \(J\) is \(C^{1,1}\)
- \(x \mapsto J(t, x)\) convex for \(t\) fixed
- \(t \mapsto J(t, x)\) non-decreasing for \(x\) fixed
- \(\partial_x \dot{J}(t, x) \geq 0\)
Free Boundary (No-Trade Region)

\[ [0, T] \times [0, 1] = A \cup B \cup C \]

with

- \( A = \{(t, x); \partial_x J(t, x) < 0\} = \{(t, x); 0 \leq t < \tau_\ell(x)\} \)
- \( B = \{(t, x); 0 \leq \partial_x J(t, x) \leq S\} = \{(t, x); \tau_\ell(x) \leq t \leq \tau_m(x)\} \)
- \( C = \{(t, x); \partial_x J(t, x) = S\} = \{(t, x); \tau_m(x) \leq t\} \)

where

- \( \tau_\ell(x) = \inf\{t > 0; \partial_x J(t, x) \geq 0\} \)
- \( \tau_m(x) = \inf\{t > 0; \partial_x J(t, x) \geq S\} \)

\[ \tau_\ell(x) \leq T(1 - x) \leq \tau_m(x) \]
Optimal Trading Strategy

- If $t > \tau_m(x_t)$ i.e. $(t, x_t) \in C$ (never happens)
  - place market orders
  $\Delta m_t > 0$ (just enough to get into $B$)
  
- If $t = \tau_m(x_t)$ i.e. $(t, x_t) \in \partial C$
  - place market orders at a rate $dm_t = -\dot{\tau}_m(x_t)dt$
  (just enough so not to exit $B$)

- If $\tau_\ell(x_t) \leq t < \tau_m(x_t)$ i.e. $(t, x_t) \in B \cup \partial A$
  - place $L_t = x_t - \tilde{L}(t)$ limit orders
  (as much as possible without getting ahead too much)

- If $t < \tau_\ell(x_t)$ i.e. $(t, x_t) \in A$ (never happens)
  - no trade
Special Case I: Large Fill Distribution

\( \nu_t(dy) = \delta_1(dy) \): the crossings, when they occur, fill all the requested limit orders.

**Theorem**

The value function solves

\[
-\dot{J}(t, x) = \min \left[ \inf_{0 \leq y \leq x} -\dot{J}(t, x), \gamma \frac{\sigma(t)^2}{2} [X(t) - x]^2 + \nu(t)[J(t, \tilde{L}(t, x)) - J(t, x)] \right]
\]

with

\[
J(t, 0) = \gamma \int_0^t \frac{\sigma(u)^2}{2} X(u)^2 du, \quad \text{and} \quad J(T, x) = Sx
\]
Special Case II: Arrival Price Benchmark

This specific model corresponds to the case $X(\tau) = 0$ for all $\tau \in [0, T]$.

**Theorem**

The value function is the unique solution of

$$-\dot{J}(t, x) = \min \left[ \inf_{0 \leq y \leq x} -J(t, x), \gamma \frac{\sigma(t)^2}{2} x^2 + \nu(t) \int_{[0,1]} [J(t, (x - y)^+) - J(t, x)] \nu_t(dy) \right]$$

with

$$J(t, 0) = \gamma \int_0^t \frac{\sigma(u)^2}{2} X(u)^2 du, \quad \text{and} \quad J(T, x) = Sx$$
When \((t, x)\) is far enough from the corners \((0, 1)\) and \((T, 0)\), \(J\) looks like a function of \(x - X(t)\) (\textit{deviation from the benchmark}).

Stationarity assumption

- \(\nu_1(dt) = \lambda dt\) for some constant \(\lambda > 0\)
- \(\nu_t(dy) = \nu(dy)\) for all \(t \in [0, T]\).
- \(\sigma(t) = \sigma\) for all \(t \in [0, T]\)

Look for an approximation of the form

\[
J(t, x) \approx \alpha + \beta x + w(x - X(t))
\]

for some function \(w\) to be determined.

True in the \textbf{Large Fill case} (use the Lambert function)
The Discrete Case and Approximation Results

- The integer \( v \) denotes the quantity of shares (expressed as a number of lots) the broker has to sell by time \( T \).
- Trades can only be in multiples of one lot.
- \( t \leftrightarrow x_t \) looks like a staircase starting from \( x_0 = 1 \) and ending at \( x_T = 0 \).
- In units of \( v \) lots, the measures \( \nu_t(dy) \) are supported by the grid \( \{1/v, 2/v, \ldots, (v - 1)/v, 1\} \).
- The process \( x = (x_t)_{0 \leq t \leq T} \) and the controls \( L = (L_t)_{0 \leq t \leq T} \) and \( m = (m_t)_{0 \leq t \leq T} \) take values in the grid \( I_v := \{0, 1/v, \ldots, (v - 1)/v, 1\} \).
- The sets of admissible controls are defined accordingly.

- Identify functions \( \varphi \) on the grid \( I_v \) with finite sequence \( (\varphi_i)_{0 \leq i \leq v} \) where \( \varphi_i = \varphi(i/v) \).
- Denote by \( l_\varphi \) the piecewise linear continuous function \( [0, 1] \ni x \leftrightarrow [l_\varphi](x) \) which coincides with \( \varphi \) on the grid \( I_v \) and which is linear on each interval \( [i/v, (i + 1)/v] \).
- \( (\varphi_i)_{0 \leq i \leq v} \) is said to be convex if \( l_\varphi \) is convex
- For any integers \( v \) and \( v' \), and functions \( \varphi \) and \( \varphi' \) on the grids \( I_v \) and \( I_{v'} \), we have:

\[
\|l_\varphi - l_{\varphi'}\|_\infty = \sup_{x \in [0,1]} |[l_\varphi](x) - [l_{\varphi'}](x)| = \sup_{x \in I_v \cup I_{v'}} |[l_\varphi](x) - [l_{\varphi'}](x)|.
\]
Characterization of the Solution

The operators $A$ and $B$ become

$$[A\varphi]_i(t) = S - \varphi_i(t) + \varphi_{i-1}(t), \quad i = 1, \ldots, v,$$

and

$$[B\varphi]_i(t) = \gamma \frac{\sigma(t)^2}{2} [X(t) - i/v]^2 + \nu(t) \min_{0 \leq \ell \leq i} \sum_{j=1}^v [\varphi_{i-j} \wedge \ell(t) - \varphi_i(t)] \nu_t(j/v)$$

so the HJB QVI remains the same:

$$\min \left[ [A\varphi]_i(t), \dot{\varphi}_i(t) + [B\varphi]_i(t) \right] = 0, \quad i = 1, \ldots, v.$$ 

As before we have existence and uniqueness of a $C^1$ functions of $t \in [0, T]$ satisfying

$$\varphi_i(t) = Si/v + \int_t^T \min_{0 \leq j \leq i} [B\varphi]_i(u) du, \quad i = 0, 1, \ldots, v.$$

Interpreting the solution $\varphi$ as a function on $[0, T] \times \mathcal{I}_v$ defined by $\varphi(t, i/v) = \varphi_i(t)$, since $\varphi_i(T) = Si/v$ and:

$$\dot{\varphi}_i(t) = - \min_{0 \leq j \leq i} [B\varphi]_j(t)$$

we get

$$\dot{\varphi}_i(t) + [B\varphi]_i(t) \geq 0, \quad i = 0, 1, \ldots, v$$

and

$$\dot{\varphi}_i(t) = \max_{0 \leq j \leq i} \partial_t \varphi_j(t)$$

so that $i \mapsto \dot{\varphi}_i(t)$ is non-decreasing and

$$-\dot{\varphi}_i(t) = \min \left[ \min_{0 \leq j \leq i} -\dot{\varphi}_j(t), [B\varphi]_i(t) \right].$$
**Theorem**

The value function $J$ of the problem can be identified to the sequence $(J_i)_{0 \leq i \leq v}$ of $C^1$ functions of $t \in [0, T]$ satisfying:

\[
\begin{aligned}
J_0(t) &= \int_t^T \frac{\gamma \sigma(u)^2}{2} X(u)^2, \\
J_i(T) &= Si/v, \\ 
\partial_t J_i(t) &= \min \left[ \partial_t J_{i-1}(t), \\
&\quad \nu(t) \sum_{j=1}^{v} [\varphi_{(i-j)} \tilde{\ell}_i(t) - \varphi_i(t)] \nu_t(j) + \frac{\gamma \sigma(t)^2}{2} [X(t) - i/v]^2 \right]
\end{aligned}
\]

where

\[
\tilde{\ell}_i(t) = \min \{\ell; \varphi_\ell(t) = \min_{0 \leq j \leq i} \varphi_j(t)\}
\]
Optimal Solution in the Discrete Case

- $\tau^m_i = \inf\{t \in [0, T]; J_i(t) - J_{i-1}(t) < S/v\}$
- $\tau^l_i = \inf\{t \in [0, T]; J_i(t) - J_{i-1}(t) < 0\}$
- $\tau^m_i \leq Tx(i - \frac{1}{2}) < \tau^l_i$