European Call on the difference between two indexes
Calendar Spread Options

- Single Commodity at two different times
  \[ \mathbb{E}\{(I(T_2) - I(T_1) - K)^+\} \]
- Mathematically easier (only one underlier)

**Amaranth** largest (and fatal) positions

- Shoulder Natural Gas Spread (play on inventories)
- **Long** March Gas / **Short** April Gas
  - Depletion stops in March / injection starts in April
  - Can be fatal: *widow maker spread*
There is a long injection season from the spring through the fall when natural gas is injected and stored in caverns for use during the long winter to meet the higher residential demand, as in Figure 2.1. The figure illustrates the U.S. Department of Energy's total (lower 48 states) working underground storage for natural gas inventories over 2006. Inventories stop being drawn down in March and begin to rise in April. As we will see in Section 2.1.3.2, the summer and fall futures contracts, when storage is rising, trade at a discount to the winter contracts, when storage peaks and levels off. Thus, the markets provide a return for storing natural gas. A storage operator can purchase summer futures and sell winter futures, the difference being the return for storage. At maturity of the summer contract, the storage owner can move the delivered physical gas into storage and release it when the winter contract matures. Storage is worth more if such spread bets are steep between near and far months.

2.1.3 Risk Management Instruments

Futures and forward contracts, swaps, spreads and options are the most standard tools for speculation and risk management in the natural gas market.
November 2006 bets were particularly large compared to the rest, as Amaranth accumulated the largest ever long position in the November futures contract in the month preceding its downfall. Regarding the Fund’s overall strategy, Burton and Strasburg (2006a) write that Amaranth was generally long winter contracts and short summer and fall ones, a winning bet since 2004. Other sources affirm that Amaranth was long the far-end of the curve and short the front-end, and their positions lost value when far-forward gas contracts fell more than near-term contracts did in September 2006.

From these bets, Amaranth believed a stormy and exceptionally cold winter in 2006 would result in excess usage of natural gas in the winter and a shortage in March of the following year. Higher demand would result in a possible stockout by the end of February and higher March prices. Yet April prices would fall as supply increases at the start of the injection season. In this scenario, there is theoretically no ceiling on how much the price of the March contract can rise relative to the rest of the curve. Fischer (2006), natural gas trader at Chicago-based hedge fund Citadel Investment Group, believes Amaranth bet on similar hurricane patterns in the previous two years. As a result, the extreme event that hurt Amaranth was that nothing happened—there was no Hurricane Katrina or similar.

![Shoulder Month Spread](image-url)
Cross Commodity

- Crush Spread: between Soybean and soybean products (meal & oil)
- Crack Spread:
  - gasoline crack spread between Crude and Unleaded
  - heating oil crack spread between Crude and HO
- Spark spread

\[ S_t = F_E(t) - H_{\text{eff}} F_G(t) \]

\( H_{\text{eff}} \) Heat Rate
Present value of profits for future power generation (case of one fuel)

\[
\mathbb{E}\left\{ \int_0^T D(0, t)(\tilde{F}_P(t, \tau) - H \cdot \tilde{F}_G(t, \tau) - K)^+ \, dt \right\}
\]

where

- \( \tau > 0 \) fixed (small)
- \( D(0, t) \) discount factor to compute present values
- \( \tilde{F}_P(t, \tau) \) (resp. \( \tilde{F}_G(t, \tau) \)) price at time \( t \) of a power (resp. gas) contract with delivery \( t + \tau \)
- \( H \) Heat Rate
- \( K \) Operation and Maintenance cost (sometimes denoted \( O&M \))
Basket of Spread Options

**Deterministic** discounting (with constant interest rate)

\[ D(t, T) = e^{-r(T-t)} \]

Interchange **expectation** and **integral**

\[ \int_{0}^{T} e^{-rt} \mathbb{E}\{(\tilde{F}_{P}(t, \tau) - H \ast \tilde{F}_{G}(t, \tau) - K)^{+}\} \, dt \]

Continuous **stream of spread options**

**In Practice**

- **Discretize time**, say daily

\[ \sum_{t=0}^{T} e^{-rt} \mathbb{E}\{(\tilde{F}_{P}(t, \tau) - H \ast \tilde{F}_{G}(t, \tau) - K)^{+}\} \]

- **Bin** Daily Production in **Buckets** \( B_k \)'s (e.g. \( 5 \times 16, 2 \times 16, 7 \times 8 \), settlement locations, .....).

\[ \sum_{t=0}^{T} e^{-r(T-t)} \sum_{k} \mathbb{E}\{(\tilde{F}_{P}^{(k)}(t, \tau) - H^{(k)} \ast \tilde{F}_{G}^{(k)}(t, \tau) - K^{(k)})^{+}\} \]

**Basket of Spark Spread Options**
\[ p = e^{-rT} \mathbb{E}\{(I_2(T) - I_1(T) - K)^+\} \]

- Underlying indexes are spot prices
  - Geometric Brownian Motions (\(K = 0\) Margrabe)
  - Geometric Ornstein-Uhlenbeck (OK for Gas)
  - Geometric Ornstein-Uhlenbeck with jumps (OK for Power)
- Underlying indexes are forward/futures prices
  - HJM-type models with deterministic coefficients

**Problem**

Finding closed form formula and/or fast/sharp approximation for

\[ \mathbb{E}\{(\alpha e^{\gamma X_1} - \beta e^{\delta X_2} - \kappa)^+\} \]

for a Gaussian vector \((X_1, X_2)\) of \(N(0, 1)\) random variables with correlation \(\rho\).

**Sensitivities?**
Easy Case: Exchange Option & Margrabe Formula

\[
p = e^{-rT} \mathbb{E} \{(S_2(T) - S_1(T))^+ \}
\]

- \(S_1(T)\) and \(S_2(T)\) log-normal
- \(p\) given by a formula à la Black-Scholes

\[
p = x_2 N(d_1) - x_1 N(d_0)
\]

with

\[
d_1 = \frac{\ln(x_2/x_1)}{\sigma \sqrt{T}} + \frac{1}{2} \sigma \sqrt{T} \quad d_0 = \frac{\ln(x_2/x_1)}{\sigma \sqrt{T}} - \frac{1}{2} \sigma \sqrt{T}
\]

and:

\[
x_1 = S_1(0), \quad x_2 = S_2(0), \quad \sigma^2 = \sigma_1^2 - 2 \rho \sigma_1 \sigma_2 + \sigma_2^2
\]

- Deltas are also given by "closed form formulae".

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Proof of Margrabe Formula

\[ p = e^{-rT}E_Q \{ (S_2(T) - S_1(T))^+ \} = e^{-rT}E_Q \left\{ \left( \frac{S_2(T)}{S_1(T)} - 1 \right)^+ S_1(T) \right\} \]

- \( Q \) risk-neutral probability measure
- Define (Girsanov) \( P \) by:

\[
\left. \frac{dP}{dQ} \right|_{\mathcal{F}_T} = S_1(T) = \exp \left( -\frac{1}{2} \mu_1^2 T + \mu_1 \hat{W}_1(T) \right)
\]

- Under \( P \),
  - \( \hat{W}_1(t) - \sigma_1 t \) and \( \hat{W}_2(t) \)
  - \( S_2/S_1 \) is geometric Brownian motion under \( P \) with volatility

\[
\sigma^2 = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2
\]

\[ p = S_1(0)E_P \left\{ \left( \frac{S_2(T)}{S_1(T)} - 1 \right)^+ \right\} \]

**Black-Scholes** formula with \( K = 1, \sigma \) as above.
Pricing Calendar Spreads in Forward Models

Model

\[ dF(t, T) = F(t, T)[\mu(t, T)dt + \sum_{k=1}^{n} \sigma_k(t, T)dW_k(t)] \]

\( \mu(t, T) \) and \( \sigma_k(t, T) \) deterministic so

**forward prices are log-normal**

**Calendar Spread** involves prices of two forward contracts with different maturities

\[ S_1(t) = F(t, T_1) \quad \text{and} \quad S_2(t) = F(t, T_2), \]

Price at time \( t \) of a calendar spread option with maturity \( T \) and strike \( K \)

\[ \mathbb{E}\{(F(T, T_2) - F(T, T_1) - K)^*\} \]
Use formula for
\[ \mathbb{E}\{ (\alpha e^{\gamma X_1} - \beta e^{\delta X_2} - \kappa)^+ \} \]

with
\[ \alpha = e^{-r(T-t)} F(t, T_2), \quad \beta = \sqrt{\sum_{k=1}^{n} \int_t^T \sigma_k(s, T_2)^2 ds}, \]
\[ \gamma = e^{-r(T-t)} F(t, T_1), \quad \text{and} \quad \delta = \sqrt{\sum_{k=1}^{n} \int_t^T \sigma_k(s, T_1)^2 ds} \]

and \[ \kappa = e^{-r(T-t)} (\mu \equiv 0 \text{ per risk-neutral dynamics}) \]
\[ \rho = \frac{1}{\beta \delta} \sum_{k=1}^{n} \int_t^T \sigma_k(s, T_1) \sigma_k(s, T_2) \, ds \]
Pricing Spark Spreads in Forward Models

Cross-commodity
- subscript \( e \) for forward prices, times-to-maturity, volatility functions, ... relative to electric power
- subscript \( g \) for quantities pertaining to natural gas.

Pay-off

\[
(F_e(T, T_e) - H \cdot F_g(T, T_g) - K)^+.
\]

- \( T < \min\{T_e, T_g\} \)
- Heat rate \( H \)
- Strike \( K \) given by O&M costs

Natural
- **Buyer** owner of a power plant that transforms gas into electricity,
- **Protection** against low electricity prices and/or high gas prices.
Joint Dynamics of the Commodities

\[
\begin{align*}
    dF_e(t, T_e) &= F_e(t, T_e)[\mu_e(t, T_e)dt + \sum_{k=1}^{n} \sigma_{e,k}(t, T_e)dW_k(t)] \\
    dF_g(t, T_g) &= F_g(t, T_g)[\mu_g(t, T_g)dt + \sum_{k=1}^{n} \sigma_{g,k}(t, T_g)dW_k(t)]
\end{align*}
\]

- Each commodity has its own volatility factors
- between The two dynamics share the **same** driving Brownian motion processes $W_k$, hence **correlation**.
Fitting Join Cross-Commodity Models

on any given day $t$ we have
- electricity forward contract prices for $N^{(e)}$ times-to-maturity
  $\tau_{1}^{(e)} < \tau_{2}^{(e)} , \ldots < \tau_{N^{(e)}}^{(e)}$
- natural gas forward contract prices for $N^{(g)}$ times-to-maturity
  $\tau_{1}^{(g)} < \tau_{2}^{(g)} , \ldots < \tau_{N^{(g)}}^{(g)}$

Typically $N^{(e)} = 12$ and $N^{(g)} = 36$ (possibly more).

- Estimate instantaneous vols $\sigma^{(e)}(t)$ & $\sigma^{(g)}(t)$ 30 days rolling window
- For each day $t$, the $N = N^{(e)} + N^{(g)}$ dimensional random vector $X(t)$

$$
X(t) = \begin{bmatrix}
\left( \frac{\log \tilde{F}_{e}(t+1, \tau_{j}^{(e)}) - \log \tilde{F}_{e}(t, \tau_{j}^{(e)})}{\sigma^{(e)}(t)} \right)_{j=1, \ldots, N^{(e)}} \\
\left( \frac{\log \tilde{F}_{g}(t+1, \tau_{j}^{(g)}) - \log \tilde{F}_{g}(t, \tau_{j}^{(g)})}{\sigma^{(g)}(t)} \right)_{j=1, \ldots, N^{(g)}}
\end{bmatrix}
$$

- Run PCA on historical samples of $X(t)$
- Choose small number $n$ of factors
- for $k = 1, \ldots, n$,
  - first $N^{(e)}$ coordinates give the electricity volatilities $\tau \mapsto \sigma^{(e)}_{k}(\tau)$ for $k = 1, \ldots, n$
  - remaining $N^{(g)}$ coordinates give the gas volatilities $\tau \mapsto \sigma^{(g)}_{k}(\tau)$.

Skip gory details
Pricing a Spark Spread Option

Price at time $t$

$$p_t = e^{-r(T-t)} \mathbb{E}_t \{ (F_e(T, T_e) - H \ast F_g(T, T_g) - K)^+ \}$$

$F_e(T, T_e)$ and $F_g(T, T_g)$ are log-normal under the pricing measure calibrated by PCA

$$F_e(T, T_e) = F_e(t, T_e) \exp \left[ -\frac{1}{2} \sum_{k=1}^{n} \int_t^T \sigma_{e,k}(s, T_e)^2 ds + \sum_{k=1}^{n} \int_t^T \sigma_{e,k}(s, T_e) dW_k(s) \right]$$

and:

$$F_g(T, T_g) = F_g(t, T_g) \exp \left[ -\frac{1}{2} \sum_{k=1}^{n} \int_t^T \sigma_{g,k}(s, T_g)^2 ds + \sum_{k=1}^{n} \int_t^T \sigma_{g,k}(s, T_g) dW_k(s) \right]$$

Set

$$S_1(t) = H \ast F_g(t, T_g) \quad \text{and} \quad S_2(t) = F_e(t, T_e)$$
Pricing a Spark Spread Option

Use the constants

\[ \alpha = e^{-r(T-t)} F_e(t, T_e), \quad \text{and} \quad \beta = \sqrt{\sum_{k=1}^{n} \int_t^T \sigma_{e,k}(s, T_e)^2 \, ds} \]

for the first log-normal distribution,

\[ \gamma = H e^{-r(T-t)} F_g(t, T_g), \quad \text{and} \quad \delta = \sqrt{\sum_{k=1}^{n} \int_t^T \sigma_{g,k}(s, T_g)^2 \, ds} \]

for the second one, \( \kappa = e^{-r(T-t)} K \) and

\[ \rho = \frac{1}{\beta \delta} \int_t^T \sum_{k=1}^{n} \sigma_{e,k}(s, T_e) \sigma_{g,k}(s, T_g) \, ds \]

for the correlation coefficient.
Fourier Approximations (Madan, Carr, Dempster, …)
Bachelier approximation
Zero-strike approximation
Kirk approximation
Upper and Lower Bounds

Can we also approximate the Greeks?
Bachelier Approximation

- Generate $x_1^{(1)}, x_2^{(1)}, \ldots, x_N^{(1)}$ from $N(\mu_1, \sigma_1^2)$
- Generate $x_1^{(2)}, x_2^{(2)}, \ldots, x_N^{(2)}$ from $N(\mu_1, \sigma_1^2)$
- Correlation $\rho$
- Look at the distribution of

$$e^{x_1^{(2)}} - e^{x_1^{(1)}}, e^{x_2^{(2)}} - e^{x_2^{(1)}}, \ldots, e^{x_N^{(2)}} - e^{x_N^{(1)}}$$
Log-Normal Samples

- $X_1$
- $\exp(X_1)$
- $X_2$
- $\exp(X_2)$

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Histogram of the Difference between two Log-normals
Bachelier Approximation

- Assume \((S_2(T) - S_1(T))\) is Gaussian
- Match the first two moments

\[
p = \left( m(T) - Ke^{-rT} \right) \Phi \left( \frac{m(T) - Ke^{-rT}}{s(T)} \right) + s(T) \varphi \left( \frac{m(T) - Ke^{-rT}}{s(T)} \right)
\]

with:

\[
m(T) = (x_2 - x_1)e^{(\mu - r)T}
\]

\[
s^2(T) = e^{2(\mu - r)T} \left[ x_1^2 \left( e^{\sigma_1^2T} - 1 \right) - 2x_1x_2 \left( e^{\rho\sigma_1\sigma_2T} - 1 \right) + x_2^2 \left( e^{\sigma_2^2T} - 1 \right) \right]
\]

Easy to compute the Greeks!
Zero-Strike Approximation

\[ p = e^{-rT} \mathbb{E}\{ (S_2(T) - S_1(T) - K)^+ \} \]

- Assume \( S_2(T) = F_E(T) \) is log-normal
- Replace \( S_1(T) = H \ast F_G(T) \) by \( \tilde{S}_1(T) = S_1(T) + K \)
- Assume \( S_2(T) \) and \( S_1(T) \) are jointly log-normal
- Use Margrabe formula for \( p = e^{-rT} \mathbb{E}\{ (S_2(T) - \tilde{S}_1(T))^+ \} \)

Use the Greeks from Margrabe formula!
\[ \hat{p}^K = x_2 \Phi \left( \ln \left( \frac{x_2}{x_1 + Ke^{-rT}} \right) + \frac{\sigma^K}{2} \right) - (x_1 + Ke^{-rT}) \Phi \left( \frac{\ln \left( \frac{x_2}{x_1 + Ke^{-rT}} \right)}{\sigma^K} - \frac{\sigma^K}{2} \right) \]

where

\[ \sigma^K = \sqrt{\sigma_2^2 - 2\rho \sigma_1 \sigma_2 x_1 + Ke^{-rT}} + \sigma_1^2 \left( \frac{x_1}{x_1 + Ke^{-rT}} \right)^2 \]

Exactly what we called ”Zero Strike Approximation”!!!
Upper and Lower Bounds

\[ \Pi(\alpha, \beta, \gamma, \delta, \kappa, \rho) = \mathbb{E} \left\{ \left( \alpha e^{\beta X_1 - \beta^2/2} - \gamma e^{\delta X_2 - \delta^2/2} - \kappa \right)^+ \right\} \]

where
- \( \alpha, \beta, \gamma, \delta \) and \( \kappa \) real constants
- \( X_1 \) and \( X_2 \) are jointly Gaussian \( \mathcal{N}(0, 1) \)
- correlation \( \rho \)
  \[
  \alpha = x_2 e^{-q_2 T} \quad \beta = \sigma_2 \sqrt{T} \quad \gamma = x_1 e^{-q_1 T} \quad \delta = \sigma_1 \sqrt{T} \quad \text{and} \quad \kappa = K e^{-rT}.
  \]
\[ \mathbb{E}\{X^+\} = \sup_{0 \leq Y \leq 1} \mathbb{E}\{XY\} \]

So in particular

\[ \mathbb{E}\{X^+\} \geq \sup_{u_1, u_2, d \in \mathbb{R}} \mathbb{E}\{X \mathbf{1}_{\{u_1 X_1 + u_2 X_2 \leq d\}}\} \]

and we apply this to

\[ X = \alpha e^{\beta x_1 - \beta^2 / 2} - \gamma e^{\delta x_2 - \delta^2 / 2} - \kappa \]

so everything can be computed!
A Precise Lower Bound

\[ \hat{\rho} = x_2 e^{-q_2 T} \Phi \left( d^* + \sigma_2 \cos(\theta^* + \phi) \sqrt{T} \right) - x_1 e^{-q_1 T} \Phi \left( d^* + \sigma_1 \sin \theta^* \sqrt{T} \right) - Ke^{-r T} \Phi(d^*) \]

where

- \( \theta^* \) is the solution of

\[ \frac{1}{\delta \cos \theta} \ln \left( -\frac{\beta \kappa \sin(\theta + \phi)}{\gamma [\beta \sin(\theta + \phi) - \delta \sin \theta]} \right) - \frac{\delta \cos \theta}{2} = \frac{1}{\beta \cos(\theta + \phi)} \ln \left( -\frac{\delta \kappa \sin \theta}{\alpha [\beta \sin(\theta + \phi) - \delta \sin \theta]} \right) - \frac{\beta \cos(\theta + \phi)}{2} \]

- the angle \( \phi \) is defined by setting \( \rho = \cos \phi \)

- \( d^* \) is defined by

\[ d^* = \frac{1}{\sigma \cos(\theta^* - \psi) \sqrt{T}} \ln \left( \frac{x_2 e^{-q_2 T} \sigma_2 \sin(\theta^* + \phi)}{x_1 e^{-q_1 T} \sigma_1 \sin \theta^*} \right) - \frac{1}{2} \left( \sigma_2 \cos(\theta^* + \phi) + \sigma_1 \cos \theta^* \right) \sqrt{T} \]

- the angles \( \phi \) and \( \psi \) are chosen in \([0, \pi]\) such that:

\[ \cos \phi = \rho \quad \text{and} \quad \cos \psi = \frac{\sigma_1 - \rho \sigma_2}{\sigma}, \]
Remarks on this Lower Bound

• \( \hat{p} \) is equal to the true price \( p \) when
  - \( K = 0 \)
  - \( x_1 = 0 \)
  - \( x_2 = 0 \)
  - \( \rho = -1 \)
  - \( \rho = +1 \)

• Margrabe formula when \( K = 0 \) because

\[
\theta^* = \pi + \psi = \pi + \arccos \left( \frac{\sigma_1 - \rho \sigma_2}{\sigma} \right).
\]

with:

\[
\sigma = \sqrt{\sigma_1^2 - 2 \rho \sigma_1 \sigma_2 + \sigma_2^2}
\]
The portfolio comprising at each time $t \leq T$

$$\Delta_1 = -e^{-q_1 T} \Phi \left( d^* + \sigma_1 \cos \theta^* \sqrt{T} \right)$$

and

$$\Delta_2 = e^{-q_2 T} \Phi \left( d^* + \sigma_2 \cos(\theta^* + \phi) \sqrt{T} \right)$$

units of each of the underlying assets is a sub-hedge

its value at maturity is a.s. a lower bound for the pay-off
\begin{itemize}
  \item $\vartheta_1$ and $\vartheta_2$ sensitivities w.r.t. volatilities $\sigma_1$ and $\sigma_2$
  \item $\chi$ sensitivity w.r.t. correlation $\rho$
  \item $\kappa$ sensitivity w.r.t. strike price $K$
  \item $\Theta$ sensitivity w.r.t. maturity time $T$
\end{itemize}

\begin{align*}
\vartheta_1 &= x_1 e^{-q_1 T} \varphi \left( d^* + \sigma_1 \cos \theta^* \sqrt{T} \right) \cos \theta^* \sqrt{T} \\
\vartheta_2 &= -x_2 e^{-q_2 T} \varphi \left( d^* + \sigma_2 \cos(\theta^* + \phi) \sqrt{T} \right) \cos(\theta^* + \phi) \sqrt{T} \\
\chi &= -x_1 e^{-q_1 T} \varphi \left( d^* + \sigma_1 \cos \theta^* \sqrt{T} \right) \frac{\sin \theta^*}{\sin \phi} \sqrt{T} \\
\kappa &= -\Phi(d^*) e^{-rT} \\
\Theta &= \frac{\sigma_1 \vartheta_1 + \sigma_2 \vartheta_2}{2T} - q_1 x_1 \Delta_1 - q_2 x_2 \Delta_2 - rK \kappa
\end{align*}
Behavior of the tracking error as the number of re-hedging times increases. The model data are $x_1 = 100, x_2 = 110, \sigma_1 = 10\%, \sigma_2 = 15\%$ and $T = 1, \rho = 0.9, K = 30$ (left) and $\rho = 0.6, K = 20$ (right).
Stylized Version

- **Leasing an Energy Asset**
  - Fossil Fuel Power Plant
  - Oil Refinery
  - Pipeline

- **Owner of the Agreement**
  - Decides *when* and *how* to use the asset (e.g. run the power plant)
  - Has someone else do the leg work
The Classical (Real Option) Approach

- Lifetime of the plant \([T_1, T_2]\)
- \(C\) capacity of the plant (in MWh)
- \(H\) heat rate of the plant (in MMBtu/MWh)
- \(P_t\) price of power on day \(t\)
- \(G_t\) price of fuel (gas) on day \(t\)
- \(K\) fixed Operating Costs
- Value of the Plant (ORACLE)

\[
C \sum_{t=T_1}^{T_2} e^{-rt} \mathbb{E}\{(P_t - HG_t - K)^+\}
\]

String of Spark Spread Options
Plant Operation Model: the Finite Mode Case

- Markov process (state of the world) \( X_t = (X_t^{(1)}, X_t^{(2)}, \cdots) \)
  (e.g. \( X_t^{(1)} = P_t, \quad X_t^{(2)} = G_t, \quad X_t^{(3)} = O_t \) for a dual plant)

- Plant characteristics
  - \( \mathbb{Z}_M \triangleq \{0, \cdots, M - 1\} \) modes of operation of the plant
  - \( H_0, H_1 \cdots, H_{M-1} \) heat rates
  - \( \{C(i, j)\}_{(i,j)\in\mathbb{Z}_M} \) regime switching costs \( C(i, j) = C(i, \ell) + C(\ell, j) \)
  - \( \psi_i(t, x) \) reward at time \( t \) when world in state \( x \), plant in mode \( i \)

- Operation of the plant (control) \( u = (\xi, T) \) where
  - \( \xi_k \in \mathbb{Z}_M \triangleq \{0, \cdots, M - 1\} \) successive modes
  - \( 0 \leq \tau_{k-1} \leq \tau_k \leq T \) switching times

- \( T \) (horizon) length of the tolling agreement

- Total reward

\[
H(x, i, [0, T]; u)(\omega) \triangleq \int_0^T -\psi_{us}(s, X_s) \, ds - \sum_{\tau_k < T} C(u_{\tau_{k-1}}, u_{\tau_k})
\]
Stochastic Control Problem

- \((\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t^X), \mathbb{P})\) (risk neutral) stochastic basis
- \(\mathcal{U}(t)\) acceptable controls on \([t, T]\)
  - adapted càdlàg \(\mathbb{Z}_M\)-valued processes \(u\) of a.s. finite variation on \([t, T]\)

**Optimal Switching Problem**

\[
J(t, x, i) = \sup_{u \in \mathcal{U}(t)} J(t, x, i; u),
\]

where

\[
J(t, x, i; u) = \mathbb{E}\left[H(x, i, [t, T]; u) \mid X_t = x, u_t = i\right]
\]

\[
= \mathbb{E}\left[\int_0^T -\psi_{us}(s, X_s) \, ds - \sum_{\tau_k < T} C(u_{\tau_k-}, u_{\tau_k}) \mid X_t = x, u_t = i\right]
\]
Iterative Optimal Stopping

\( U^k(t) \triangleq \{ (\xi, T) \in U(t) : \tau_\ell = T \text{ for } \ell \geq k + 1 \} \)

Admissible strategies on \([t, T]\) with at most \(k\) switches

\[
J^k(t, x, i) \triangleq \text{esssup}_{u \in U^k(t)} \mathbb{E} \left[ \int_t^T -\psi_{us}(s, X_s) \, ds - \sum_{t \leq \tau_k < T} C(u_{\tau_k -}, u_{\tau_k}) \mid X_t = x, u_t = i \right].
\]

Alternative recursive construction

\[
J^0(t, x, i) \triangleq \mathbb{E} \left[ \int_t^T -\psi_i(s, X_s) \, ds \mid X_t = x \right],
\]

\[
J^k(t, x, i) \triangleq \sup_{\tau \in S_t} \mathbb{E} \left[ \int_t^T -\psi_i(s, X_s) \, ds + \mathcal{M}^{k,i}(\tau, X_\tau) \mid X_t = x \right].
\]

Intervention operator \(\mathcal{M}\)

\[
\mathcal{M}^{k,i}(t, x) \triangleq \max_{j \neq i} \left\{ -C_{i,j} + J^{k-1}(t, x, j) \right\}.
\]

Studied mathematically by Hamadène - Jeanblanc (\(M = 2\)).
Alternative Formulations

- Variational Formulation and Viscosity Solutions of PDEs
- System of Reflected Backward Stochastic Differential Equations (BSDEs)
Discrete Time Dynamic Programming

- Time Step $\Delta t = T / M^\#$
- Time grid $S^\Delta = \{m\Delta t, m = 0, 1, \ldots, M^\#\}$
- Switches are allowed in $S^\Delta$

**DPP**

For $t_1 = m\Delta t$, $t_2 = (m + 1)\Delta t$ consecutive times

$$J^k(t_1, X_{t_1}, i) = \max \left( \mathbb{E} \left[ \int_{t_1}^{t_2} -\psi_i(s, X_s) \, ds + J^k(t_2, X_{t_2}, i) \mid \mathcal{F}_{t_1} \right] , \mathcal{M}^{k,i}(t_1, X_{t_1}) \right)$$

$$\simeq \left( \psi_i(t_1, X_{t_1}) \Delta t + \mathbb{E} \left[ J^k(t_2, X_{t_2}, i) \mid \mathcal{F}_{t_1} \right] \right) \lor \left( \max_{j \neq i} \left\{ -C_{i,j} + J^{k-1}(t_1, X_{t_1}, j) \right\} \right).$$

(1)

Tsitsiklis - van Roy
Recall

\[ J^k(m\Delta t, x, i) = \mathbb{E}\left[ \sum_{j=m}^{\tau_k} \psi_i(j \Delta t, X_{j\Delta t}) \Delta t + M^{k,i}(\tau^k \Delta t, X_{\tau^k \Delta t}) \mid X_{m\Delta t} = x \right]. \]

Analogue for \( \tau^k \):

\[ \tau^k(m\Delta t, x_{m\Delta t}, i) = \begin{cases} \tau^k((m + 1)\Delta t, x^\ell_{(m+1)\Delta t}, i), & \text{no switch;} \\ m, & \text{switch,} \end{cases} \]

and the set of paths on which we switch is given by \( \{\ell : \hat{j}^\ell(m\Delta t; i) \neq i\} \) with

\[ \hat{j}^\ell(t_1; i) = \arg \max_j \left( -C_{i,j} + J^{k-1}(t_1, x^\ell_{t_1}, j), \psi_i(t_1, x^\ell_{t_1}) \Delta t + \hat{E}_{t_1}[J^k(t_2, \cdot, i)](x^\ell_{t_1}) \right). \]

The full recursive pathwise construction for \( J^k \) is

\[ J^k(m\Delta t, x^\ell_{m\Delta t}, i) = \begin{cases} \psi_i(m\Delta t, x^\ell_{m\Delta t}) \Delta t + J^k((m + 1)\Delta t, x^\ell_{(m+1)\Delta t}, i), & \text{no switch;} \\ -C_{i,j} + J^{k-1}(m\Delta t, x^\ell_{m\Delta t}, j), & \text{switch to } j. \end{cases} \]
Remarks

- Regression used solely to update the optimal stopping times $\tau^k$
- Regressed values never stored
- Helps to eliminate potential biases from the regression step.
Algorithm

1. Select a set of basis functions \((B_j)\) and algorithm parameters \(\Delta t, M^\#, N^p, \bar{K}, \delta\).

2. Generate \(N^p\) paths of the driving process: \(\{x^\ell_{m\Delta t}, m = 0, 1, \ldots, M^\#, \ell = 1, 2, \ldots, N^p\}\) with fixed initial condition \(x^\ell_0 = x_0\).

3. Initialize the value functions and switching times \(J^k(T, x^\ell_T, i) = 0, \tau^k(T, x^\ell_T, i) = M^\# \forall i, k\).

4. Moving backward in time with \(t = m\Delta t, m = M^#, \ldots, 0\) repeat the Loop:
   - Compute inductively the layers \(k = 0, 1, \ldots, \bar{K}\) (evaluate \(\mathbb{E}[J^k(m\Delta t + \Delta t, \cdot, i) | \mathcal{F}_{m\Delta t}]\) by linear regression of \(\{J^k(m\Delta t + \Delta t, x^\ell_{m\Delta t + \Delta t}, i)\}\) against \(\{B_j(x^\ell_{m\Delta t})\}_{j=1}^{N^B}\), then add the reward \(\psi_i(m\Delta t, x^\ell_{m\Delta t}) \cdot \Delta t\)
   - Update the switching times and value functions

5. end Loop.

6. Check whether \(\bar{K}\) switches are enough by comparing \(J^{\bar{K}}\) and \(J^{\bar{K}-1}\) (they should be equal).

Observe that during the main loop we only need to store the buffer \(J(t, \cdot), \ldots, J(t + \delta, \cdot)\); and \(\tau(t, \cdot), \ldots, \tau(t + \delta, \cdot)\).
Example 1

\[ dX_t = 2(10 - X_t) \, dt + 2 \, dW_t, \quad X_0 = 10, \]

- Horizon \( T = 2, \)
- Switch separation \( \delta = 0.02. \)
- Two regimes
- Reward rates \( \psi_0(X_t) = 0 \) and \( \psi_1(X_t) = 10(X_t - 10) \)
- Switching cost \( C = 0.3. \)
$J^k(t, x, 0)$ as a function of $t$
Exercise Boundaries

$k = 2$ (left)  
$k = 7$ (right)

NB: Decreasing boundary around $t = 0$ is an artifact of the Monte Carlo.
State process and boundaries

Cumulative wealth

Time Units

Carmona
Energy Markets, Munich
Example 2: Comparisons

Spark spread $X_t = (P_t, G_t)$

\[
\begin{align*}
\log(P_t) & \sim \text{OU}(\kappa = 2, \theta = \log(10), \sigma = 0.8) \\
\log(G_t) & \sim \text{OU}(\kappa = 1, \theta = \log(10), \sigma = 0.4)
\end{align*}
\]

- $P_0 = 10$, $G_0 = 10$, $\rho = 0.7$
- Agreement Duration $[0, 0.5]$
- Reward functions
  \[
  \begin{align*}
  \psi_0(X_t) &= 0 \\
  \psi_1(X_t) &= 10(P_t - G_t) \\
  \psi_2(X_t) &= 20(P_t - 1.1G_t)
  \end{align*}
  \]
- Switching costs
  \[
  C_{i,j} = 0.25|i - j|
  \]
**Numerical Comparison**

<table>
<thead>
<tr>
<th>Method</th>
<th>Mean</th>
<th>Std. Dev</th>
<th>Time (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Explicit FD</td>
<td>5.931</td>
<td>—</td>
<td>25</td>
</tr>
<tr>
<td>LS Regression</td>
<td>5.903</td>
<td>0.165</td>
<td>1.46</td>
</tr>
<tr>
<td>TvrR Regression</td>
<td>5.276</td>
<td>0.096</td>
<td>1.45</td>
</tr>
<tr>
<td>Kernel</td>
<td>5.916</td>
<td>0.074</td>
<td>3.8</td>
</tr>
<tr>
<td>Quantization</td>
<td>5.658</td>
<td>0.013</td>
<td>400*</td>
</tr>
</tbody>
</table>

**Table:** Benchmark results for Example 2.
Example 3: Dual Plant & Delay

\[
\begin{align*}
\log(P_t) & \sim OU(\kappa = 2, \theta = \log(10), \sigma = 0.8), \\
\log(G_t) & \sim OU(\kappa = 1, \theta = \log(10), \sigma = 0.4), \\
\log(O_t) & \sim OU(\kappa = 1, \theta = \log(10), \sigma = 0.4),
\end{align*}
\]

- \(P_0 = G_0 = O_0 = 10\), \(\rho_{pg} = 0.5\), \(\rho_{po} = 0.3\), \(\rho_{go} = 0\)
- Agreement Duration \(T = 1\)
- Reward functions

\[
\begin{aligned}
\psi_0(X_t) & \equiv 0 \\
\psi_1(X_t) & = 5 \cdot (P_t - G_t) \\
\psi_2(X_t) & = 5 \cdot (P_t - O_t) \\
\psi_3(X_t) & = 5 \cdot (3P_t - 4G_t) \\
\psi_4(X_t) & = 5 \cdot (3P_t - 4O_t).
\end{aligned}
\]

- Switching costs \(C_{i,j} \equiv 0.5\)
- Delay \(\delta = 0, 0.01, 0.03\) (up to ten days)
Numerical Results

Setting | No Delay | $\delta = 0.01$ | $\delta = 0.03$
---|---|---|---
Base Case | 13.22 | 12.03 | 10.87
Jumps in $P_t$ | 23.33 | 22.00 | 20.06
Regimes 0-3 only | 11.04 | 10.63 | 10.42
Regimes 0-2 only | 9.21 | 9.16 | 9.14
Gas only: 0, 1, 3 | 9.53 | 7.83 | 7.24

**Table:** LS scheme with 400 steps and 16000 paths.

**Remarks**
- High $\delta$ lowers profitability by over 20%.
Example 4: Exhaustible Resources

Include $l_t$ current level of resources left ($l_t$ non-increasing process).

$$J(t, x, c, i) = \sup_{\tau, j} \mathbb{E} \left[ \int_t^\tau -\psi_i(s, X_s) \, ds + J(\tau, X_\tau, l_\tau, j) - C_{i,j} \mid X_t = x, l_t = c \right].$$ (5)

- Resource depletion (boundary condition) $J(t, x, 0, i) \equiv 0$.
- Not really a control problem $l_t$ can be computed on the fly

Mining example of Brennan and Schwartz varying the initial copper price $X_0$

<table>
<thead>
<tr>
<th>Method/ $X_0$</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>BS ’85</td>
<td>1.45</td>
<td>4.35</td>
<td>8.11</td>
<td>12.49</td>
<td>17.38</td>
<td>22.68</td>
</tr>
<tr>
<td>PDE FD</td>
<td>1.42</td>
<td>4.21</td>
<td>8.04</td>
<td>12.43</td>
<td>17.21</td>
<td>22.62</td>
</tr>
<tr>
<td>RMC</td>
<td>1.33</td>
<td>4.41</td>
<td>8.15</td>
<td>12.44</td>
<td>17.52</td>
<td>22.41</td>
</tr>
</tbody>
</table>
Extensions

- Extension to **Gas Storage** valuation
- Extension to **Hydro** valuation
- Improve the theoretical results
  - Need to improve delays
  - Need **convergence analysis**
  - Need better analysis of **exercise boundaries**
  - Need to implement duality upper bounds
    - we have approximate value functions
    - we have approximate exercise boundaries
    - so we have lower bounds
Extending the Analysis Adding Access to a Financial Market

Porchet-Touzi

- Same (Markov) factor process $X_t = (X_t^{(1)}, X_t^{(2)}, \cdots)$ as before
- Same plant characteristics as before
- Same operation control $u = (\xi, T)$ as before
- Same maturity $T$ (end of tolling agreement) as before
- **Reward** for operating the plant

$$H(x, i, T; u)(\omega) \overset{\triangle}{=} \int_0^T -\psi_{u_s}(s, X_s) \, ds - \sum_{\tau_k < T} C(u_{\tau_k-}, u_{\tau_k})$$
Hedging/Investing in Financial Market

Access to a financial market (possibly incomplete)

- $y$ initial wealth
- $\pi_t$ investment portfolio
- $Y_T^{y,\pi}$ corresponding terminal wealth from investment

**Utility function** $U(y) = -e^{-\gamma y}$

Maximal expected utility

$$v(y) = \sup_{\pi} \mathbb{E}\{U(Y_T^{y,\pi})\}$$
Indifference Pricing

With the power plant (tolling contract)

\[ V(x, i, y) = \sup_{u, \pi} \mathbb{E}\{ U(Y_T^y, \pi) + H(x, i, T; u) \} \]

INDIFFERENCE PRICING

\[ \overline{\rho} = p(x, i, y) = \sup\{ \rho \geq 0; V(x, i, y) \geq v(y) \} \]

Analysis of

- BSDE formulation
- PDE formulation
Spread Options, Swings, and Asset Valuation