Energy Markets II:
Spread Options, Weather Derivatives & Asset Valuation

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European Call on the difference between two indexes
Calendar Spread Options

- Single Commodity at two different times
  \[ \mathbb{E}\{(I(T_2) - I(T_1) - K)^+\} \]
- Mathematically easier (only one underlier)

**European Call on the difference between two indexes**

- **Calendar Spread**
- **Amaranth** largest (and fatal) positions
  - Shoulder Natural Gas Spread (play on inventories)
  - **Long** March Gas
  - **Short** April Gas
    - Depletion stops in March, injection starts in April
    - Can be fatal: emph\textbf{widow maker spread}
There is a long injection season from spring through the fall when natural gas is injected and stored in caverns for use during the long winter to meet the higher residential demand, as in FIGURE 2.1. The figure illustrates the U.S. Department of Energy's total (lower 48 states) working underground storage for natural gas inventories over 2006. Inventories stop being drawn down in March and begin to rise in April. As we will see in Section 2.1.3.2, the summer and fall futures contracts, when storage is rising, trade at a discount to the winter contracts, when storage peaks and levels off. Thus, the markets provide a return for storing natural gas. A storage operator can purchase summer futures and sell winter futures, the difference being the return for storage. At maturity of the summer contract, the storage owner can move the delivered physical gas into storage and release it when the winter contract matures. Storage is worth more if such spread bets are steep between near and far months.

2.1.3 Risk Management Instruments

Futures and forward contracts, swaps, spreads and options are the most standard tools for speculation and risk management in the natural gas market. Commodities market U.S. Natural Gas Inventories 2005-6

(from Raj Hatharamani ORFE Senior Thesis)
November 2006 bets were particularly large compared to the rest, as Amaranth accumulated the largest ever long position in the November futures contract in the month preceding its downfall. Regarding the Fund’s overall strategy, Burton and Strasburg (2006a) write that Amaranth was generally long winter contracts and short summer and fall ones, a winning bet since 2004. Other sources affirm that Amaranth was long the far-end of the curve and short the front-end, and their positions lost value when far-forward gas contracts fell more than near-term contracts did in September 2006.

From these bets, Amaranth believed a stormy and exceptionally cold winter in 2006 would result in excess usage of natural gas in the winter and a shortage in March of the following year. Higher demand would result in a possible stockout by the end of February and higher March prices. Yet April prices would fall as supply increases at the start of the injection season. In this scenario, there is theoretically no ceiling on how much the price of the March contract can rise relative to the rest of the curve. Fischer (2006), natural gas trader at Chicago-based hedge fund Citadel Investment Group, believes Amaranth bet on similar hurricane patterns in the previous two years. As a result, the extreme event that hurt Amaranth was that nothing happened—there was no Hurricane Katrina or similar event.
More Spread Options

- **Cross Commodity**
  - Crush Spread: between Soybean and soybean products (meal & oil)
  - Crack Spread:
    - gasoline crack spread between Crude and Unleaded
    - heating oil crack spread between Crude and HO
  - **Spark spread**

\[ S_t = F_E(t) - H_{eff} F_G(t) \]

\( H_{eff} \) **Heat Rate**
Synthetic Generation

Present value of profits for future power generation (case of one fuel)

\[ \mathbb{E}\left\{ \int_{0}^{T} D(0, t)(\tilde{F}_P(t, \tau) - H \cdot \tilde{F}_G(t, \tau) - K)^+ \, dt \right\} \]

where

- \( \tau > 0 \) fixed (small)
- \( D(0, t) \) discount factor to compute present values
- \( \tilde{F}_P(t, \tau) \) (resp. \( \tilde{F}_G(t, \tau) \)) price at time \( t \) of a power (resp. gas) contract with delivery \( t + \tau \)
- \( H \) Heat Rate
- \( K \) Operation and Maintenance cost (sometimes denoted \( O&M \))
Basket of Spread Options

**Deterministic** discounting (with constant interest rate)

\[ D(t, T) = e^{-r(T-t)} \]

Interchange *expectation* and *integral*

\[ \int_0^T e^{-rt} \mathbb{E}\{(\tilde{F}_P(t, \tau) - H \ast \tilde{F}_G(t, \tau) - K)^+\} \, dt \]

Continuous *stream of spread options*

**In Practice**

- Discretize time, say daily

\[ \sum_{t=0}^{T} e^{-rt} \mathbb{E}\{(\tilde{F}_P(t, \tau) - H \ast \tilde{F}_G(t, \tau) - K)^+\} \]

- Bin Daily Production in **Buckets** \( B_k \)'s (e.g. 5 \( \times \) 16, 2 \( \times \) 16, 7 \( \times \) 8, settlement locations, .....).

\[ \sum_{t=0}^{T} e^{-r(T-t)} \sum_{k} \mathbb{E}\{(\tilde{F}^{(k)}_P(t, \tau) - H^{(k)} \ast \tilde{F}^{(k)}_G(t, \tau) - K^{(k)})^+\} \]
\[ p = e^{-rT} \mathbb{E}\{(I_2(T) - I_1(T) - K)^+\} \]

- Underlying indexes are spot prices
  - Geometric Brownian Motions (\(K = 0\) Margrabe)
  - Geometric Ornstein-Uhlenbeck (OK for Gas)
  - Geometric Ornstein-Uhlenbeck with jumps (OK for Power)
- Underlying indexes are forward/futures prices
  - HJM-type models with deterministic coefficients

**Problem**

finding closed form formula and/or fast/sharp approximation for

\[ \mathbb{E}\{(\alpha e^{\gamma X_1} - \beta e^{\delta X_2} - \kappa)^+\} \]

for a Gaussian vector \((X_1, X_2)\) of \(N(0, 1)\) random variables with correlation \(\rho\).

**Sensitivities?**
Easy Case : Exchange Option & Margrabe Formula

\[ p = e^{-rT} \mathbb{E}\{(S_2(T) - S_1(T))^+\} \]

- \( S_1(T) \) and \( S_2(T) \) log-normal
- \( p \) given by a formula à la Black-Scholes

\[ p = x_2 \Phi(d_1) - x_1 \Phi(d_0) \]

with

\[ d_1 = \frac{\ln(x_2/x_1)}{\sigma \sqrt{T}} + \frac{1}{2} \sigma \sqrt{T} \quad d_0 = \frac{\ln(x_2/x_1)}{\sigma \sqrt{T}} - \frac{1}{2} \sigma \sqrt{T} \]

and:

\[ x_1 = S_1(0), \quad x_2 = S_2(0), \quad \sigma^2 = \sigma_1^2 - 2\rho \sigma_1 \sigma_2 + \sigma_2^2 \]

- Deltas are also given by "closed form formulae".
Proof of Margrabe Formula

\[ p = e^{-rT} \mathbb{E}_Q \{ (S_2(T) - S_1(T))^+ \} = e^{-rT} \mathbb{E}_Q \left\{ \left( \frac{S_2(T)}{S_1(T)} - 1 \right)^+ S_1(T) \right\} \]

- \( \mathbb{Q} \) risk-neutral probability measure
- Define (Girsanov) \( \mathbb{P} \) by:

\[ \frac{d\mathbb{P}}{d\mathbb{Q}} \bigg|_{\mathcal{F}_T} = S_1(T) = \exp \left( -\frac{1}{2} \sigma_1^2 T + \sigma_1 \hat{W}_1(T) \right) \]

- Under \( \mathbb{P} \),
  - \( \hat{W}_1(t) - \sigma_1 t \) and \( \hat{W}_2(t) \)
  - \( S_2/S_1 \) is geometric Brownian motion under \( \mathbb{P} \) with volatility

\[ \sigma^2 = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2 \]

\[ p = S_1(0) \mathbb{E}_\mathbb{P} \left\{ \left( \frac{S_2(T)}{S_1(T)} - 1 \right)^+ \right\} \]

**Black-Scholes** formula with \( K = 1 \), \( \sigma \) as above.
Real Option Approach

- Lifetime of the plant \([T_1, T_2]\)
- \(C\) capacity of the plant (in MWh)
- \(H\) heat rate of the plant (in MMBtu/MWh)
- \(P_t\) price of power on day \(t\)
- \(G_t\) price of fuel (gas) on day \(t\)
- \(K\) fixed Operating Costs
- Value of the Plant (ORACLE)

\[ C \sum_{t=T_1}^{T_2} e^{-rt} \mathbb{E}\{(P_t - HG_t - K)^+\} \]

String of Spark Spread Options
(Flash Back)

The Calpine - Morgan Stanley Deal

- Calpine needs to refinance USD 8 MM by November 2004
- **Jan. 2004**: Deutsche Bank: no traction on the offering
- **Feb. 2004**: *The Street* thinks Calpine is ”heading South”
- **March 2004**: Morgan Stanley offers a (complex) structured deal
  - A strip of spark spread options on 14 Calpine plants
  - A similar bond offering

**How were the options priced?**

- By Morgan Stanley ?
- By Calpine ?
Calpine Debt

The graph shows the debt of Calpine with the y-axis representing the debt in USD and the x-axis representing the years from 2005 to 2027. The debt peaks in 2007 and 2011.
Calpine Debt with Deutsche Bank Financing

Debt Distribution for Calpine with Deutsche Bank Refinancing

Year:
- 2006: $503
- 2008: $926
- 2010: $2440
- 2012: $3654
- 2015: $900
- 2017: $363
- 2019: $200
- 2021: $900
- 2023: $900
- 2025: $900

Debt ($Millions):
Debt Distribution for Calpine with Morgan Stanley Refinancing

- Year 2005: Debt $250 Million
- Year 2007: Debt $503 Million
- Year 2009: Debt $805 Million
- Year 2011: Debt $3959 Million
- Year 2013: Debt $280 Million
- Year 2015: Debt $900 Million
- Year 2017: Debt $1618 Million
- Year 2019: Debt $363 Million
- Year 2021: Debt $200 Million
- Year 2023: Debt $900 Million
- Year 2025: Debt $900 Million
- Year 2027: Debt $900 Million
A Possible Model

Assume that Calpine owns only one plant

**MS guarantees its spark spread will be at least** \( \kappa \) **for** \( M \) **years**

Approach à la **Leland’s Theory of the Value of the Firm**

\[
V = v - p_0 + \sup_{\tau \leq T} \mathbb{E} \left\{ \int_0^\tau e^{-rt} \delta_t \, dt \right\}
\]

where

\[
\delta_t = \begin{cases} 
(P_t - H \ast G_t - K) \vee \kappa - c_t & \text{if } 0 \leq t \leq M \\
(P_t - H \ast G_t - K)^+ - c_t & \text{if } M \leq t \leq T 
\end{cases}
\]

and

- \( v \) current value of firm’s assets
- \( p_0 \) option premium
- \( M \) length of the option life
- \( \kappa \) strike of the option
- \( c_t \) cost of servicing the existing debt
Expected Bankruptcy Time as function of Coupon

Default Time

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Plant Value

Plant Value as function of Coupon

- M = 0.1
- M = 0.2
- M = 0.5

COUPON

Plant Value

0 500 1000 1500 2000 2500

0 2 4 6 8 10 12 14
Debt Value as function of Coupon

Graph showing debt value varying with coupon, with different lines for M=0.1, M=0.2, and M=0.5.
Involves prices of two forward contracts with different maturities, say $T_1$ and $T_2$

$$S_1(t) = F(t, T_1) \quad \text{and} \quad S_2(t) = F(t, T_2),$$

**Remember** forward prices are log-normal

Price at time $t$ of a calendar spread option with maturity $T$ and strike $K$

$$\alpha = e^{-r[T-t]}F(t, T_2), \quad \beta = \sqrt{\sum_{k=1}^{n} \int_{t}^{T} \sigma_k(s, T_2)^2 ds},$$

$$\gamma = e^{-r[T-t]}F(t, T_1), \quad \text{and} \quad \delta = \sqrt{\sum_{k=1}^{n} \int_{t}^{T} \sigma_k(s, T_1)^2 ds}$$

and $\kappa = e^{-r(T-t)} \ (\mu \equiv 0 \ \text{per risk-neutral dynamics})$

$$\rho = \frac{1}{\beta \delta} \sum_{k=1}^{n} \int_{t}^{T} \sigma_k(s, T_1)\sigma_k(s, T_2) \ ds$$
Cross-commodity
- subscript \( e \) for forward prices, times-to-maturity, volatility functions, ... relative to electric power
- subscript \( g \) for quantities pertaining to natural gas.

Pay-off

\[
(F_e(T, T_e) - H * F_g(T, T_g) - K)^+. 
\]

- \( T < \min\{T_e, T_g\} \)
- Heat rate \( H \)
- Strike \( K \) given by O& M costs

Natural
- **Buyer** owner of a power plant that transforms gas into electricity,
- **Protection** against low electricity prices and/or high gas prices.
Joint Dynamics of the Commodities

\[
\begin{align*}
\text{d}F_{e}(t, T_{e}) &= F_{e}(t, T_{e})[\mu_{e}(t, T_{e})dt + \sum_{k=1}^{n} \sigma_{e,k}(t, T_{e})dW_{k}(t)] \\
\text{d}F_{g}(t, T_{g}) &= F_{g}(t, T_{g})[\mu_{g}(t, T_{g})dt + \sum_{k=1}^{n} \sigma_{g,k}(t, T_{g})dW_{k}(t)]
\end{align*}
\]

- Each commodity has its own volatility factors
- between The two dynamics share the same driving Brownian motion processes $W_{k}$, hence **correlation**.
on any given day $t$ we have

- electricity forward contract prices for $N^{(e)}$ times-to-maturity
  $\tau^{(e)}_1 < \tau^{(e)}_2, \ldots < \tau^{(e)}_{N^{(e)}}$

- natural gas forward contract prices for $N^{(g)}$ times-to-maturity
  $\tau^{(g)}_1 < \tau^{(g)}_2, \ldots < \tau^{(g)}_{N^{(g)}}$

Typically $N^{(e)} = 12$ and $N^{(g)} = 36$ (possibly more).

- Estimate instantaneous vols $\sigma^{(e)}(t) \& \sigma^{(g)}(t)$ 30 days rolling window

For each day $t$, the $N = N^{(e)} + N^{(g)}$ dimensional random vector $X(t)$

$$X(t) = \begin{bmatrix}
\left( \frac{\log \tilde{F}_e(t+1, \tau^{(e)}_j) - \log \tilde{F}_e(t, \tau^{(e)}_j)}{\sigma^{(e)}(t)} \right)_{j=1,\ldots,N^{(e)}} \\
\left( \frac{\log \tilde{F}_g(t+1, \tau^{(g)}_j) - \log \tilde{F}_g(t, \tau^{(g)}_j)}{\sigma^{(g)}(t)} \right)_{j=1,\ldots,N^{(g)}}
\end{bmatrix}$$

- Run PCA on historical samples of $X(t)$
- Choose small number $n$ of factors
- for $k = 1, \ldots, n$,
  - first $N^{(e)}$ coordinates give the electricity volatilities $\tau \leftrightarrow \sigma^{(e)}_k(\tau)$ for $k = 1, \ldots, n$
  - remaining $N^{(g)}$ coordinates give the gas volatilities $\tau \leftrightarrow \sigma^{(g)}_k(\tau)$.

Skip gory details
Pricing a Spark Spread Option

Price at time $t$

$$p_t = e^{-r(T-t)}E_t \{ (F_e(T, T_e) - H \star F_g(T, T_g) - K)^+ \}$$

$F_e(T, T_e)$ and $F_g(T, T_g)$ are log-normal under the pricing measure calibrated by PCA

$$F_e(T, T_e) = F_e(t, T_e) \exp \left[-\frac{1}{2} \sum_{k=1}^{n} \int_{t}^{T} \sigma_{e,k}(s, T_e)^2 ds + \sum_{k=1}^{n} \int_{t}^{T} \sigma_{e,k}(s, T_e)dW_k(s) \right]$$

and:

$$F_g(T, T_g) = F_g(t, T_g) \exp \left[-\frac{1}{2} \sum_{k=1}^{n} \int_{t}^{T} \sigma_{g,k}(s, T_g)^2 ds + \sum_{k=1}^{n} \int_{t}^{T} \sigma_{g,k}(s, T_g)dW_k(s) \right]$$

Set

$$S_1(t) = H \star F_g(t, T_g) \quad \text{and} \quad S_2(t) = F_e(t, T_e)$$
Use the constants

\[ \alpha = e^{-r(T-t)} F_e(t, T_e), \quad \text{and} \quad \beta = \sqrt{\sum_{k=1}^{n} \int_{t}^{T} \sigma_{e,k}(s, T_e)^2 \, ds} \]

for the first log-normal distribution,

\[ \gamma = He^{-r(T-t)} F_g(t, T_g), \quad \text{and} \quad \delta = \sqrt{\sum_{k=1}^{n} \int_{t}^{T} \sigma_{g,k}(s, T_g)^2 \, ds} \]

for the second one, \( \kappa = e^{-r(T-t)} K \) and

\[ \rho = \frac{1}{\beta \delta} \int_{t}^{T} \sum_{k=1}^{n} \sigma_{e,k}(s, T_e) \sigma_{g,k}(s, T_g) \, ds \]

for the correlation coefficient.
Approximations

- Fourier Approximations (Madan, Carr, Dempster, …)
- Bachelier approximation
- Zero-strike approximation
- Kirk approximation
- Upper and Lower Bounds

Can we also approximate the **Greeks**?
Bachelier Approximation

- Generate $x_1^{(1)}, x_2^{(1)}, \ldots, x_N^{(1)}$ from $N(\mu_1, \sigma_1^2)$
- Generate $x_1^{(2)}, x_2^{(2)}, \ldots, x_N^{(2)}$ from $N(\mu_1, \sigma_1^2)$
- Correlation $\rho$
- Look at the distribution of

$$e^{x_1^{(2)}} - e^{x_1^{(1)}}, e^{x_2^{(2)}} - e^{x_2^{(1)}}, \ldots, e^{x_N^{(2)}} - e^{x_N^{(1)}}$$
Log-Normal Samples

- Graphs showing the distribution of log-normal samples for variables X1 and X2.
- The histograms display the frequency distribution of the samples.
- The x-axes represent the logarithm of the variables, and the y-axes represent the frequency.

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Histogram of the Difference between two Log-normals
Bachelier Approximation

- Assume \((S_2(T) - S_1(T))\) is Gaussian
- Match the first two moments

\[
p = \left( m(T) - Ke^{-rT} \right) \phi \left( \frac{m(T) - Ke^{-rT}}{s(T)} \right) + s(T) \varphi \left( \frac{m(T) - Ke^{-rT}}{s(T)} \right)
\]

with:

\[
m(T) = (x_2 - x_1)e^{(\mu - r)T}
\]

\[
s^2(T) = e^{2(\mu - r)T} \left[ x_1^2 \left( e^{\sigma_1^2 T} - 1 \right) - 2x_1x_2 \left( e^{\rho \sigma_1 \sigma_2 T} - 1 \right) + x_2^2 \left( e^{\sigma_2^2 T} - 1 \right) \right]
\]

Easy to compute the Greeks!
Zero-Strike Approximation

\[ p = e^{-rT} \mathbb{E}\{(S_2(T) - S_1(T) - K)^+\} \]

- Assume \( S_2(T) = F_E(T) \) is log-normal
- Replace \( S_1(T) = H \ast F_G(T) \) by \( \tilde{S}_1(T) = S_1(T) + K \)
- Assume \( S_2(T) \) and \( \tilde{S}_1(T) \) are jointly log-normal
- Use Margrabe formula for \( p = e^{-rT} \mathbb{E}\{(S_2(T) - \tilde{S}_1(T))^+\} \)

Use the Greeks from Margrabe formula!
Zero-Strike Approximation

\[ p = e^{-rT} \mathbb{E}\{(S_2(T) - S_1(T) - K)^+\} \]

- Assume \( S_2(T) = F_E(T) \) is log-normal
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Use the Greeks from Margrabe formula!
Zero-Strike Approximation

\[ p = e^{-rT} \mathbb{E}\{ (S_2(T) - S_1(T) - K)^+ \} \]

- Assume \( S_2(T) = F_E(T) \) is log-normal
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- Assume \( S_2(T) \) and \( \tilde{S}_1(T) \) are jointly log-normal
- Use Margrabe formula for \( p = e^{-rT} \mathbb{E}\{ (S_2(T) - \tilde{S}_1(T))^+ \} \)

Use the Greeks from Margrabe formula!
\[ \hat{\rho}^K = x_2 \Phi \left( \ln \left( \frac{x_2}{x_1 + K e^{-rT}} \right) + \frac{\sigma^K}{2} \right) - (x_1 + K e^{-rT}) \Phi \left( \ln \left( \frac{x_2}{x_1 + K e^{-rT}} \right) - \frac{\sigma^K}{2} \right) \]

where

\[ \sigma^K = \sqrt{\sigma^2 - 2 \rho \sigma_1 \sigma_2 \frac{x_1}{x_1 + K e^{-rT}} + \sigma_1^2 \left( \frac{x_1}{x_1 + K e^{-rT}} \right)^2} \]

Exactly what we called ”Zero Strike Approximation”!!!
Upper and Lower Bounds

$$\Pi(\alpha, \beta, \gamma, \delta, \kappa, \rho) = \mathbb{E} \left\{ \left( \alpha e^{\beta X_1 - \beta^2/2} - \gamma e^{\delta X_2 - \delta^2/2} - \kappa \right)^+ \right\}$$

where

- $\alpha, \beta, \gamma, \delta$ and $\kappa$ real constants
- $X_1$ and $X_2$ are jointly Gaussian $N(0, 1)$
- correlation $\rho$

$$\alpha = x_2 e^{-q_2 T} \quad \beta = \sigma_2 \sqrt{T} \quad \gamma = x_1 e^{-q_1 T} \quad \delta = \sigma_1 \sqrt{T} \quad \text{and} \quad \kappa = K e^{-r T}.$$
A Precise Lower Bound

\[ \hat{\rho} = x_2 e^{-q_2 T} \Phi \left( d^* + \sigma_2 \cos(\theta^* + \phi) \sqrt{T} \right) - x_1 e^{-q_1 T} \Phi \left( d^* + \sigma_1 \sin \theta^* \sqrt{T} \right) - Ke^{-r T} \Phi(d^*) \]

where

- \( \theta^* \) is the solution of
  \[
  \frac{1}{\delta \cos \theta} \ln \left( -\frac{\beta \kappa \sin(\theta + \phi)}{\gamma [\beta \sin(\theta + \phi) - \delta \sin \theta]} \right) - \frac{\delta \cos \theta}{2} = \frac{1}{\beta \cos(\theta + \phi)} \ln \left( -\frac{\delta \kappa \sin \theta}{\alpha [\beta \sin(\theta + \phi) - \delta \sin \theta]} \right) - \frac{\beta \cos(\theta + \phi)}{2}
  \]
- the angle \( \phi \) is defined by setting \( \rho = \cos \phi \)
- \( d^* \) is defined by
  \[
  d^* = \frac{1}{\sigma \cos(\theta^* - \psi) \sqrt{T}} \ln \left( \frac{x_2 e^{-q_2 T} \sigma_2 \sin(\theta^* + \phi)}{x_1 e^{-q_1 T} \sigma_1 \sin \theta^*} \right) - \frac{1}{2} \left( \sigma_2 \cos(\theta^* + \phi) + \sigma_1 \cos \theta \right)
  \]
- the angles \( \phi \) and \( \psi \) are chosen in \([0, \pi]\) such that:
  \[
  \cos \phi = \rho \quad \text{and} \quad \cos \psi = \frac{\sigma_1 - \rho \sigma_2}{\sigma},
  \]
Remarks on this Lower Bound

- \( \hat{p} \) is equal to the true price \( p \) when
  - \( K = 0 \)
  - \( x_1 = 0 \)
  - \( x_2 = 0 \)
  - \( \rho = -1 \)
  - \( \rho = +1 \)

- Margrabe formula when \( K = 0 \) because

\[
\theta^* = \pi + \psi = \pi + \arccos \left( \frac{\sigma_1 - \rho \sigma_2}{\sigma} \right).
\]

with:

\[
\sigma = \sqrt{\sigma_1^2 - 2 \rho \sigma_1 \sigma_2 + \sigma_2^2}
\]
Remarks on this Lower Bound

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* Margrabe formula when \( K = 0 \) because

\[
\theta^* = \pi + \psi = \pi + \arccos\left(\frac{\sigma_1 - \rho \sigma_2}{\sigma}\right).
\]

with:

\[
\sigma = \sqrt{\sigma_1^2 - 2\rho \sigma_1 \sigma_2 + \sigma_2^2}
\]
The portfolio comprising at each time $t \leq T$

\[ \Delta_1 = -e^{-q_1T} \Phi \left( d^* + \sigma_1 \cos \theta^* \sqrt{T} \right) \]

and

\[ \Delta_2 = e^{-q_2T} \Phi \left( d^* + \sigma_2 \cos(\theta^* + \phi) \sqrt{T} \right) \]

units of each of the underlying assets is a sub-hedge

*its value at maturity is a.s. a lower bound for the pay-off*
The Other Greeks

- $\vartheta_1$ and $\vartheta_2$ sensitivities w.r.t. volatilities $\sigma_1$ and $\sigma_2$
- $\chi$ sensitivity w.r.t. correlation $\rho$
- $\kappa$ sensitivity w.r.t. strike price $K$
- $\Theta$ sensitivity w.r.t. maturity time $T$

\[\vartheta_1 = x_1 e^{-q_1 T \varphi \left( d^* + \sigma_1 \cos \theta^* \sqrt{T} \right)} \cos \theta^* \sqrt{T}\]

\[\vartheta_2 = -x_2 e^{-q_2 T \varphi \left( d^* + \sigma_2 \cos(\theta^* + \phi) \sqrt{T} \right)} \cos(\theta^* + \phi) \sqrt{T}\]

\[\chi = -x_1 e^{-q_1 T \varphi \left( d^* + \sigma_1 \cos \theta^* \sqrt{T} \right)} \sigma_1 \frac{\sin \theta^*}{\sin \phi} \sqrt{T}\]

\[\kappa = -\Phi (d^*) e^{-rT}\]

\[\Theta = \frac{\sigma_1 \vartheta_1 + \sigma_2 \vartheta_2}{2T} - q_1 x_1 \Delta_1 - q_2 x_2 \Delta_2 - rK \kappa\]
The Other Greeks

- \( \vartheta_1 \) and \( \vartheta_2 \) sensitivities w.r.t. volatilities \( \sigma_1 \) and \( \sigma_2 \)
- \( \chi \) sensitivity w.r.t. correlation \( \rho \)
- \( \kappa \) sensitivity w.r.t. strike price \( K \)
- \( \Theta \) sensitivity w.r.t. maturity time \( T \)

\[
\begin{align*}
\vartheta_1 &= x_1 e^{-q_1 T} \varphi \left( d^* + \sigma_1 \cos \theta^* \sqrt{T} \right) \cos \theta^* \sqrt{T} \\
\vartheta_2 &= -x_2 e^{-q_2 T} \varphi \left( d^* + \sigma_2 \cos(\theta^* + \phi) \sqrt{T} \right) \cos(\theta^* + \phi) \sqrt{T} \\
\chi &= -x_1 e^{-q_1 T} \varphi \left( d^* + \sigma_1 \cos \theta^* \sqrt{T} \right) \sigma_1 \frac{\sin \theta^*}{\sin \phi} \sqrt{T} \\
\kappa &= -\Phi (d^*) e^{-rT} \\
\Theta &= \frac{\sigma_1 \vartheta_1 + \sigma_2 \vartheta_2}{2T} - q_1 x_1 \Delta_1 - q_2 x_2 \Delta_2 - rK \kappa
\end{align*}
\]
Behavior of the tracking error as the number of re-hedging times increases. The model data are $x_1 = 100$, $x_2 = 110$, $\sigma_1 = 10\%$, $\sigma_2 = 15\%$ and $T = 1$. $\rho = 0.9$, $K = 30$ (left) and $\rho = 0.6$, $K = 20$ (right).
Generalization: European Basket Option

Black-Scholes Set-Up

- Multidimensional model
- $n$ stocks $S_1, \ldots, S_n$
- Risk neutral dynamics

\[
\frac{dS_i(t)}{S_i(t)} = rdt + \sum_{j=1}^{n} \sigma_{ij} dB_j(t),
\]

- initial values $S_1(0), \ldots, S_n(0)$
- $B_1, \ldots, B_n$ independent standard Brownian motions
- Correlation through matrix $(\sigma_{ij})$
Vector of weights \((w_i)_{i=1,...,n}\) (most often \(w_i \geq 0\))

Basket option struck at \(K\) at maturity \(T\) given by payoff

\[
\left( \sum_{i=1}^{n} w_i S_i(T) - K \right)^+ 
\]

(*Asian Options*)

Risk neutral valuation: price at time 0

\[
p = e^{-rT} \mathbb{E} \left\{ \left( \sum_{i=1}^{n} w_i S_i(T) - K \right)^+ \right\}
\]
Existing Literature

- **Jarrow and Rudd**
  - Replace true distribution by simpler distribution with same first moments
  - Edgeworth (Charlier) expansions
  - Bachelier approximation when Gaussian distribution used

- **SemiParametric** Bounds (known marginals)

- **Fully NonParametric** Bounds
  - Intervals too large
  - Used only to rule out arbitrage

- Replacing Arithmetic Averages by Geometric Averages (**Musiela**)

Carmona  |  Energy Markets
Reformulation of the Problem

- Change $w_i$ if necessary to absorb exponent mean
- Change $w_i$ if necessary to introduce variance in exponent
- Replace $K$ by $-w_0 e^{G_0 - \text{var}\{G_0\}/2}$ with $G_0 \sim N(0, 0)$
- Set $x_i = |w_i|$ and $\epsilon_i = \text{sign}(w_i)$

Our original problem becomes: **Compute**

$$\mathbb{E}\{X^+\}$$

for

$$X = \sum_{i=0}^{n} \epsilon_i x_i e^{G_i - \text{Var}(G_i)/2}.$$
What Are We Looking For?

- Explicit formulae in close form
- Compute Greeks as well

\[ n = 1 \]

- **Black Scholes** Formula
- **Margrabe** Formula
Two Optimization Problems

For any $X \in L^1$,

$$\sup_{0 \leq Y \leq 1} \mathbb{E}\{XY\} = \mathbb{E}\{X^+\} = \inf_{X = Z_1 - Z_2, Z_1 \geq 0, Z_2 \geq 0} \mathbb{E}\{Z_1\}.$$
Lower Bound Strategy

\[ \sup_{0 \leq Y \leq 1} \mathbb{E}\{XY\} = \mathbb{E}\{X^+\} \]

- Compute sup in LHS restricting \( Y \)
- We choose \( Y = 1_{\{u \cdot G \leq d\}} \) for \( u \in \mathbb{R}^{n+1} \) and \( d \in \mathbb{R} \)
where \( G = (G_0, G_1, \ldots, G_n) \) and \( u \cdot G = u_0 G_0 + u_1 G_1 + \ldots + u_n G_n \)

Can we compute?

\[ p^* = \sup_{u,d} \mathbb{E}\{X1_{\{u \cdot G \leq d\}}\} \]

We sure can!

\[ \mathbb{E}\{X1_{\{u \cdot G \leq d\}}\} = \sum_{i=0}^{n} \mathbb{E}\left\{ \epsilon_i x_i \mathbb{E}\{e^{G_i - \text{Var}(G_i)/2}|u \cdot G\}1_{\{u \cdot G \leq d\}} \right\} \]
\[ p_* = \sup_{d \in \mathbb{R}} \sup_{u \cdot \Sigma u = 1} \sum_{i=0}^{n} \epsilon_i x_i \Phi (d + (\Sigma u)_i) \]

\[ = \sup_{d \in \mathbb{R}} \sup_{\|v\| = 1} \sum_{i=0}^{n} \epsilon_i x_i \Phi \left( d + \sigma_i (\sqrt{Cv})_i \right). \]

where

\[ C = D \Sigma D \quad \text{and} \quad D = \text{diag}(1/\sigma_i) \]

and

\[ \varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad \text{and} \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du. \]
First Order Conditions

Lagrangian $\mathcal{L}$:

$$
\mathcal{L}(v, d) = \sum_{i=0}^{n} \epsilon_i x_i \Phi \left( d + \sigma_i (\sqrt{C}v)_i \right) - \frac{\mu}{2} \left( \|v\|^2 - 1 \right).
$$

where $d^*$ and $v^*$ satisfy the following first order conditions

$$
\sum_{i=0}^{n} \epsilon_i x_i \sigma_i \sqrt{C}_{ij} \varphi \left( d^* + \sigma_i (\sqrt{C}v^*)_i \right) - \mu v^*_j = 0 \quad \text{for } j = 0, \ldots, n
$$

$$
\sum_{i=0}^{n} \epsilon_i x_i \varphi \left( d^* + \sigma_i (\sqrt{C}v^*)_i \right) = 0
$$

$$
\|v^*\| = 1.
$$
for each \( k \) in \( \{0, 1, \ldots, n\} \)

\[
X = \sum_{i \neq k} \epsilon_i x_i e^{G_i - \text{Var}(G_i)/2} - \lambda_i^k x_k e^{G_k - \text{Var}(G_k)/2}
\]

\[
= \sum_{i \neq k} \left( \epsilon_i x_i e^{G_i - \text{Var}(G_i)/2} - \lambda_i^k x_k e^{G_k - \text{Var}(G_k)/2} \right)^+
\]

\[
- \sum_{i \neq k} \left( \epsilon_i x_i e^{G_i - \text{Var}(G_i)/2} - \lambda_i^k x_k e^{G_k - \text{Var}(G_k)/2} \right)^-
\]

if \( \sum_{i \neq k} \lambda_i^k = -\epsilon_k \)
In formula

\[ \mathbb{E}\{X^+\} = \inf_{X=Z_1-Z_2, Z_1 \geq 0, Z_2 \geq 0} \mathbb{E}\{Z_1\}. \]

Restrict \( Z_1 \) to

\[ \sum_{i \neq k} \left( \varepsilon_i x_i e^{G_i - \text{Var}(G_i)/2} - \lambda_i^k \hat{x}_k e^{G_k - \text{Var}(G_k)/2} \right)^+ \]

where \( k = 0, \ldots, n \), \( \sum_{i \neq k} \lambda_i^k = -\varepsilon_k \) and \( \lambda_i^k \varepsilon_i > 0 \) for all \( i \neq k \).
Upper Bound

\[ p^* = \min_{0 \leq k \leq n} \left\{ \sum_{i=0}^{n} \varepsilon_i x_i \Phi \left( d^k + \varepsilon_i \sigma_i^k \right) \right\} \]

where \( d^k \) is given by the following first order conditions

\[
\frac{\varepsilon_i}{\sigma_i^k} \ln \left( \frac{\varepsilon_i x_i}{\lambda_i^k x_k} \right) - \frac{\varepsilon_i \sigma_i^k}{2} = \frac{\varepsilon_j}{\sigma_j^k} \ln \left( \frac{\varepsilon_j x_j}{\lambda_j^k x_k} \right) - \frac{\varepsilon_j \sigma_j^k}{2} = d^k \quad \text{for } i, j \neq k
\]

\[
\sum_{i \neq k} \lambda_i^k = -\tilde{\varepsilon}_k
\]

\[
\lambda_i^k \varepsilon_i > 0 \quad \text{for } i \neq k.
\]
If for all $i, j = 0, \ldots, n$, 

$$
\sum_{ij} = \varepsilon_i \varepsilon_j \sigma_i \sigma_j,
$$

then 

$$
p_* = p^*.
$$
Error Bound

\[ 0 \leq p^* - p_* \leq \sqrt{\frac{2}{\pi}} \min_{0 \leq k \leq n} \left\{ \sum_{i=0}^{n} x_i \sigma_i^k \right\}. \]

where

\[ \sigma_i^k = \sqrt{\text{Var}(\{G_i - G_k\})} \]
FIGURE 1. Lower and upper bound on the price for a basket option on 50 stocks (each one having a weight of \( \frac{1}{50} \)) as a function of K. '+' denote Monte Carlo results.

Functions \( u_k \) for \( k = 10 \) \( j \) with \( j = 1, 2, \ldots, 25 \), \( x^* \) would be ...
Lower and upper bound on the price of an Asian option. The dotted line represents the geometric average approximation.
Computation of (Approximate) Greeks

\[ \Delta_{*i} = \frac{\partial p_\ast}{\partial x_i} = \varepsilon_i \Phi \left( d^* + \sigma_i (\sqrt{Cv^*})_i \right) \]

\[ \text{Vega}_{*i} = \frac{\partial p_\ast}{\partial \sigma_i} \sqrt{T} = \varepsilon_i x_i (\sqrt{Cv^*})_i \varphi \left( d^* + \sigma_i (\sqrt{Cv^*})_i \right) \sqrt{T} \]

\[ \chi_{*ij} = \frac{\partial p_\ast}{\partial \rho_{ij}} = \frac{1}{2} \sum_{k=0}^{n} \varepsilon_k x_k \left( \sigma_i C_{kj}^{-\frac{1}{2}} v_j^* + \sigma_j C_{ki}^{-\frac{1}{2}} v_i^* \right) \varphi \left( d^* + \sigma_k (\sqrt{Cv^*})_k \right) \]

\[ \Theta_{*} = \frac{\partial p_\ast}{\partial T} = \frac{1}{2T} \sum_{k=0}^{n} \varepsilon_k x_k \sigma_k (\sqrt{Cv^*})_k \varphi \left( d^* + \sigma_k (\sqrt{Cv^*})_k \right) . \]
Second Order Derivatives

\[ \Gamma_{ij}^* = \varepsilon_i \varepsilon_j \frac{\varphi \left( d^* + \sigma_i \left( \sqrt{Cv^*} \right)_i \right) \varphi \left( d^* + \sigma_j \left( \sqrt{Cv^*} \right)_j \right)}{\sum_{k=0}^{n} \varepsilon_k x_k \sigma_k \left( \sqrt{Cv^*} \right)_k \varphi \left( d^* + \sigma_k \left( \sqrt{Cv^*} \right)_k \right)} , \]

then

\[ -\Theta^* + \frac{1}{2T} \sum_{i=0}^{n} \sum_{j=0}^{n} \Sigma_{ij} x_i x_j \Gamma_{*ij} = 0. \]
Option Payoff

\[ \left( \sum_{i=1}^{n} w_i S_i(T) - K \right)^+ 1_{\{ \inf_{t \leq T} S_1(t) \geq H \}}. \]

Option price is

\[ \mathbb{E} \left\{ \left( \sum_{i=0}^{n} \varepsilon_i x_i e^{G_i(1)-\frac{1}{2} \sigma_i^2} 1_{\{ \inf_{\theta \leq 1} x_1 e^{G_1(\theta)-\frac{1}{2} \sigma_1^2 \theta} \geq H \}} \right)^+ \right\}, \]

where

- \( \varepsilon_1 = +1, \sigma_1 > 0 \) and \( H < x_1 \)
- \( \{ G(\theta); \theta \leq 1 \} \) is a \((n+1)\)-dimensional Brownian motion starting from 0 with covariance \( \Sigma \).
Use lower bound.

\[ p_* = \sup_{d,u} \mathbb{E} \left\{ \sum_{i=0}^{n} \varepsilon_i x_i e^{G_i(1) - \frac{1}{2} \sigma_i^2} \mathbf{1} \left\{ \inf_{\theta \leq 1} x_1 e^{G_1(\theta) - \frac{1}{2} \sigma_1^2 \theta} \geq H; u \cdot G(1) \leq d \right\} \right\}. \]

Girsanov implies

\[ p_* = \sup_{d,u} \sum_{i=0}^{n} \varepsilon_i x_i \mathbb{P} \left\{ \inf_{\theta \leq 1} G_1(\theta) \right\} + \left( \sum_{i=1} - \sigma_1^2 / 2 \right) \theta \geq \ln \left( \frac{H}{x_1} \right); u \cdot G(1) \leq d - (\Sigma u)_i \right\}. \]
### Numerical Results

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Stylized Version

- **Leasing an Energy Asset**
  - Fossil Fuel Power Plant
  - Oil Refinery
  - Pipeline

- **Owner of the Agreement**
  - Decides *when* and *how* to use the asset (e.g. run the power plant)
  - Has someone else do the leg work
Markov process (state of the world) $X_t = (X_t^{(1)}, X_t^{(2)}, \cdots)$ (e.g. $X_t^{(1)} = P_t$, $X_t^{(2)} = G_t$, $X_t^{(3)} = O_t$ for a dual plant)

Plant characteristics
- $\mathbb{Z}_M \triangleq \{0, \cdots, M - 1\}$ modes of operation of the plant
- $H_0, H_1, \cdots, H_{M-1}$ heat rates
- $\{C(i, j)\}_{(i, j) \in \mathbb{Z}_M}$ regime switching costs ($C(i, j) = C(i, \ell) + C(\ell, j)$)
- $\psi_i(t, x)$ reward at time $t$ when world in state $x$, plant in mode $i$

Operation of the plant (control) $u = (\xi, T)$ where
- $\xi_k \in \mathbb{Z}_M \triangleq \{0, \cdots, M - 1\}$ successive modes
- $0 \leq \tau_{k-1} \leq \tau_k \leq T$ switching times
- $T$ (horizon) length of the tolling agreement
- Total reward

$$H(x, i, [0, T]; u)(\omega) \triangleq \int_0^T \psi_{us}(s, X_s) \, ds - \sum_{\tau_k < T} C(u_{\tau_k-}, u_{\tau_k})$$
Stochastic Control Problem

\[ \mathcal{U}(t) \] acceptable controls on \([t, T]\)
(adapted càdlàg \(\mathbb{Z}_M\)-valued processes \(u\) of a.s. finite variation on \([t, T]\))

Optimal Switching Problem

\[
J(t, x, i) = \sup_{u \in \mathcal{U}(t)} J(t, x, i; u),
\]

where

\[
J(t, x, i; u) = \mathbb{E} \left[ H(x, i, [t, T]; u) \mid X_t = x, u_t = i \right]
\]
\[= \mathbb{E} \left[ \int_0^T \psi_{u_s}(s, X_s) \, ds - \sum_{\tau_k < T} C(u_{\tau_k-}, u_{\tau_k}) \mid X_t = x, u_t = i \right] \]
Consider problem with **at most** $k$ mode switches

$$
U^k(t) \triangleq \{(\xi, T) \in U(t): \tau_\ell = T \text{ for } \ell \geq k + 1\}
$$

Admissible strategies on $[t, T]$ with at most $k$ switches

$$
J^k(t, x, i) \triangleq \text{esssup}_{u \in U^k(t)} \mathbb{E}\left[ \int_t^T \psi_{us}(s, X_s) \, ds - \sum_{t \leq \tau_k < T} C(u_{\tau_k-}, u_{\tau_k}) \bigg| X_t = x, u_t = i \right].
$$
Alternative Recursive Construction

\[ J^0(t, x, i) \triangleq \mathbb{E} \left[ \int_t^T \psi_i(s, X_s) \, ds \middle| X_t = x \right], \]

\[ J^k(t, x, i) \triangleq \sup_{\tau \in \mathcal{S}_t} \mathbb{E} \left[ \int_t^\tau \psi_i(s, X_s) \, ds + \mathcal{M}^{k,i}(\tau, X_\tau) \middle| X_t = x \right]. \]

**Intervention operator** \( \mathcal{M} \)

\[ \mathcal{M}^{k,i}(t, x) \triangleq \max_{j \neq i} \left\{ -C_{i,j} + J^{k-1}(t, x, j) \right\}. \]

Hamadène - Jeanblanc (M=2)
Variational Formulation

Notation

- \( \mathcal{L}_X X \) space-time generator of Markov process \( X_t \) in \( \mathbb{R}^d \)
- \( \mathcal{M}\phi(t, x, i) = \max_{j \neq i} \{-C_{i,j} + \phi(t, x, j)\} \) intervention operator

Assume

- \( \phi(t, x, i) \) in \( C^{1,2}(([0, T] \times \mathbb{R}^d) \setminus D) \cap C^{1,1}(D) \)
- \( D = \bigcup_i\{(t, x) : \phi(t, x, i) = \mathcal{M}\phi(t, x, i)\} \)
- (QVI) for all \( i \in \mathbb{Z}_M \):
  1. \( \phi \geq \mathcal{M}\phi \),
  2. \( \mathbb{E}^x \int_0^T 1_{\phi \leq \mathcal{M}\phi} \, dt = 0 \),
  3. \( \mathcal{L}_X \phi(t, x, i) + \psi_i(t, x) \leq 0 \), \( \phi(T, x, i) = 0 \),
  4. \( \left( \mathcal{L}_X \phi(t, x, i) + \psi_i(t, x) \right) \left( \phi(t, x, i) - \mathcal{M}\phi(t, x, i) \right) = 0 \).

Conclusion

\( \phi \) is the optimal value function for the switching problem
Reflected Backward SDE’s

Assume

- \( X_0 = x \) & \( \exists (Y^x, Z^x, A) \) adapted to \((\mathcal{F}^X_t)\)

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y^x_t|^2 + \int_0^T \|Z^x_t\|^2 dt + |A_T|^2 \right] < \infty
\]

and

\[
Y^x_t = \int_t^T \psi_i(s, X^x_s) \, ds + A_T - A_t - \int_t^T Z_s \cdot dW_s,
\]

\[
Y^x_t \geq \mathcal{M}^{k,i}(t, X^x_t),
\]

\[
\int_0^T (Y^x_t - \mathcal{M}^{k,i}(t, X^x_t)) \, dA_t = 0, \quad A_0 = 0.
\]

Conclusion: if \( Y^x_0 = J^k(0, x, i) \) then

\[
Y^x_t = J^k(t, X^x_t, i)
\]
QVI for optimal switching: coupled system of reflected BSDE’s for \((Y^i)_{i \in \mathbb{Z}_M}\),

\[
Y_t^i = \int_t^T \psi_i(s, X_s) \, ds + A_T^i - A_t^i - \int_t^T Z_s^i \cdot dW_s,
\]

\[
Y_t^i \geq \max_{j \neq i} \{-C_{i,j} + Y_t^j\}.
\]

Existence and uniqueness Directly for \(M > 2\)?

\(M = 2\), Hamadène - Jeanblanc use difference process \(Y^1 - Y^2\).
Discrete Time Dynamic Programming

- Time Step $\Delta t = T/M$
- Time grid $S^\Delta = \{m\Delta t, m = 0, 1, \ldots, M\}$
- Switches are allowed in $S^\Delta$

DPP

For $t_1 = m\Delta t$, $t_2 = (m + 1)\Delta t$ consecutive times

$$J^k(t_1, X_{t_1}, i) = \max \left( \mathbb{E} \left[ \int_{t_1}^{t_2} \psi_i(s, X_s) \, ds + J^k(t_2, X_{t_2}, i) \, | \, \mathcal{F}_{t_1} \right], \mathcal{M}^{k,i}(t_1, X_{t_1}) \right)$$

$$\simeq \left( \psi_i(t_1, X_{t_1}) \, \Delta t + \mathbb{E} \left[ J^k(t_2, X_{t_2}, i) \, | \, \mathcal{F}_{t_1} \right] \right) \lor \left( \max_{j \neq i} \left\{ -C_{i,j} + J^{k-1}(t_1, X_{t_1}, j) \right\} \right).$$

(1)

Tsitsiklis - van Roy
Recall

\[ J^k(m\Delta t, x, i) = \mathbb{E}\left[ \sum_{j=m}^{\tau^k} \psi_i(j\Delta t, X_{j\Delta t}) \Delta t + M^{k,i}(\tau^k \Delta t, X_{\tau^k\Delta t}) \mid X_{m\Delta t} = x \right]. \]

Analogue for \( \tau^k \):

\[
\tau^k(m\Delta t, x^\ell_{m\Delta t}, i) = \begin{cases} 
\tau^k((m+1)\Delta t, x^\ell_{(m+1)\Delta t}, i), & \text{no switch;} \\
 m, & \text{switch,}
\end{cases}
\tag{2}
\]

and the set of paths on which we switch is given by \( \{ \ell : \hat{j}^\ell(m\Delta t; i) \neq i \} \) with

\[
\hat{j}^\ell(t_1; i) = \arg \max_j \left( -C_{i,j} + J^{k-1}(t_1, x^\ell_{t_1}, j), \psi_i(t_1, x^\ell_{t_1}) \Delta t + \hat{E}_{t_1}[J^k(t_2, \cdots, i)](x^\ell_{t_1}) \right).
\tag{3}
\]

The full recursive pathwise construction for \( J^k \) is

\[
J^k(m\Delta t, x^\ell_{m\Delta t}, i) = \begin{cases} 
\psi_i(m\Delta t, x^\ell_{m\Delta t}) \Delta t + J^k((m+1)\Delta t, x^\ell_{(m+1)\Delta t}, i), & \text{no switch;} \\
- C_{i,j} + J^{k-1}(m\Delta t, x^\ell_{m\Delta t}, j), & \text{switch to } j.
\end{cases}
\tag{4}
\]
Remarks

Regression used solely to update the optimal stopping times $\tau^k$
Regressed values never stored
Helps to eliminate potential biases from the regression step.
Algorithm

1. Select a set of basis functions \((B_j)\) and algorithm parameters \(\Delta t, M^\#_t, N^p, \bar{K}, \delta\).

2. Generate \(N^p\) paths of the driving process: \(\{x_{m\Delta t}^\ell, m = 0, 1, \ldots, M^\#_t, \ell = 1, 2, \ldots, N^p\}\) with fixed initial condition \(x_0^\ell = x_0\).

3. Initialize the value functions and switching times \(J^k(T, x_T^\ell, i) = 0, \tau^k(T, x_T^\ell, i) = M^\#_t \forall i, k\).

4. Moving backward in time with \(t = m\Delta t, m = M^\#_t, \ldots, 0\) repeat the Loop:
   - Compute inductively the layers \(k = 0, 1, \ldots, \bar{K}\) (evaluate \(E[J^k(m\Delta t + \Delta t, \cdot, i)|F_{m\Delta t}]\) by linear regression of \(\{J^k(m\Delta t + \Delta t, x_{m\Delta t+\Delta t}^\ell, i)\}\) against \(\{B_j(x_m^\ell)\}_{j=1}^{NB}\), then add the reward \(\psi_i(m\Delta t, x_m^\ell) \cdot \Delta t\)
   - Update the switching times and value functions

5. end Loop.

6. Check whether \(\bar{K}\) switches are enough by comparing \(J^{\bar{K}}\) and \(J^{\bar{K}-1}\) (they should be equal).

Observe that during the main loop we only need to store the buffer \(J(t, \cdot), \ldots, J(t + \delta, \cdot); \) and \(\tau(t, \cdot), \cdots, \tau(t + \delta, \cdot)\).
• Bouchard - Touzi
• Gobet - Lemor - Warin
Example 1

\[ dX_t = 2(10 - X_t) \, dt + 2 \, dW_t, \quad X_0 = 10, \]

- Horizon \( T = 2, \)
- Switch separation \( \delta = 0.02. \)
- Two regimes
- Reward rates \( \psi_0(X_t) = 0 \) and \( \psi_1(X_t) = 10(X_t - 10) \)
- Switching cost \( C = 0.3. \)
$J^k(t, x, 0)$ as a function of $t$
Exercise Boundaries

\[ k = 2 \text{ (left)} \]
\[ k = 7 \text{ (right)} \]

NB: Decreasing boundary around \( t = 0 \) is an artifact of the Monte Carlo.
Example 2: Comparisons

Spark spread \( X_t = (P_t, G_t) \)

\[
\begin{cases}
\log(P_t) \sim OU(\kappa = 2, \theta = \log(10), \sigma = 0.8) \\
\log(G_t) \sim OU(\kappa = 1, \theta = \log(10), \sigma = 0.4)
\end{cases}
\]

- \( P_0 = 10, \ G_0 = 10, \ \rho = 0.7 \)
- Agreement Duration \([0, 0.5]\)
- Reward functions

\[
\begin{align*}
\psi_0(X_t) &= 0 \\
\psi_1(X_t) &= 10(P_t - G_t) \\
\psi_2(X_t) &= 20(P_t - 1.1 \ G_t)
\end{align*}
\]

- Switching costs

\( C_{i,j} = 0.25|i - j| \)
## Numerical Comparison

<table>
<thead>
<tr>
<th>Method</th>
<th>Mean</th>
<th>Std. Dev</th>
<th>Time (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Explicit FD</td>
<td>5.931</td>
<td>–</td>
<td>25</td>
</tr>
<tr>
<td>LS Regression</td>
<td>5.903</td>
<td>0.165</td>
<td>1.46</td>
</tr>
<tr>
<td>TvrR Regression</td>
<td>5.276</td>
<td>0.096</td>
<td>1.45</td>
</tr>
<tr>
<td>Kernel</td>
<td>5.916</td>
<td>0.074</td>
<td>3.8</td>
</tr>
<tr>
<td>Quantization</td>
<td>5.658</td>
<td>0.013</td>
<td>400*</td>
</tr>
</tbody>
</table>

**Table:** Benchmark results for Example 2.
Example 3: Dual Plant & Delay

\[
\begin{align*}
\log(P_t) &\sim OU(\kappa = 2, \theta = \log(10), \sigma = 0.8), \\
\log(G_t) &\sim OU(\kappa = 1, \theta = \log(10), \sigma = 0.4), \\
\log(O_t) &\sim OU(\kappa = 1, \theta = \log(10), \sigma = 0.4).
\end{align*}
\]

- \( P_0 = G_0 = O_0 = 10, \rho_{pg} = 0.5, \rho_{po} = 0.3, \rho_{go} = 0 \)
- Agreement Duration \( T = 1 \)
- Reward functions

\[
\begin{align*}
\psi_0(X_t) &= 0 \\
\psi_1(X_t) &= 5 \cdot (P_t - G_t) \\
\psi_2(X_t) &= 5 \cdot (P_t - O_t), \\
\psi_3(X_t) &= 5 \cdot (3P_t - 4G_t) \\
\psi_4(X_t) &= 5 \cdot (3P_t - 4O_t).
\end{align*}
\]

- Switching costs \( C_{i,j} \equiv 0.5 \)
- Delay \( \delta = 0, 0.01, 0.03 \) (up to ten days)
Numerical Results

<table>
<thead>
<tr>
<th>Setting</th>
<th>No Delay</th>
<th>$\delta = 0.01$</th>
<th>$\delta = 0.03$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Base Case</td>
<td>13.22</td>
<td>12.03</td>
<td>10.87</td>
</tr>
<tr>
<td>Jumps in $P_t$</td>
<td>23.33</td>
<td>22.00</td>
<td>20.06</td>
</tr>
<tr>
<td>Regimes 0-3 only</td>
<td>11.04</td>
<td>10.63</td>
<td>10.42</td>
</tr>
<tr>
<td>Regimes 0-2 only</td>
<td>9.21</td>
<td>9.16</td>
<td>9.14</td>
</tr>
<tr>
<td>Gas only: 0, 1, 3</td>
<td>9.53</td>
<td>7.83</td>
<td>7.24</td>
</tr>
</tbody>
</table>

Table: LS scheme with 400 steps and 16000 paths.

Remarks

- High $\delta$ lowers profitability by over 20%.
Example 4: Exhaustible Resources

Include $l_t$ current level of resources left ($l_t$ non-increasing process).

$$J(t, x, c, i) = \sup_{\tau, j} \mathbb{E} \left[ \int_t^\tau \psi_i(s, X_s) \, ds + J(\tau, X_\tau, l_\tau, j) - C_{i,j} \mid X_t = x, l_t = c \right].$$  

(5)

- Resource depletion (boundary condition) $J(t, x, 0, i) \equiv 0$.
- Not really a control problem $l_t$ can be computed on the fly

Mining example of Brennan and Schwartz varying the initial copper price $X_0$

<table>
<thead>
<tr>
<th>Method/ $X_0$</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>BS ’85</td>
<td>1.45</td>
<td>4.35</td>
<td>8.11</td>
<td>12.49</td>
<td>17.38</td>
<td>22.68</td>
</tr>
<tr>
<td>PDE FD</td>
<td>1.42</td>
<td>4.21</td>
<td>8.04</td>
<td>12.43</td>
<td>17.21</td>
<td>22.62</td>
</tr>
<tr>
<td>RMC</td>
<td>1.33</td>
<td>4.41</td>
<td>8.15</td>
<td>12.44</td>
<td>17.52</td>
<td>22.41</td>
</tr>
</tbody>
</table>
Accomodate **outages**

Include switch separation as a form of **delay**

Was extended *(R.C. - M. Ludkovski)* to treat

- Gas Storage
- Hydro Plants

**Porchet-Touzi**
What Remains to be Done

- Need to improve delays
- Need **convergence analysis**
- Need better analysis of **exercise boundaries**
- Need to implement duality upper bounds
  - we have approximate value functions
  - we have approximate exercise boundaries
  - so we have lower bounds
  - need to extend **Meinshausen-Hambly** to optimal switching set-up
Extending the Analysis Adding Access to a Financial Market

Porchet-Touzi

- Same (Markov) factor process $X_t = (X_t^{(1)}, X_t^{(2)}, \cdots)$ as before
- Same plant characteristics as before
- Same operation control $u = (\xi, T)$ as before
- Same maturity $T$ (end of tolling agreement) as before
- **Reward** for operating the plant

$$H(x, i, T; u)(\omega) \triangleq \int_0^T \psi_{u_s}(s, X_s) \, ds - \sum_{\tau_k < T} C(u_{\tau_k-}, u_{\tau_k})$$
Access to a financial market (possibly incomplete)

- $y$ initial wealth
- $\pi_t$ investment portfolio
- $Y_T^{y,\pi}$ corresponding terminal wealth from investment

**Utility function** $U(y) = -e^{-\gamma y}$

Maximum expected utility

$$\nu(y) = \sup_{\pi} \mathbb{E}\{U(Y_T^{y,\pi})\}$$
With the power plant (tolling contract)

\[ V(x, i, y) = \sup_{u, \pi} \mathbb{E}\{ U(Y_{T}^{\pi}, T + H(x, i, T; u)) \} \]

**INDIFFERENCE PRICING**

\[ \bar{p} = p(x, i, y) = \sup\{ p \geq 0; V(x, i, y - p) \geq v(y) \} \]

Analysis of
- BSDE formulation
- PDE formulation