# PRICING AND HEDGING MULTIVARIATE CONTINGENT CLAIMS

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ABSTRACT. This paper provides with approximate formulas that generalize Black-Scholes formula in all dimensions. Pricing and hedging of multivariate contingent claims are achieved by computing lower and upper bounds. These bounds are given in closed form in the same spirit as the classical one-dimensional Black-Scholes formula. Lower bounds perform remarkably well. Like in the one-dimensional case, Greeks are also available in closed form. We discuss an extension to basket options with barrier.

## 1. INTRODUCTION

This paper provides with approximate formulas that generalize Black-Scholes formula in all dimensions. The classical Black-Scholes formula gives in closed form the price of a call option on a single stock whose dynamics is a geometric Brownian motion. Its use has spread to fixed income markets to price caps and floors in Libor models or swaptions in Swap models when volatilities are deterministic.

Many options however have multivariate payoffs. Although the mathematical theory does not present any particular difficulties, actual computations of prices cannot be done in closed form any more. Financial practitioners have to resort to numerical integration, simulations or approximations. In high dimensions, numerical integration and simulation methods may be too slow for practical purposes. Many areas of computational finance require robust and accurate algorithms to price these options.

In this paper we give approximate formulas that are fast, easy to implement and yet very accurate. These formulas are based on rigorous lower and upper bounds. These bounds are derived under two assumptions. First, we restrict ourselves to a special class of multivariate payoffs. Throughout payoffs are of the European type (options can only be exercised at maturity) and when exercised these options pay a *linear combination* of asset prices. This wide class includes basket options (i.e., options on a basket of stocks), spread options (i.e., options on the difference between two stocks or indices) and more generally rainbow options but also discrete-time average Asian options and also combination of those like Asian spread options (i.e., options on the difference between time averages of two stocks or indices.) Second, we work in the so-called multidimensional Black-Scholes model. In this model, assets follow a multidimensional geometric Brownian motion dynamics. In other words, all volatilities are constants. As usual, to extend the results for deterministic time dependent volatilities one just has to replace volatilities by their root-mean-square over the option life.

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To continue the discussion, let us fix some notations. In a multidimensional Black-Scholes model with $n$ stocks $S_1, \ldots, S_n$, risk neutral dynamics are given by

$$\frac{dS_i(t)}{S_i(t)} = r dt + \sum_{j=1}^{n} \sigma_{ij} dB_j(t),$$

with some initial values $S_1(0), \ldots, S_n(0)$. $B_1, \ldots, B_n$ are independent standard Brownian motions. Correlations among different stocks are captured through the matrix $(\sigma_{ij})$. Given a vector of weights $(w_i)_{i=1,\ldots,n}$, we are interested, for instance, in valuing the following basket option struck at $K$ whose payoff at maturity $T$ is

$$\left( \sum_{i=1}^{n} w_i S_i(T) - K \right)^+ .$$

Risk neutral valuation gives the price at time 0 as the following expectation

$$p = e^{-rT} \mathbb{E} \left\{ \left( \sum_{i=1}^{n} w_i S_i(T) - K \right)^+ \right\}. \tag{1}$$

Deriving formulas in closed form for such options with multivariate payoffs has already been tackled in the financial literature. For example, Jarrow and Rudd in [2] provide a general method based on Edgeworth (sometimes also called Charlier) expansions. Their idea is to replace the integration over the multidimensional log-normal distribution by an integration over another distribution with the same moments of low order so that this last integration can be done in closed form. In the case where the new distribution is Gaussian, this approximation is often called the Bachelier approximation since it gives back formulas alike those derived by Bachelier.

Another take on this problem (introduced in [4]) is to replace arithmetic averages by their corresponding geometric averages. The latter have the nice property of being log-normally distributed; they therefore lead to formula alike the Black-Scholes formula. See, for example, [3] pp. 218-225 for a presentation of these results. This method assumes however that the weights $(w_i)_{i=1,\ldots,n}$ are all positive. Our method does not require this assumption and will prove to be more accurate.

There are two difficulties in computing (1): the lack of tractability of the multivariate log-normal distribution on the one hand and the non linearity of the function $x \mapsto x^+$ on the other. Whereas [2] and [4] circumvent the first difficulty, our approach relies on finding optimal one-dimensional approximations thanks to properties of the function $x \mapsto x^+$. In one dimension, computations can then be carried out explicitly.

Approximations in closed form are given in Proposition 4 and 6 below. Various price sensitivities, the so-called Greeks, are given in Proposition 9, 10, 11 and 12. Section 3 shows actual numerical results as well as an extension to multivariate barrier options.

## 2. APPROXIMATE LOWER AND UPPER BOUNDS

As we have just explained, our goal is to compute $\mathbb{E}\{X^+\}$ where $X$ is the random variable

$$X = \sum_{i=0}^{n} \varepsilon_i x_i e^{G_i - \frac{1}{2}\mathrm{Var}(G_i)}.$$

$(G_i)_{i=0,\dots,n}$ is a mean zero Gaussian vector of size $n+1$ and covariance matrix $\Sigma$. $\varepsilon_i = \pm 1$ and $x_i > 0$ for all $i = 0, \dots, n$. In view of (1), this is just $\varepsilon_i = \mathrm{sgn}(w_i)$ and $x_i = |w_i| S_i(0)$. Note that entries of $\Sigma$ are dimensionless, that is, they are "volatilities squared $\times$ time to maturity".

Without loss of generality, we suppose that not all of the $\varepsilon_i$ have the same sign. If this were the case, computing $\mathbb{E}\{X^+\}$ would not present any difficulty. Note also that $\Sigma$ is symmetric positive semi-definite but not necessarily definite. Before we explain our approximation method we need the following definition and proposition.

**Definition 1.** *For every $i, j, k = 0, \dots, n$, we let*

$$\Sigma_{ij}^k = \Sigma_{ij} - \Sigma_{ik} - \Sigma_{kj} + \Sigma_{kk}$$

*and*

$$\sigma_i = \sqrt{\Sigma_{ii}} \qquad \sigma_i^k = \sqrt{\Sigma_{ii}^k}.$$

**Proposition 1.** *For every $k = 0, \dots, n$, let $(G_i^k)_{i=0,\dots,n}$ be a mean zero Gaussian vector with covariance $\Sigma^k$. Then,*

$$\mathbb{E}\{X^+\} = \mathbb{E}\left\{ \left( \sum_{i=0}^n \varepsilon_i x_i e^{G_i^k - \frac{1}{2} Var\left(G_i^k\right)} \right)^+ \right\}.$$

*Proof.* This is an easy consequence of Girsanov's transform. Indeed,

$$\begin{aligned} \mathbb{E}\{X^+\} &= \mathbb{E}\left\{ e^{G_k - \frac{1}{2}\mathrm{Var}(G_k)} \left( \sum_{i=0}^n \varepsilon_i x_i e^{G_i - G_k - \frac{1}{2}(\mathrm{Var}(G_i) - \mathrm{Var}(G_k))} \right)^+ \right\} \\ &= \mathbb{E}_{Q^k}\left\{ \left( \sum_{i=0}^n \varepsilon_i x_i e^{G_i - G_k - \frac{1}{2}(\mathrm{Var}(G_i) - \mathrm{Var}(G_k))} \right)^+ \right\}, \end{aligned}$$

where probability measure $Q^k$ is defined by its Radon-Nikodým derivative

$$\frac{dQ^k}{dP} = e^{G_k - \frac{1}{2}\mathrm{Var}(G_k)}.$$

Under $Q^k$, $(G_i - G_k)_{0 \le i \le n}$ is again a Gaussian vector. Its covariance matrix is $\Sigma^k$. ∎

Without loss of generality, we will also assume that for every $k = 0, \dots, n$, $\Sigma^k \neq 0$. Indeed if such were the case, Proposition 1 above would give us the price without any further computation.

### 2.1. Two optimization problems.

The following proposition will provide us with bounds.

**Proposition 2.** *For any $X \in L^1$,*

$$\sup_{0 \le Y \le 1} \mathbb{E}\{XY\} = \mathbb{E}\{X^+\} = \inf_{X = Z_1 - Z_2, Z_1 \ge 0, Z_2 \ge 0} \mathbb{E}\{Z_1\}. \tag{2}$$

*Proof.* On the left-hand side, letting $0 \le Y \le 1$,

$$\mathbb{E}\{XY\} = \mathbb{E}\{X^+ Y\} - \mathbb{E}\{X^- Y\} \le \mathbb{E}\{X^+\}$$

and taking $Y = \mathbf{1}_{\{X \ge 0\}}$ shows that the supremum is actually attained. On the right-hand side, it is well known that if $X = Z_1 - Z_2$ with both $Z_1$ and $Z_2$ non negative, then $Z_1 \ge X^+$. ∎

These two optimization problems are dual of each other in the sense of linear programming.

2.2. **Derivation of the lower bound.** Our lower bound is obtained by restricting the set over which the supremum in (2) is computed. We choose $Y$ of the form[1] $\mathbf{1}_{\{u\cdot G\leq d\}}$ where $u\in\mathbb{R}^{n+1}$ and $d\in\mathbb{R}$ are arbitrary. Let us let

$$p_* = \sup_{u,d}\mathbb{E}\left\{X\mathbf{1}_{\{u\cdot G\leq d\}}\right\}.$$

The next two propositions give further information on $p_*$. First, we need the following definition.

**Definition 2.** *Let $D$ to be the $(n+1)\times(n+1)$ diagonal matrix whose $i$th diagonal element is $1/\sigma_i$ if $\sigma_i\neq 0$ and 0 otherwise. Let $C$ to be such that*

$$C = D\Sigma D.$$

*$C$ is also a positive semi-definite matrix and we denote by $\sqrt{C}$ a square root of it (i.e., $C=\sqrt{C}\sqrt{C}^T$.)*

**Proposition 3.**

$$p_* = \sup_{d\in\mathbb{R}}\sup_{u\cdot\Sigma u=1}\sum_{i=0}^n\varepsilon_i x_i\Phi\left(d+(\Sigma u)_i\right) = \sup_{d\in\mathbb{R}}\sup_{\|v\|=1}\sum_{i=0}^n\varepsilon_i x_i\Phi\left(d+\sigma_i(\sqrt{C}v)_i\right).$$

Here and throughout the paper, we use the notation $\varphi(x)$ and $\Phi(x)$ for the density and the cumulative distribution function of the standard Gaussian distribution, i.e.,

$$\varphi(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\qquad\text{and}\qquad\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-u^2/2}du.$$

*Proof.*

$$\begin{aligned}
p_* &= \sup_{d\in\mathbb{R}}\sup_{u\in\mathbb{R}^{n+1}}\mathbb{E}\left\{\mathbb{E}\left\{X|u\cdot G\right\}\mathbf{1}_{\{u\cdot G\leq d\}}\right\}\\
&= \sup_{d\in\mathbb{R}}\sup_{u\in\mathbb{R}^{n+1}}\sum_{i=0}^n\varepsilon_i x_i\mathbb{E}\left\{e^{\frac{\mathrm{Cov}(G_i,u\cdot G)}{u\cdot\Sigma u}u\cdot G-\frac{1}{2}\frac{\mathrm{Cov}(G_i,u\cdot G)^2}{u\cdot\Sigma u}}\mathbf{1}_{\{u\cdot G\leq d\}}\right\}\\
&= \sup_{d\in\mathbb{R}}\sup_{u\cdot\Sigma u=1}\sum_{i=0}^n\varepsilon_i x_i\mathbb{E}\left\{e^{\mathrm{Cov}(G_i,u\cdot G)u\cdot G-\frac{1}{2}\mathrm{Cov}(G_i,u\cdot G)^2}\mathbf{1}_{\{u\cdot G\leq d\}}\right\}\\
&= \sup_{d\in\mathbb{R}}\sup_{u\cdot\Sigma u=1}\sum_{i=0}^n\varepsilon_i x_i\Phi\left(d+(\Sigma u)_i\right).
\end{aligned}$$

By defining $D^{-1}$ to be the $(n+1)\times(n+1)$ diagonal matrix whose $i$th diagonal element is $\sigma_i$, we easily check that $\Sigma=D^{-1}\sqrt{C}\sqrt{C}^TD^{-1}$. Therefore by taking $v=D^{-1}\sqrt{C}^Tu$, we have the second equality of the proposition. ∎

To actually compute this supremum, it is interesting to look at the Lagrangian $\mathcal{L}$:

$$\mathcal{L}(v,d)=\sum_{i=0}^n\varepsilon_i x_i\Phi\left(d+\sigma_i(\sqrt{C}v)_i\right)-\frac{\mu}{2}\left(\|v\|^2-1\right).$$

[1] $\cdot$ denotes the usual inner product of $\mathbb{R}^{n+1}$.

**Proposition 4.**

$$p_* = \sum_{i=0}^{n} \varepsilon_i x_i \Phi\left(d^* + \sigma_i(\sqrt{C}v^*)_i\right)$$

*where $d^*$ and $v^*$ satisfy the following first order conditions*

$$\sum_{i=0}^{n} \varepsilon_i x_i \sigma_i \sqrt{C}_{ij} \varphi\left(d^* + \sigma_i(\sqrt{C}v^*)_i\right) - \mu v_j^* \quad = \quad 0 \quad \textit{for } j = 0, \ldots, n \tag{3}$$

$$\sum_{i=0}^{n} \varepsilon_i x_i \varphi\left(d^* + \sigma_i(\sqrt{C}v^*)_i\right) \quad = \quad 0 \tag{4}$$

$$\|v^*\| \quad = \quad 1. \tag{5}$$

Note that expression for $p_*$ is as close to the classical Black-Scholes formula as one could hope. To conclude this subsection, we give a necessary condition for $d^*$ to be finite. It is interesting when it comes to numerical computations but it also ensures us that lower bounds are not trivial. We need to make a non-degeneracy assumption. Recall that the matrix $C$ was introduced in Definition 2. Through its definition, $C$ may have columns and rows of zeros. We are now assuming that the square matrix $\tilde{C}$ obtained by removing these rows and columns is non-singular. $\tilde{C}$ is well defined because we assumed that none of the $\Sigma^k$ (and therefore $\Sigma$) were actually the zero matrix.

**Condition 1.**

$$\det(\tilde{C}) \neq 0$$

**Proposition 5.** *Under Condition 1,*

$$p_* > \mathbb{E}\left\{X\right\}^+, \quad \textit{or equivalently} \quad |d^*| < +\infty.$$

*Proof.* Assume for instance that $\mathbb{E}\left\{X\right\} \geq 0$. We want to show that $p_* > \mathbb{E}\left\{X\right\}$. Let us let $f_v(d) = \sum_{i=0}^{n} \varepsilon_i x_i \Phi\left(d + \sigma_i(\sqrt{C}v)_i\right)$. First note that for any $v$, $\lim_{d\to+\infty} f_v(d) = \mathbb{E}\left\{X\right\}$. We are going to show the claim by showing that there exists a unit vector $v$ such that $f_v'(d) < 0$ when $d$ is near $+\infty$. Under Condition 1, $Range(\sqrt{C}) = \bigoplus_{i=0}^{n} \sigma_i \mathbb{R} \neq \{0\}$ and we can pick a unit vector $v$ such that $\sigma_i > 0 \Rightarrow \varepsilon_i(\sqrt{C}v)_i > 0$. For such a $v$, write $f_v'$ as

$$f_v'(d) = \varphi(d)\left\{\sum_{i:\varepsilon_i=+1} x_i e^{-d\sigma_i(\sqrt{C}v)_i - \frac{1}{2}\sigma_i^2(\sqrt{C}v)_i^2} - \sum_{i:\varepsilon_i=-1} x_i e^{-d\sigma_i(\sqrt{C}v)_i - \frac{1}{2}\sigma_i^2(\sqrt{C}v)_i^2}\right\}.$$

By denoting $\underline{\sigma} = \min_{i:\varepsilon_i=+1} \sigma_i(\sqrt{C}v)_i \geq 0$ and $\overline{\sigma} = \max_{i:\varepsilon_i=-1} \sigma_i(\sqrt{C}v)_i \leq 0$, we get the following bound, valid for $d \geq 0$:

$$f_v'(d) \leq \varphi(d)\left\{\left(\sum_{i:\varepsilon_i=+1} x_i\right) e^{-d\underline{\sigma} - \frac{1}{2}\underline{\sigma}^2} - \left(\sum_{i:\varepsilon_i=-1} x_i\right) e^{-d\overline{\sigma} - \frac{1}{2}\overline{\sigma}^2}\right\}.$$

Without loss of generality, we can assume that $\underline{\sigma}$ and $\overline{\sigma}$ are not simultaneously zero and the above upper bound is strictly negative for $d$ large enough. The case where $\mathbb{E}\left\{X\right\} \leq 0$ is treated analogously by showing that $f_v' > 0$ around $-\infty$. ∎

2.3. **Derivation of the upper bound.** Our upper bound is obtained by restricting the set over which the infimum in (2) is computed. For every $k = 0, \dots, n$, let $\mathcal{E}_k = \{i : \sigma_i^k \neq 0\} \neq \emptyset$. Let us also let $\tilde{x}_k = |\sum_{i \notin \mathcal{E}_k} \varepsilon_i x_i|$ and $\tilde{\varepsilon}_k = \operatorname{sgn}(\sum_{i \notin \mathcal{E}_k} \varepsilon_i x_i)$. Without loss of generality, we can assume $\tilde{x}_k > 0$. Then, choose reals $(\lambda_i^k)_{i \in \mathcal{E}_k}$ such that $\sum_{i \in \mathcal{E}_k} \lambda_i^k = -\tilde{\varepsilon}_k$ and rewrite $X$ as

$$\begin{aligned} X &= \sum_{i \in \mathcal{E}_k} \varepsilon_i x_i e^{G_i - \frac{1}{2}\mathrm{Var}(G_i)} - \lambda_i^k \tilde{x}_k e^{G_k - \frac{1}{2}\mathrm{Var}(G_k)} \\ &= \sum_{i \in \mathcal{E}_k} \left( \varepsilon_i x_i e^{G_i - \frac{1}{2}\mathrm{Var}(G_i)} - \lambda_i^k \tilde{x}_k e^{G_k - \frac{1}{2}\mathrm{Var}(G_k)} \right)^+ \\ &\quad - \sum_{i \in \mathcal{E}_k} \left( \varepsilon_i x_i e^{G_i - \frac{1}{2}\mathrm{Var}(G_i)} - \lambda_i^k \tilde{x}_k e^{G_k - \frac{1}{2}\mathrm{Var}(G_k)} \right)^- . \end{aligned}$$

The family of random variables $Z_1$ that we choose consists of those of the form

$$\sum_{i \in \mathcal{E}_k} \left( \varepsilon_i x_i e^{G_i - \frac{1}{2}\mathrm{Var}(G_i)} - \lambda_i^k \tilde{x}_k e^{G_k - \frac{1}{2}\mathrm{Var}(G_k)} \right)^+$$

where $k = 0, \dots, n$, $\sum_{i \in \mathcal{E}_k} \lambda_i^k = -\tilde{\varepsilon}_k$ and $\lambda_i^k \varepsilon_i > 0$ for all $i \in \mathcal{E}_k$. Because all the $\varepsilon_i$ do not have the same sign, the set of such $\lambda^k$ is nonempty for each $k$.

**Proposition 6.**

$$p^* = \min_{0 \le k \le n} \left\{ \sum_{i=0}^{n} \varepsilon_i x_i \Phi\left( d^k + \varepsilon_i \sigma_i^k \right) \right\} \tag{6}$$

*where $d^k$ is given by the following first order conditions*

$$\begin{aligned} \frac{\varepsilon_i}{\sigma_i^k} \ln\left( \frac{\varepsilon_i x_i}{\lambda_i^k \tilde{x}_k} \right) - \frac{\varepsilon_i \sigma_i^k}{2} &= \frac{\varepsilon_j}{\sigma_j^k} \ln\left( \frac{\varepsilon_j x_j}{\lambda_j^k \tilde{x}_k} \right) - \frac{\varepsilon_j \sigma_j^k}{2} = d^k \quad \text{for } i, j \in \mathcal{E}_k \\ \sum_{i \in \mathcal{E}_k} \lambda_i^k &= -\tilde{\varepsilon}_k \\ \lambda_i^k \varepsilon_i &> 0 \quad \text{for } i \in \mathcal{E}_k. \end{aligned}$$

Again, note that expression for $p^*$ is as close to the classical Black-Scholes formula as one could hope.

*Proof.*

$$\begin{aligned} p^* &= \min_{0 \le k \le n} \inf_{\sum_{i \in \mathcal{E}_k} \lambda_i^k = -\tilde{\varepsilon}_k} \mathbb{E} \left\{ \sum_{i \in \mathcal{E}_k} \left( \varepsilon_i x_i e^{G_i - \frac{1}{2}\mathrm{Var}(G_i)} - \lambda_i^k \tilde{x}_k e^{G_k - \frac{1}{2}\mathrm{Var}(G_k)} \right)^+ \right\} \\ &= \min_{0 \le k \le n} \inf_{\sum_{i \in \mathcal{E}_k} \lambda_i^k = -\tilde{\varepsilon}_k} \sum_{i \in \mathcal{E}_k} \varepsilon_i x_i \Phi\left( \frac{\varepsilon_i}{\sigma_i^k} \ln\left( \frac{\varepsilon_i x_i}{\lambda_i^k \tilde{x}_k} \right) + \frac{\varepsilon_i \sigma_i^k}{2} \right) \\ &\qquad\qquad - \lambda_i^k \tilde{x}_k \Phi\left( \frac{\varepsilon_i}{\sigma_i^k} \ln\left( \frac{\varepsilon_i x_i}{\lambda_i^k \tilde{x}_k} \right) - \frac{\varepsilon_i \sigma_i^k}{2} \right) . \end{aligned}$$

Forming the Lagrangian $\mathcal{L}^k$

$$\begin{aligned}\mathcal{L}^k(\lambda^k) &= \sum_{i\in\mathcal{E}_k} \varepsilon_i x_i \Phi\left(\frac{\varepsilon_i}{\sigma_i^k}\ln\left(\frac{\varepsilon_i x_i}{\lambda_i^k \tilde{x}_k}\right)+\frac{\varepsilon_i\sigma_i^k}{2}\right)\\ &\qquad -\lambda_i^k\tilde{x}_k\Phi\left(\frac{\varepsilon_i}{\sigma_i^k}\ln\left(\frac{\varepsilon_i x_i}{\lambda_i^k \tilde{x}_k}\right)-\frac{\varepsilon_i\sigma_i^k}{2}\right)-\mu\left(\sum_{i\in\mathcal{E}_k}\lambda_i^k+\tilde{\varepsilon}_k\right),\end{aligned}$$

we find the first order conditions

$$\frac{\partial\mathcal{L}^k}{\partial\lambda_i^k} = -\tilde{x}_k\Phi\left(\frac{\varepsilon_i}{\sigma_i^k}\ln\left(\frac{\varepsilon_i x_i}{\lambda_i^k \tilde{x}_k}\right)-\frac{\varepsilon_i\sigma_i^k}{2}\right)-\mu=0,$$

from which we deduce that the arguments of $\Phi$ must all equal each other. ∎

2.4. **Cases of equality.** It is easily seen that when $n=1$, lower and upper bounds both reduce to the Black-Scholes formula and therefore give the true value. Let us stress that $n=1$ not only contains the classical call and put options but also the exchange option of Margrabe. The following proposition gives other cases where the lower and upper bounds are in fact equal to the true value.

**Proposition 7.** *If for all* $i,j=0,\ldots,n$,

$$\Sigma_{ij}=\varepsilon_i\varepsilon_j\sigma_i\sigma_j,$$

*then*

$$p_*=p^*.$$

*Proof.* Exactly as in Proposition 1, note that for any $k$,

$$p_*=\sup_{d\in\mathbb{R}}\sup_{u\cdot\Sigma^k u=1}\sum_{i=0}^n\varepsilon_i x_i\Phi\left(d+(\Sigma^k u)_i\right).$$

Therefore, for any $k$,

$$\sup_{u\cdot\Sigma^k u=1}\sum_{i=0}^n\varepsilon_i x_i\Phi\left(d^k+(\Sigma^k u)_i\right)\le p_*\le p^*\le\sum_{i=0}^n\varepsilon_i x_i\Phi\left(d^k+\varepsilon_i\sigma_i^k\right). \tag{7}$$

Let us choose $k$ such that $\sigma_k=\min_{0\le i\le n}\sigma_i$. Note that under the hypothesis, $\Sigma_{ij}^k=(\varepsilon_i\sigma_i-\varepsilon_k\sigma_k)(\varepsilon_j\sigma_j-\varepsilon_k\sigma_k)$. Notice further that since all the $\varepsilon_i$ do not have the same sign, we can define the following vector $u$:

$$u_i=\frac{\mathrm{sgn}(\varepsilon_i\sigma_i-\varepsilon_k\sigma_k)}{\sum_{j=0}^n|\varepsilon_j\sigma_j-\varepsilon_k\sigma_k|}.$$

One trivially checks that $u\cdot\Sigma^k u=1$ and that $\left(\Sigma^k u\right)_i=\varepsilon_i\sigma_i-\varepsilon_k\sigma_k$. Because of the way we chose $k$,

$$\varepsilon_i\sigma_i-\varepsilon_k\sigma_k=\varepsilon_i|\varepsilon_i\sigma_i-\varepsilon_k\sigma_k|=\varepsilon_i\sigma_i^k.$$

This proves that the inequalities in (7) are in fact equalities. ∎

2.5. **Bound on the gap.** Although an estimate of the gap is readily available as soon as lower and upper bounds are computed, it is interesting to have an *a priori* bound on the gap $p^* - p_*$.

**Proposition 8.**

$$0 \leq p^* - p_* \leq \sqrt{\frac{2}{\pi}} \min_{0\leq k\leq n} \left\{ \sum_{i=0}^{n} x_i \sigma_i^k \right\}.$$

*Proof.*

$$\begin{aligned} p^* - p_* &\leq \min_{0\leq k\leq n} \sum_{i=0}^{n} \varepsilon_i x_i \Phi\left(d^k + \varepsilon_i \sigma_i^k\right) - \max_{0\leq k\leq n} \sum_{i=0}^{n} \varepsilon_i x_i \Phi\left(d^k + (\Sigma^k u^*)_i\right) \\ &\leq \min_{0\leq k\leq n} \sum_{i=0}^{n} \varepsilon_i x_i \left\{ \Phi\left(d^k + \varepsilon_i \sigma_i^k\right) - \Phi\left(d^k + (\Sigma^k u^*)_i\right) \right\}. \end{aligned}$$

By Cauchy-Schwarz inequality[2],

$$\left|(\Sigma^k u^*)_i\right| = \left| \sum_{j=0}^{n} \sum_{l=0}^{n} \Sigma_{lj}^k u_j^* \delta_{il} \right| \leq \sqrt{\sum_{j=0}^{n} \sum_{l=0}^{n} \Sigma_{lj}^k u_j^* u_l^*} \sqrt{\sum_{j=0}^{n} \sum_{l=0}^{n} \Sigma_{lj}^k \delta_{il} \delta_{ij}} = \sigma_i^k.$$

It follows that

$$p^* - p_* \leq \min_{0\leq k\leq n} \sum_{i=0}^{n} x_i \|\varphi\|_\infty \left(\sigma_i^k - (\Sigma^k u^*)_i\right) \leq 2\|\varphi\|_\infty \min_{0\leq k\leq n} \sum_{i=0}^{n} x_i \sigma_i^k,$$

which is the desired upper bound on the gap. ∎

2.6. **Computation of the Greeks.**

*Lower bound.* To compute partial derivatives with respect to the coefficients of $C$ (i.e., the various correlation parameters), we again need to make a non-degeneracy assumption. Assume Condition 1 holds true. Then, $\sqrt{\tilde{C}}$ is also non-singular and we define $C^{-\frac{1}{2}}$ to be the $(n+1) \times (n+1)$ matrix obtained with the entries of $\sqrt{\tilde{C}}^{-1}$ and putting back the rows and columns of zeros that we first removed from $C$.

**Proposition 9.** *Under Condition 1,*

$$\begin{aligned} \Delta_{*i} = \frac{\partial p_*}{\partial x_i} &= \varepsilon_i \Phi\left(d^* + \sigma_i(\sqrt{C}v^*)_i\right) \\ Vega_{*i} = \frac{\partial p_*}{\partial \sigma_i}\sqrt{T} &= \varepsilon_i x_i (\sqrt{C}v^*)_i \varphi\left(d^* + \sigma_i(\sqrt{C}v^*)_i\right)\sqrt{T} \\ \chi_{*ij} = \frac{\partial p_*}{\partial \rho_{ij}} &= \frac{1}{2}\sum_{k=0}^{n} \varepsilon_k x_k \left(\sigma_i C_{kj}^{-\frac{1}{2}} v_j^* + \sigma_j C_{ki}^{-\frac{1}{2}} v_i^*\right) \varphi\left(d^* + \sigma_k(\sqrt{C}v^*)_k\right) \\ \Theta_* = \frac{\partial p_*}{\partial T} &= \frac{1}{2T}\sum_{k=0}^{n} \varepsilon_k x_k \sigma_k (\sqrt{C}v^*)_k \varphi\left(d^* + \sigma_k(\sqrt{C}v^*)_k\right). \end{aligned}$$

[2] $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.

*Proof.* First order derivatives are easily computable thanks to the following observation.

$$\frac{dp_*}{dx_i} = \frac{\partial p_*}{\partial x_i} + \frac{\partial p_*}{\partial d}\frac{\partial d}{\partial x_i} + \nabla_v p_* \cdot \frac{\partial v}{\partial x_i} = \frac{\partial p_*}{\partial x_i} + \frac{\mu}{2}\frac{\partial \|v^*\|^2}{\partial x_i} = \frac{\partial p_*}{\partial x_i}$$

because $p_*$ satisfies the first order conditions (3-5) at $(d^*, v^*)$. ∎

Second order derivatives are more difficult to obtain since the previous trick is no longer possible. There exist however simple and natural approximations that satisfy the multidimensional Black-Scholes equation.

**Proposition 10.** *Let*

$$\Gamma_{*ij} = \varepsilon_i \varepsilon_j \frac{\varphi\left(d^* + \sigma_i(\sqrt{C}v^*)_i\right)\varphi\left(d^* + \sigma_j(\sqrt{C}v^*)_j\right)}{\sum_{k=0}^n \varepsilon_k x_k \sigma_k (\sqrt{C}v^*)_k \varphi\left(d^* + \sigma_k(\sqrt{C}v^*)_k\right)},$$

*then*

$$-\Theta_* + \frac{1}{2T}\sum_{i=0}^n \sum_{j=0}^n \Sigma_{ij} x_i x_j \Gamma_{*ij} = 0.$$

The unusual term "$1/2T$" comes from our convention on $\Sigma$.

*Proof.* It suffices to show that:

$$-\left(\sum_{k=0}^n \varepsilon_k x_k \sigma_k \varphi_k (\sqrt{C}v^*)_k\right)^2 + \sum_{i=0}^n \sum_{j=0}^n \varepsilon_i \varepsilon_j \Sigma_{ij} x_i x_j \varphi_i \varphi_j = 0$$

where we used the short-hand notation $\varphi_k = \varphi\left(d^* + \sigma_k(\sqrt{C}v^*)_k\right)$. Simply note that because of (3),

$$\begin{aligned}\sum_{i=0}^n \sum_{j=0}^n \varepsilon_i \varepsilon_j \Sigma_{ij} x_i x_j \varphi_i \varphi_j &= \sum_{i=0}^n \sum_{j=0}^n \sum_{k=0}^n \varepsilon_i \varepsilon_j \sigma_i \sigma_j \sqrt{C}_{ik} \sqrt{C}_{kj} x_i x_j \varphi_i \varphi_j \\ &= \mu \sum_{j=0}^n \varepsilon_j \sigma_j x_j \varphi_j (\sqrt{C}v^*)_j.\end{aligned}$$

Again because of (3) and (5), we have:

$$\mu = \sum_{i=0}^n \varepsilon_i \sigma_i x_i \varphi_i (\sqrt{C}v^*)_i.$$

This completes the proof. ∎

*Upper bound.* We now turn ourselves to the case of upper bounds. The function $\min$ is only almost everywhere differentiable. Therefore the next two propositions have to be understood in an almost sure sense. $k^*$ denotes the value for which the minimum is achieved in (6).

**Proposition 11.**

$$\begin{aligned}
\Delta_i^* = \frac{\partial p^*}{\partial x_i} &= \varepsilon_i \Phi\left(d^{k^*} + \varepsilon_i \sigma_i^{k^*}\right) \\
Vega_i^* = \frac{\partial p^*}{\partial \sigma_i}\sqrt{T} &= x_i \frac{\sigma_i - \rho_{ik^*}\sigma_{k^*}}{\sigma_i^{k^*}} \varphi\left(d^{k^*} + \varepsilon_i \sigma_i^{k^*}\right)\sqrt{T} \\
\chi_{ij}^* = \frac{\partial p^*}{\partial \rho_{ij}} &= -\delta_{jk^*} x_i \frac{\rho_{ik^*}\sigma_i \sigma_{k^*}}{\sigma_i^{k^*}} \varphi\left(d^{k^*} + \varepsilon_i \sigma_i^{k^*}\right) \\
\Theta^* = \frac{\partial p^*}{\partial T} &= \frac{1}{2T}\sum_{l=0}^{n} x_l \sigma_l^{k^*} \varphi\left(d^{k^*} + \varepsilon_l \sigma_l^{k^*}\right)
\end{aligned}$$

*with the convention* $0/0 = 0$*.*

*Proof.* We proceed in the same way as for the lower bound. ∎

**Proposition 12.** *Let*

$$\Gamma_{ij}^* = \begin{cases} \frac{\varphi\left(d^{k^*} + \varepsilon_i \sigma_i^{k^*}\right)}{x_i \sigma_i^{k^*}} \delta_{ij} & \textit{if } i \in \mathcal{E}_{k^*} \\ 0 \quad \textit{for all } j & \textit{if } i \notin \mathcal{E}_{k^*}, \end{cases}$$

*then*

$$-\Theta^* + \frac{1}{2T}\sum_{i=0}^{n}\sum_{j=0}^{n} \Sigma_{ij}^{k^*} x_i x_j \Gamma_{ij}^* = 0.$$

*Proof.* Straightforward. ∎

## 3. Numerical examples and Extensions

3.1. **Basket options.** As a first example, we shall consider the case of a basket option. For simplicity, let us suppose that there are $n$ stocks whose initial values are all \$1 and whose volatilities are also all the same, equal to $\sigma$. Correlation between any two distinct stocks is $\rho$. This amounts to the following:

$$\varepsilon = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ -1 \end{bmatrix}, \quad x = \begin{bmatrix} 1/n \\ \vdots \\ 1/n \\ K \end{bmatrix} \quad \text{and} \quad \Sigma = \sigma^2 T \begin{bmatrix} 1 & \rho & \cdots & \rho & 0 \\ \rho & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \rho & 0 \\ \rho & \cdots & \rho & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

The option has maturity 1 year. We present the results in Figure 1 when $n = 50$ and for different volatilities ($\sigma = 10\%, 20\%, 30\%$) and different correlation parameters ($\rho = 30\%, 50\%, 70\%$.) We plot lower and upper bounds against strike $K$. For the sake of comparison we also plot results of brute force Monte Carlo simulations with 100,000 simulated paths.

Agreement of the lower bound with Monte Carlo results is excellent, Monte Carlo results being sometimes slightly below the lower bound. Obviously, upper bounds are really not as good as lower bounds.

In view of these plots one can make two comments. The gap between lower and upper bound tends to decrease as correlation increases, which is in total agreement with Proposition 7. On the other hand the gap increases with the volatility, this, in turn, could be suspected from Proposition 8.

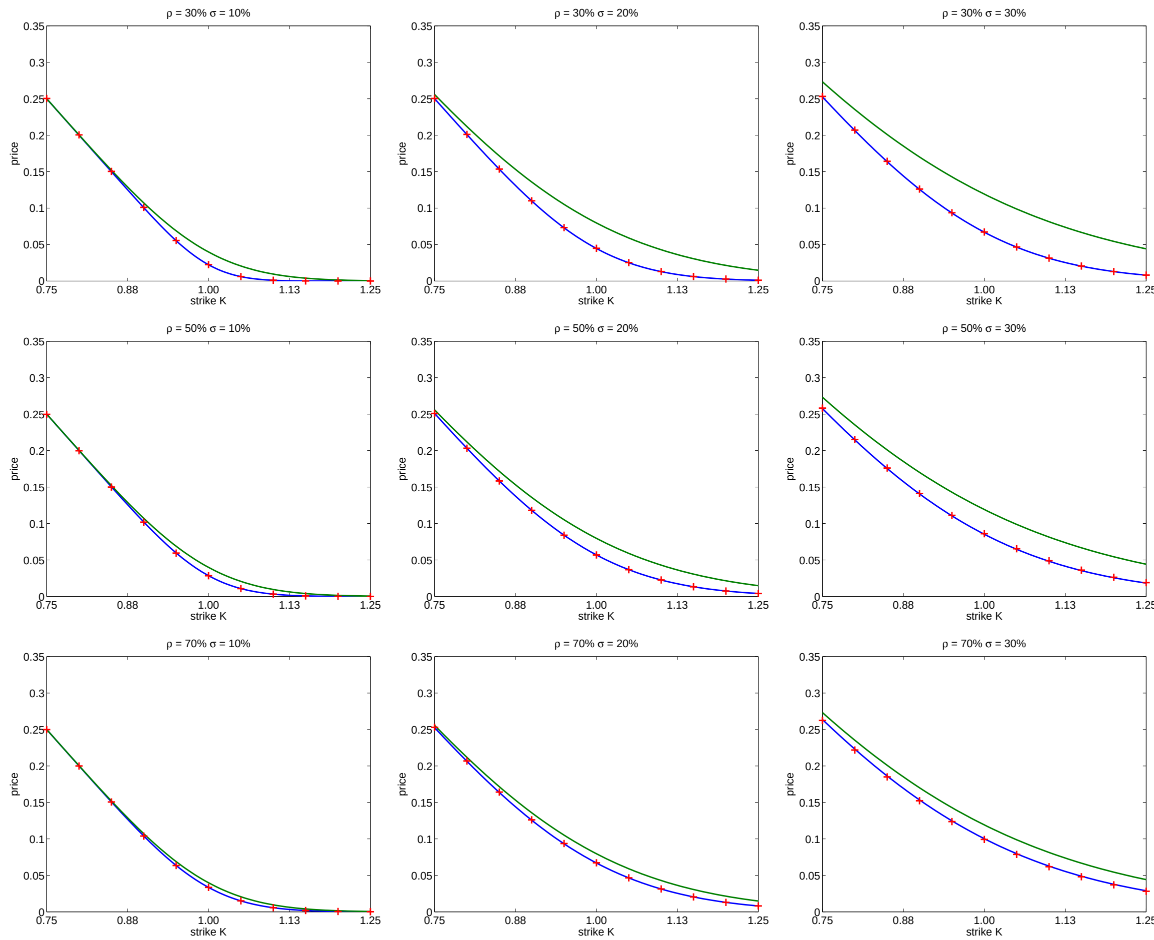


FIGURE 1. Lower and upper bound on the price for a basket option on 50 stocks (each one having a weight of $1/50$) as a function of $K$. "+" denote Monte Carlo results.

3.2. **Discrete-time average Asian options.** In the case of Asian option, we compare the lower bound with another often used approximative lower bound for Asian option. This lower bound is obtained by replacing an arithmetic average by a geometric one (see, for example, [4].) Results are reported in Figure 2. Again, we take an option with 1 year to expiry and an initial value for the stock of \$1. Averaging is performed over 50 equally spaced dates. Results are given for different stock volatilities ($\sigma = 10\%, 20\%, 30\%$.) The lower bound is uniformly better than the geometric average approximation.

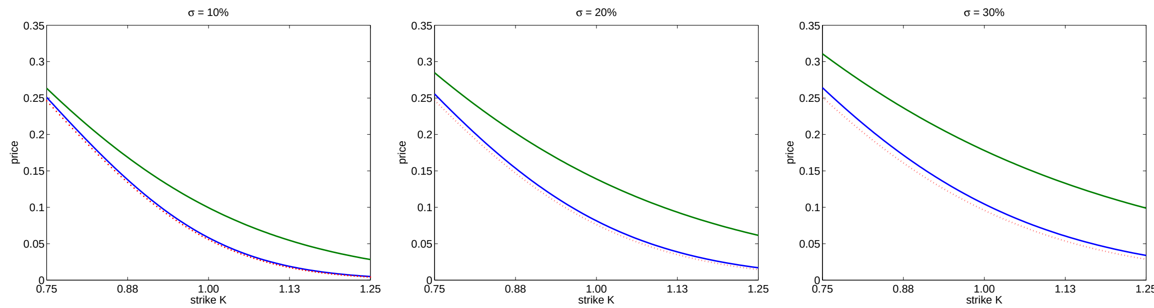


FIGURE 2. Lower and upper bound on the price of an Asian option. The dotted line represents the geometric average approximation.

3.3. **Basket options with barrier.** In this subsection, we show how to extend the previous results to the case of a basket option with a down-and-out barrier condition on the first stock of the basket. More specifically, the option payoff is

$$\left(\sum_{i=1}^{n} w_i S_i(T) - K\right)^+ \mathbf{1}_{\left\{\inf_{t\le T} S_1(t)\ge H\right\}}.$$

With the notation used so far, the option price is[3]

$$\mathbb{E}\left\{\left(\sum_{i=0}^{n} \varepsilon_i x_i e^{G_i(1)-\frac{1}{2}\sigma_i^2}\mathbf{1}_{\left\{\inf_{\theta\le 1} x_1 e^{G_1(\theta)-\frac{1}{2}\sigma_1^2\theta}\ge H\right\}}\right)^+\right\},$$

where $\{G(\theta);\theta\le 1\}$ is a $(n+1)$-dimensional Brownian motion starting from 0 with covariance $\Sigma$. We propose to approximate the option price and its replicating strategy with an optimal lower bound.

$$p_* = \sup_{d,u}\mathbb{E}\left\{\sum_{i=0}^{n} \varepsilon_i x_i e^{G_i(1)-\frac{1}{2}\sigma_i^2}\mathbf{1}_{\left\{\inf_{\theta\le 1} x_1 e^{G_1(\theta)-\frac{1}{2}\sigma_1^2\theta}\ge H; u\cdot G(1)\le d\right\}}\right\}.$$

Using Girsanov's theorem, this rewrites

$$p_* = \sup_{d,u}\sum_{i=0}^{n}\varepsilon_i x_i \mathbb{P}\left\{\inf_{\theta\le 1} G_1(\theta) + \left(\Sigma_{i1}-\sigma_1^2/2\right)\theta \ge \ln\left(\frac{H}{x_1}\right); u\cdot G(1)\le d-(\Sigma u)_i\right\}.$$

Let us define a new standard Brownian motion $\{W(\theta);\theta\le 1\}$ independent of $\{G_1(\theta);\theta\le 1\}$ by

$$u\cdot G(\theta) = \sqrt{u\cdot\Sigma u - \frac{(\Sigma u)_1^2}{\sigma_1^2}}W(\theta) + \frac{(\Sigma u)_1}{\sigma_1^2}G_1(\theta)$$

[3]Assuming, without loss of generality, that $\varepsilon_1 = +1$, $\sigma_1 > 0$ and $H < x_1$.

We choose to normalize $u$ by setting $|(\Sigma u)_1| = \sigma_1^2$. Letting $\lambda_i = \Sigma_{i1} - \sigma_1^2/2$ and $Y = d - (\Sigma u)_i - \sqrt{u\cdot\Sigma u - \sigma_1^2}W(1)$, we get

$$\begin{aligned} p_* &= \max\Bigg\{\sup_{d\in\mathbb{R}}\sup_{(\Sigma u)_1=\sigma_1^2}\sum_{i=0}^n \varepsilon_i x_i \mathbb{P}\left\{\inf_{\theta\le 1} G_1(\theta)+\lambda_i\theta \ge \ln\left(\frac{H}{x_1}\right); G_1(1)\le Y\right\};\\ &\qquad \sup_{d\in\mathbb{R}}\sup_{(\Sigma u)_1=-\sigma_1^2}\sum_{i=0}^n \varepsilon_i x_i \mathbb{P}\left\{\inf_{\theta\le 1} G_1(\theta)+\lambda_i\theta \ge \ln\left(\frac{H}{x_1}\right); G_1(1)\ge Y\right\}\Bigg\}. \end{aligned}$$

To compute these probabilities we use the following result (see, for example, [3] pp. 470.)

**Lemma 1.** *Let $B$ be standard Brownian motion and $X(\theta) = \sigma B(\theta) + \lambda\theta$, then for $y \le 0$,*

$$\mathbb{P}\left\{\inf_{\theta\le 1} X(\theta)\ge y; X(1)\le x\right\} = \begin{cases} \Phi\left(\frac{-y+\lambda}{\sigma}\right) - e^{\frac{2\lambda y}{\sigma^2}}\Phi\left(\frac{y+\lambda}{\sigma}\right) - \Phi\left(\frac{-x+\lambda}{\sigma}\right) \\ \quad + e^{\frac{2\lambda y}{\sigma^2}}\Phi\left(\frac{-x+\lambda+2y}{\sigma}\right) \quad \textit{if } y\le x \\ 0 \quad \textit{otherwise,} \end{cases}$$

$$\mathbb{P}\left\{\inf_{\theta\le 1} X(\theta)\ge y; X(1)\ge x\right\} = \begin{cases} \Phi\left(\frac{-x+\lambda}{\sigma}\right) - e^{\frac{2\lambda y}{\sigma^2}}\Phi\left(\frac{-x+\lambda+2y}{\sigma}\right) \quad \textit{if } y\le x \\ \Phi\left(\frac{-y+\lambda}{\sigma}\right) - e^{\frac{2\lambda y}{\sigma^2}}\Phi\left(\frac{y+\lambda}{\sigma}\right) \quad \textit{otherwise.} \end{cases}$$

Therefore, we have, by first conditioning on $Y$:[4]

$$\begin{aligned} &\mathbb{P}\left\{\inf_{\theta\le 1} G_1(\theta)+\lambda_i\theta \ge \ln\left(\frac{H}{x_1}\right); G_1(1)\le Y\right\} = \\ &\quad \mathbb{E}\Bigg\{\Bigg[\Phi\left(\frac{\lambda_i - \ln(H/x_1)}{\sigma_1}\right) - \left(\frac{H}{x_1}\right)^{\frac{2\lambda_i}{\sigma_1^2}}\Phi\left(\frac{\lambda_i+\ln(H/x_1)}{\sigma_1}\right) - \Phi\left(\frac{-Y}{\sigma_1}\right) \\ &\qquad\qquad + \left(\frac{H}{x_1}\right)^{\frac{2\lambda_i}{\sigma_1^2}}\Phi\left(\frac{-Y+2\ln(H/x_1)}{\sigma_1}\right)\Bigg]\mathbf{1}_{\{\ln(H/x_1)\le Y+\lambda_i\}}\Bigg\} = \\ &\quad \left[\Phi\left(\frac{\lambda_i - \ln(H/x_1)}{\sigma_1}\right) - \left(\frac{H}{x_1}\right)^{\frac{2\lambda_i}{\sigma_1^2}}\Phi\left(\frac{\lambda_i+\ln(H/x_1)}{\sigma_1}\right)\right]\Phi\left(\frac{d-(\Sigma u)_i+\lambda_i-\ln(H/x_1)}{\sqrt{u\cdot\Sigma u-\sigma_1^2}}\right) \\ &\quad -\Phi_2\left(-\frac{d-(\Sigma u)_i}{\sqrt{u\cdot\Sigma u}}, \frac{d-(\Sigma u)_i+\lambda_i-\ln(H/x_1)}{\sqrt{u\cdot\Sigma u-\sigma_1^2}}, -\sqrt{1-\frac{\sigma_1^2}{u\cdot\Sigma u}}\right) \\ &\quad + \left(\frac{H}{x_1}\right)^{\frac{2\lambda_i}{\sigma_1^2}}\Phi_2\left(-\frac{d-(\Sigma u)_i-2\ln(H/x_1)}{\sqrt{u\cdot\Sigma u}}, \frac{d-(\Sigma u)_i+\lambda_i-\ln(H/x_1)}{\sqrt{u\cdot\Sigma u-\sigma_1^2}}, -\sqrt{1-\frac{\sigma_1^2}{u\cdot\Sigma u}}\right). \end{aligned}$$

[4] $\Phi_2$ denotes the cumulative distribution function of the standard bivariate Gaussian distribution, i.e.,

$$\Phi_2(x,y,\rho) = \frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{x}\int_{-\infty}^{y}\exp\left(-\frac{1}{2}\frac{u^2-2\rho uv+v^2}{1-\rho^2}\right)dudv.$$

Similarly,

$$\mathbb{P}\left\{\inf_{\theta\le 1} G_1(\theta)+\lambda_i\theta \ge \ln\left(\frac{H}{x_1}\right); G_1(1)\ge Y\right\} =$$

$$\left[\Phi\left(\frac{\lambda_i-\ln(H/x_1)}{\sigma_1}\right)-\left(\frac{H}{x_1}\right)^{\frac{2\lambda_i}{\sigma_1^2}}\Phi\left(\frac{\lambda_i+\ln(H/x_1)}{\sigma_1}\right)\right]\Phi\left(\frac{d-(\Sigma u)_i-\lambda_i+\ln(H/x_1)}{\sqrt{u\cdot\Sigma u-\sigma_1^2}}\right)$$

$$+\Phi_2\left(\frac{d-(\Sigma u)_i}{\sqrt{u\cdot\Sigma u}},-\frac{d-(\Sigma u)_i-\lambda_i+\ln(H/x_1)}{\sqrt{u\cdot\Sigma u-\sigma_1^2}},-\sqrt{1-\frac{\sigma_1^2}{u\cdot\Sigma u}}\right)$$

$$-\left(\frac{H}{x_1}\right)^{\frac{2\lambda_i}{\sigma_1^2}}\Phi_2\left(\frac{d-(\Sigma u)_i-2\ln(H/x_1)}{\sqrt{u\cdot\Sigma u}},-\frac{d-(\Sigma u)_i-\lambda_i+\ln(H/x_1)}{\sqrt{u\cdot\Sigma u-\sigma_1^2}},-\sqrt{1-\frac{\sigma_1^2}{u\cdot\Sigma u}}\right).$$

Table 1 below gives prices for such options for various parameters. Framework and notation are the same as section 3.1. There are $n$ stocks whose initial values are \$1 and whose volatilities are all equal to $\sigma$. Correlation between any two distinct stocks is $\rho$ and options are at-the-money, i.e., $K = 1$.

| $\sigma$ | $\rho$ | $H/x_1$ | $n = 10$ | $n = 20$ | $n = 30$ |
|---|---|---|---|---|---|
| 0.4 | 0.5 | 0.7 | 0.1006 | 0.0938 | 0.0939 |
| 0.4 | 0.5 | 0.8 | 0.0811 | 0.0785 | 0.0777 |
| 0.4 | 0.5 | 0.9 | 0.0473 | 0.0455 | 0.0449 |
| 0.4 | 0.7 | 0.7 | 0.1191 | 0.1168 | 0.1165 |
| 0.4 | 0.7 | 0.8 | 0.1000 | 0.1006 | 0.0995 |
| 0.4 | 0.7 | 0.9 | 0.0608 | 0.0597 | 0.0594 |
| 0.4 | 0.9 | 0.7 | 0.1292 | 0.1291 | 0.1290 |
| 0.4 | 0.9 | 0.8 | 0.1179 | 0.1175 | 0.1173 |
| 0.4 | 0.9 | 0.9 | 0.0751 | 0.0747 | 0.0745 |
| 0.5 | 0.5 | 0.7 | 0.1154 | 0.1122 | 0.1110 |
| 0.5 | 0.5 | 0.8 | 0.0875 | 0.0844 | 0.0816 |
| 0.5 | 0.5 | 0.9 | 0.0518 | 0.0464 | 0.0458 |
| 0.5 | 0.7 | 0.7 | 0.1396 | 0.1389 | 0.1388 |
| 0.5 | 0.7 | 0.8 | 0.1103 | 0.1086 | 0.1080 |
| 0.5 | 0.7 | 0.9 | 0.0631 | 0.0619 | 0.0615 |
| 0.5 | 0.9 | 0.7 | 0.1597 | 0.1593 | 0.1592 |
| 0.5 | 0.9 | 0.8 | 0.1328 | 0.1322 | 0.1320 |
| 0.5 | 0.9 | 0.9 | 0.0786 | 0.0782 | 0.0780 |

TABLE 1. Lower bounds for a down-and-out call option on a basket of $n$ stocks.

## 4. CONCLUSION

This paper showed how to efficiently compute approximate prices and hedges of options on any linear combination of assets. Our general method allowed us to treat all these options in a common framework. Lower bounds prove to be extremely accurate. This methodology was applied to the pricing of basket, discrete-time average Asian options and basket options with barrier.

As an important by-product of this method, first and second order sensitivities are given in closed form at no extra cost. This is a clear advantage over Monte Carlo methods. Indeed, first order derivatives are also easily computable along with the price as explained, for example, in [1]. In the one-dimensional case, there is only one second order derivative (Gamma) and it can be computed by imposing that it satisfy Black-Scholes equation. In dimension $n \geq 2$, this PDE involves all $n$ second order partial derivatives, and it seems we need to compute $n-1$ of them, if we want to use the same trick.

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## REFERENCES

[1] P. Glasserman. *Monte Carlo Methods in Financial Engineering*. Springer-Verlag, New York, 2003.
[2] R. Jarrow and A. Rudd. Approximate option valuation for arbitrary stochastic processes. *J. Finan. Econ.*, 10:347–369, 1982.
[3] M. Musiela and M. Rutkowski. *Martingale Methods in Financial Modelling*. Springer Verlag, 1997.
[4] T. Vorst. Price and hedge ratios of average exchange options. *Internat. Rev. Finan. Anal.*, 1:179–193, 1992.

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