FINANCIAL MATHEMATICS

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The complexity, unpredictability and evolving nature of financial markets continues to provide an enormous challenge to mathematicians, engineers and economists to identify, analyze and quantify the issues and risks they pose. This has led to problems in stochastic analysis, simulation, differential equations, statistics and big data, and stochastic control and optimization (including dynamic game theory), all of which are reflected in the core of Financial Mathematics. Problems range from modeling a single risky stock and the risks of derivative contracts written on it, to understanding how intricate interactions between financial institutions may bring down the whole financial edifice and in turn the global economy, the problem of systemic risk. At the same time hitherto specialized markets, such as those in commodities (metals, agriculturals and energy), have become more *financialized*, which has led to our need to understand how financial reduced form models combine with supply and demand mechanisms.

In its early days, Financial Mathematics used to rest on two pillars which could be characterized roughly as *derivatives pricing* and *portfolio selection*. In this article, we outline its development into broader and more modern topics including, among others, energy and commodities markets, systemic risk, dynamic game theory and equilibrium, and understanding the impact of algorithmic and high frequency trading. We also touch on the 2008 Financial Crisis, among others, and the extent to which such increasingly frequent tremors call for more mathematics, not less in understanding and regulating financial markets and products.

1. Pricing and Development of Derivative Markets

Central to the development of quantitative finance, as distinguished from classical economics, was the modeling of uncertainty about future price fluctuations as phenomenological, rather than something that could be accurately captured by models of fundamentals, or demand and supply. The introduction of randomness into models of (initially) stock prices took off in the 1950s and 60s, particularly in the work of the economist Samuelson at MIT who adopted the continuous time Brownian-motion based tools of stochastic calculus that had been developed by physicists and mathematicians such as Einstein in 1905, Wiener in the 1920s, Lévy, Ornstein and Uhlenbeck in the 1930s, and Chandrasekhar in the 1940s. Samuelson, however, came to this technology not through physics but via a little known Ph.D. dissertation by Bachelier from 1900, which had formulated Brownian motion, for the purpose of modeling the Paris stock market, 5 years before Einstein's landmark paper in physics. Combined with the stochastic calculus developed by Itô in the early 1940s, these kinds of models became and still remain central to the analysis of a wide range of financial markets.

The Black-Scholes paradigm for equity derivatives was originally introduced in the context of Samuelson's model where stocks evolve according to geometric Brownian motions. If we consider a single stock for the sake of exposition, one assumes that the value at time t of one share is S_t which evolves according to the stochastic differential equation

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t, \tag{1}$$

with μ representing the expected growth rate, and $\sigma > 0$ the volatility. Here $(W_t)_{t\geq 0}$ is a standard Brownian motion. A contingent claim (or derivative security) on this stock is defined by its payoff at a later date T

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called maturity, and modeled as a random variable ξ whose uncertain value will be revealed at time T. The typical example is given by a European call option with maturity T and strike K, in which case, the payoff to the buyer of the option would be $\xi = (S_T - K)^+$, where we use the notation $x^+ = \max\{x, 0\}$ for the positive part of the real number x.

In order for the market model to exclude the possibility of making money with no risk (a "free lunch" usually called an arbitrage opportunity), the value at an earlier time t of owning this claim should be equal to the initial value of an investment in the stock and a safe bank account which could be managed into the same value as ξ at maturity T. Such a replicating portfolio would represent a perfect hedge, mitigating the uncertainty in the outcome of the option payoff ξ. The remarkable discovery in the early 1970s of Fisher Black, Myron Scholes and independently Robert Merton, was that it was possible to identify such a perfect replicating portfolio, and compute the initial investment needed to set it up by solving a linear partial differential equation (PDE) of parabolic type. They also provided formulas for the no-arbitrage prices of European call and put options known as Black-Scholes formulas. Later, in two papers that both appeared in 1979, the result was given its modern interpretation, first by Cox, Ross & Rubinstein in a simpler discrete model, and then by Harrison & Kreps in more general settings, which is that, if there are no arbitrage opportunities, the prices of all traded securities (stocks, futures, options) are given by computing the expectation of the present value (i.e. discounted value) of their future payoffs, but with respect to a "risk neutral" probability measure under which the growth rate μ in (1) is replaced by the riskless interest rate of the bank account.

While pricing by expectation is one of the important consequences of the Black, Scholes and Merton original works, it would be misleading and unfair to reduce their contribution to this aspect, even if it is at the origin of a wave (in retrospect it was clearly a tsunami) of interest in derivatives and the explosion of new and vibrant markets. Even though the original rationale was based on a hedging argument aimed at mitigating the uncertainties in the future outcomes of the value of the stock underlying the option, pricing, and especially pricing by expectation (whether the expectations were computed by Monte Carlo simulations or by PDEs provided by the Feynman-Kac formula), became the main motivation for many research programs.

1.1. **Stochastic Volatility Models.** It was long recognized (even by Black and Scholes and many others in the 1970s) that the lognormal distribution inherent in the geometric Brownian motion model (1) is not reflected by historical stock price data, and that volatility is not constant over time. Market participants were pricing options, even on a given day t, as if the volatility parameter σ depended upon the strike K and the time to maturity $T - t$ of the option.

To this day, two extensions of the model (1) have been used with great success to account for these stylized facts observed on empirical market data. They both involve replacing the volatility parameter by a stochastic process, so they can be viewed as stochastic volatility models. The first one is to replace equation (1) by

$$dS_t = \mu S_t \, dt + \sigma_t S_t \, dW_t, \tag{2}$$

where the volatility parameter σ is replaced by the value at time t of a stochastic process $(\sigma_t)_{t \geq 0}$ whose time evolution could be (for example) of the form

$$d\sigma_t = \lambda(\overline{\sigma} - \sigma_t) \, dt + \gamma \sqrt{\sigma_t} \, d\tilde{W}_t, \tag{3}$$

for some constants γ (known as volvol), $\overline{\sigma}$ the mean reversion level, and λ the rate of mean reversion, and where $(\tilde{W}_t)_{t \geq 0}$ is another Brownian motion which is typically negatively correlated with $(W_t)_{t \geq 0}$ to capture that when volatility rises, prices most often decline. Stochastic volatility models of this type have been (and are still) very popular. The books [5, 6] describe some of the research taking place with this approach.

Having two sources of random shocks (whether or not they are independent) creates some headaches for the quants as the no-arbitrage prices are now plentiful, and in face of the non-uniqueness of derivative prices, tricky *calibration* issues have to be resolved. So rather than dealing with the incompleteness of these stochastic volatility models, a more minimalist approach was proposed to capture the empirical properties of the option prices while at the same time keeping only one single source of shocks, and hence, completeness of the model. These models go under the name of *local volatility models*. They are based on dynamics given by Markovian stochastic differential equations of the form

$$dS_t = \mu S_t \, dt + \sigma(t, S_t) \, dW_t,$$

where $(t, s) \hookrightarrow \sigma(t, s)$ is a deterministic function, which can be computed from option prices using what is known as Dupire's formula in lieu of the geometric Brownian motion equation (1). As for the other stochastic volatility models, they are also the subject of active research.

According to the Black-Scholes theory, prices of contingent claims appear as risk adjusted expectations of the discounted cash flows triggered by the settlement of the claims. In the case of contingent claims with European exercises, the random variable giving the payoff is often given by a function of an underlying Markov process at the time of maturity of the claim. Using the Feynman-Kac formula, these types of expectations appear as solutions of PDEs of parabolic type, showing that computing prices can be done by solving PDEs. The classical machinery of the numerical analysis of linear PDEs is the cornerstone of most of the pricers of contingent claims in low dimensions. However, the increasing size of the *baskets* of instruments underlying the derivative contracts and the complexity of the exercise contingencies have limited the efficacy of the PDE solvers because of the high dimensionality. This is one of the main reasons for the increasing popularity of Monte Carlo Methods. Combined with regression ideas, they provide robust algorithms capable of pricing options (especially options with American exercises) on large portfolios avoiding in so doing the curse of dimensionality plaguing the traditional PDE methods. We refer to the book [7]. Moreover, the ease with which one can often generate Monte Carlo scenarios for the sole purpose of back testing and stress testing added to the popularity of these random simulation methods.

1.2. **Bond Pricing and Fixed Income Markets.** The early and mid-nineties saw growth of the bond markets (sudden increase in the traded volumes in Treasury, municipal, sovereign, corporate, . . . bonds), and the fixed income desks became a major source of profit for many investment banks and other financial institutions. The academic financial mathematics community took notice and a burst of research on mathematical models for fixed income instruments followed.

Parametric and non-parametric models for the term structure of interest rates (yield curves describing the evolution of forward interest rates as function of the maturity tenors of the bonds) were developed successfully from classical data analysis procedures. While spline smoothing was often used, principal component analysis of the yield data clearly points to a small number of easily identified factors and least square regressions can be used successfully to identify the term structure in parameterized families of curves. Despite the fact that infinite dimensional functional analysis and stochastic PDEs were brought to bear in order to describe the data, model calibration ended up being easier than in the case of the equity markets where the nonlinear correspondence between option prices and implied volatility surface creates challenges which, to this day, remain still mostly unsolved.

The development of fixed income markets was very rapid, and the complexity of interest rate derivatives (swaps, swaptions, floortions, captions, ...) motivated the scaling-up of mathematical models from the mere analysis of one dimensional stochastic differential equations, to the study of infinite dimensional stochastic systems and stochastic PDEs. See for example the recent textbooks [2, 3]. Current research in Financial Mathematics is geared toward the inclusion of jumps in these models and the understanding of the impact of these jumps on the calibration, pricing, and hedging procedures.

1.3. **Default Models and Credit Derivative Markets.** Buying a bond is just making a loan, issuing a bond is nothing but borrowing money. While the debt of the US government is still regarded by most (!) as default free, sovereign bonds and most corporate bonds carry a significant risk associated with the non-negligible probability that the issuer may not be able to do good on his debt, and may default by the time the principal of the loan is to be returned. Not surprisingly, models of default were first included successfully in the pricing of corporate bonds.

$\diamond$ *Structural models* of default based on the fundamentals of a firm and the competing roles of its assets and liabilities were first introduced by Merton in 1974. Their popularity was due to a rationale solidly grounded on fundamental financial principles and data reporting. However, murky data and lack of transparency have plagued the use of these models for the purpose of pricing corporate bonds and their derivatives.

⋄ On the other hand, *reduced form models* based on stochastic models for the intensity of arrival of the time of default, have gained in popularity because of the versatility of the intensity based models, and the simplicity and robustness of the calibration from Credit Default Swap (CDS) data. Indeed, one of the many reasons for the success of reduced form models is readily available data. Quotes of the spreads on CDS for most corporations are easy to get, and the fact that they are plentiful contrasts with the scarcity of corporate bond quotes which are few and far between. In fact, because the CDS market exploded, gauging the creditworthiness of a company is easily read off the CDS spreads instead of the bond spread (difference between the interest paid on a riskless government bond and a corporate bond).

⋄ A CDS is an insurance against the default of a corporation, say X. Such a CDS contract involves two counterparties, say Y and Z, the latter receiving a regular premium payment from Y as long as X does not default, and paying a lump sum to Y in case of default of X before the maturity of the CDS contract. So the existence of a CDS contract between Y and Z seems natural if the financial health of one of these two counterparties depends upon the survival of X, and the other counterparty is willing to take the opposite side of the transaction. However, none of the counterparties Y and Z *gambling* on the possibility of a serious credit event concerning X, both having different views on the likelihood of default, need to have any financial interest, direct or indirect, in X. In other words, two agents can enter into a deal involving a third entity just as a pure bet on the creditworthiness of this third entity. While originally designed as a credit insurance, CDSs ended up enhancing the overall risk in the system by the multiplication of *private* bets. As they spread like uncontrolled brush fires, they created an intricate network of complex dependencies between institutions, making it practically impossible to trace the sources of the risks.

1.4. **Securitization.** Investment is a risky business, and the risks of large portfolios of defaultable instruments (corporate bonds, loans, mortgages, CDSs) were clearly a major source of fear, at least until the spectacular growth in recent years of *securitization*. A financial institution bundles together a large number of such defaultable instruments, *slices* the portfolio according to the different levels of default risk, forming a small number (say 5) of tranches, keeps the riskiest one (called the "equity tranche"), and sells to trusting investors the remaining tranches (known as mezzanine or senior tranches) as an investment far safer than the entire portfolio itself. These instruments are called Collateralized Debt Obligations (CDOs). Pooling of risks and tranching them as re-insurance contracts to pass on to investors with differing risk profiles is natural, and the basis for insurance markets for dozens of years at least. But, unlike insurance products, CDOs were not regulated, and they enjoyed a tremendous success for most of the 2000s: credit desks multiplied, and academics and mathematicians tried to understand how practitioners were pricing them. Needless to say, no effort was made at hedging the risk exposure as not only was it not understood, but there was no reason to worry since it seemed that we could only *make money* in this game! Attempts at understanding their risk structure intensified after the serious warnings of 2004 and 2005, but they did not come early enough to seriously impact the onset of the financial crisis, most certainly triggered, or at least exacerbated by the housing freeze and the ensuing collapse of the huge market of Mortgage Backed Securities (MBS) which are complex CDOs on pools of mortgages. Part II and Part III of the Report of the Financial Crisis Inquiry Commission is an enlightening read for the connections between CDOs and the credit crisis.

The overuse of credit derivatives, particularly in the mortgage arena, contributed massively to the ongoing financial crisis. While the May 2005 ripple in the corporate CDO market served as a mild heart attack warning of worse things to come, its lessons were largely ignored as the risks abated. The attraction of unfunded returns on default protection proved too great, and the culture of unbounded bonus-based compensation for traders led to excessive risk taking that jeopardized numerous long-standing and once fiscally conservative financial institutions.

In the midst of the crisis, some questioned the role that quantitative models played in motivating or justifying these trades. A Wall Street Journal article from 2005 highlights the practice of playing CDO tranches off against each other to form a dubious hedge. In some sense this was "quantified" by calculations based on the Gaussian copula model (particularly implied correlations and its successor, the so-called base correlation) which were developed in-house by quants hired by and answering to traders. These models were developed for simplicity and speed: a single correlation parameter dictates a complex instrument dependent on hundreds of underlying and correlated risks. Many (surviving) banks have since re-structured their quant/modeling teams that they now report directly to management instead of to traders.

Numerous media outlets and commentators have blamed the crisis on financial models, and called for *less* quantification of risk. This spirit is distilled in Warren Buffet's comment " Beware of geeks bearing formulas". Indeed the caveat about the use of models to *justify a posteriori* "foolproof" hedges is justified. But this crisis, like past crises, only highlights the need for more mathematics, and quantitatively-trained people at the highest level, not least at ratings and regulatory agencies.

The real damage was done in the highly *unquantified* market for mortgage-backed securities (MBSs). Here the notions of independence and diversification through tranching was taken to ludicrous extremes. The MBS desk at major banks was relatively free of quants and quantitative analysis compared with the corporate CDO desk, even though the MBS book was many times larger. The excuse given was that these products were AAA and therefore like US Government bonds and did not need any risk analysis. As it turned out, this was a mass delusion willingly played into by banks, hedge funds and the like. The level of quantitative analysis performed by the ratings agencies was to test outcomes on a handful of scenarios, in which, typically, the worst-case scenario was that US house prices would appreciate at the rate 0.0%. The rest is history.

2. PORTFOLIO SELECTION AND INVESTMENT THEORY

A second central foundational pillar underlying modern Financial Mathematics research is the problem of optimal investment in uncertain market conditions. Typically, optimality is with respect to expected utility of portfolio value, where utility is measured by a concave increasing *utility function* as introduced by von Neumann and Morgenstern in the 1940s. A major breakthrough in applying continuous-time stochastic models to this problem is due to Robert Merton's work published in 1969 and 1971 and re-printed in [10]. In these works, Merton derived optimal strategies when stock prices have constant expected returns and volatilities and when the utility function has a specific convenient form. This remains among the few examples of explicit solutions to fully nonlinear Hamilton-Jacobi-Bellman (HJB for short) PDEs motivated by stochastic control applications.

To explain the basic analysis, suppose an investor has the choice between investing his capital in a single risky stock (or market index such as the S&P 500) or in a riskless bank account. The stock price S is uncertain and modeled as a geometric Brownian motion (1). Then the choice (or control) for the investor is π_t, the dollar amount to hold in the stock at time t, with his remaining wealth deposited in the bank earning interest at the constant rate r. If X_t denotes the portfolio value at time t, and if we assume the portfolio to be *self-financing* in the sense that no other monies flow in or out, then

$$dX_t = \pi_t \frac{dS_t}{S_t} + r(X_t - \pi_t)\, dt$$

is the equation governing his portfolio time evolution. We will take for simplicity of exposition $r = 0$, and so, from (1),

$$dX_t = \mu\pi_t\, dt + \sigma\pi_t\, dW_t.$$

Increasing the stock holding π increases the growth rate of X while also increasing its volatility. The investor is assumed to have a smooth terminal utility function $U(x)$ on $I\!R^+$ which satisfies the "usual conditions" (Inada and Asymptotic Elasticity):

$$U'(0^+) = \infty, \qquad U'(\infty) = 0, \qquad \lim_{x\to\infty} x\frac{U'(x)}{U(x)} < 1,$$

and he wants to maximize $\mathbb{E}\{U(X_T)\}$, his expected utility of wealth at a fixed time horizon T. To apply dynamic programming principles, it is standard to define the value function

$$V(t,x) = \sup_{\pi} \mathbb{E}\left\{U(X_T) \mid X_t = x\right\},$$

where the supremum is taken over admissible strategies which satisfy $\mathbb{E}\left\{\int_0^T \pi_t^2 dt\right\} < \infty$. Then $V(t,x)$ is the solution of the HJB PDE problem

$$V_t - \frac{1}{2}\lambda^2 \frac{V_x^2}{V_{xx}} = 0, \qquad V(T,x) = U(x),$$

where $\lambda := \mu/\sigma$ is known as the *Sharpe ratio*. Given the value function V, the optimal stock holding is given by

$$\pi_t^* = -\frac{\mu}{\sigma^2}\frac{V_x}{V_{xx}}(t, X_t).$$

Remarkably, Merton discovered an explicit solution when the utility is a power function:

$$U(x) = \frac{x^{1-\gamma}}{1-\gamma}, \qquad \gamma > 0, \quad \gamma \neq 1.$$

Here γ measures the concavity of the utility function and is known as the constant of relative risk-aversion. With this choice,

$$V(t,x) = \frac{x^{1-\gamma}}{1-\gamma} \exp\left(\frac{1}{2}\lambda^2 \left(\frac{1-\gamma}{\gamma}\right)(T-t)\right),$$

and more importantly,

$$\pi_t^* = \frac{\mu}{\sigma^2\gamma} X_t.$$

That is, the optimal strategy is to hold the fixed fraction $\mu/(\sigma^2\gamma)$ of current wealth in the stock, and the rest in the bank. As the stock price rises, this strategy says to sell some stock so that the fraction of the portfolio comprised of the risky asset remains the same. This fixed-mix result generalizes to multiple securities as long as they are also assumed to be (correlated) geometric Brownian motions.

Since Merton's work, the basic problem has been generalized in many directions. In particular, developments in duality theory (or martingale method) led to a revolution in thinking as to how these problems should be studied in abstract settings, culminating in very general results in the context of semimartingale models of incomplete markets. One of the most challenging problems was to extend the theory in the presence of transaction costs as this required the fine analysis of singular stochastic control problems. For recent developments, we refer to the survey article [8].

Optimal investment still generates challenging research problems as models evolve to try and incorporate realistic features such as uncertain volatility or random jumps in prices using modern technology such as forward-backward stochastic differential equations (FBSDEs), asymptotic approximations for fully nonlinear HJB PDEs, and related numerical methods.

Earlier, portfolio optimization was achieved through the well known Markowitz linear quadratic optimization problems based on the analysis of the mean vector and covariance matrix. The use of increasingly large portfolios and the introduction of ETF tracking indexes (S&P 500, Russell 2000, ...) created the need for more efficient estimation methods for large covariance matrices, typically using sparsity and robustness arguments. This problem has been the major impetus behind a significant proportion of the *big data* research in statistics.

3. Growing Research Areas

3.1. **Systemic Risk.** While the mathematical theory of risk measures, like Value at Risk (VaR), expected shortfall, or maximum drawdown, which was introduced to help policy makers and portfolio managers quantify risk and define unambiguously a form of capital requirement to preserve solvency, enjoyed immediate success among the mathematicians working on financial applications, their dynamical analogs did not benefit from the same popular support, mostly because of their complexity and the challenges in attempting to aggregate firm level risks into system behavior.

Research on systemic risk began in earnest after the September 11, 2001 attack. Then the financial crisis of 2008 brought counter party risk and the propagation of defaults to the forefront. The analysis of large complex systems where all the entities behave rationally at the individual level, but which produce calamities at the aggregate level, is a very exciting challenge for mathematicians who are now developing models for what some economists have called *rational irrationality*. See for example the recent handbook [4] for many viewpoints to this problem.

3.2. **Energy and the New Commodity Markets.** As in the case of the models of defaults used in the fixed income markets and subsequently in the credit markets, models for commodities can be roughly speaking divided into two distinct categories: reduced form models and structural models.

⋄ Commodities are physical in nature and are overwhelmingly traded on a forward basis, namely for future delivery. So, as with fixed income markets, a snapshot of the state of a market is best provided by a term structure of forwards of varying maturities. But unlike the forward rates, forward prices are actually prices of traded commodities and as a consequence, should be modeled as martingales. Despite this fundamental difference, the first models for commodities were borrowed (at least in spirit) from the models developed by sophisticated researchers for the fixed income markets. However, the shortcomings of these ad-hoc transplants created an exodus of the fundamental research in this area from reduced form models to structural models more in line with equilibrium arguments and the economic rationale of supply and demand. A case in point is the pricing of electricity contracts: Figure 1 shows the time evolution of electric power spot prices in a recently deregulated market. Clearly, none of the mathematical models used for equities, currencies or interest rates can be calibrated to be consistent with these data, and structural models involving the factors driving demand (like weather) and supply (like means of production) need to be used to give a reasonable account of the spikes. See for example the recent survey [1] and the book [12].

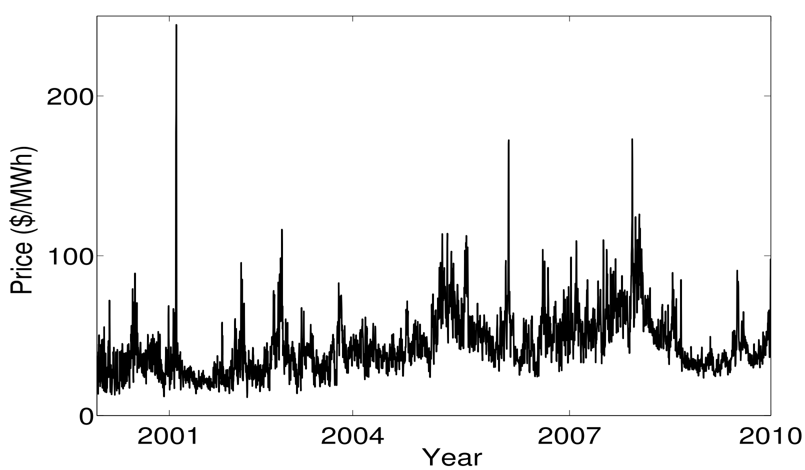

FIGURE 1. Historical daily prices of electricity from the PJM market in the North East US

⋄ The production of electricity is one of the major sources of Green House Gas emissions and various forms of market mechanisms have been touted to control these harmful externalities. The most famous is undoubtedly the implementation of the Kyoto protocol commitments in the form of mandatory cap-and-trade for CO_2 allowance certificates known as the European Union (EU) Emissions Trading Scheme (ETS). While policy issues are still muddying the final form the control of CO_2 emissions will take in the US, cap-and-trade scheme already exist in the North East of the country (RGGI) and more recently in California, giving a new impetus to the theoretical research of these markets.

$\diamond$ The proliferation of commodity indexes and the dramatic increase of investors gaining commodity exposure through ETFs tracking indexes have changed the landscape of the commodity markets and increased the correlations between commodities and equity, and among commodities included in the same indexes. Figure 2 illustrates this striking change in correlations. These changes are difficult to explain by relying solely on the fundamentals of these markets. They seem to be part of a phenomenon, known as *financialization* of the commodity markets, which has been taking place over the last 10 years, and which is now investigated by a growing number of economists, econometricians and mathematicians. Also of great interest to theoretical research is the fact that financial institutions, including hedge funds, endowment and pension funds, have realized that the road to success in the commodity markets was more often than not, relying on managing portfolios including both physical and financial assets. Once more, this combination raises new issues which are not addressed by traditional financial mathematics or financial engineering. Finally, the physical nature of energy production led to the introduction of new financial markets like the weather, freight and the emissions markets, whose design, regulation and investigation pose new mathematical challenges.

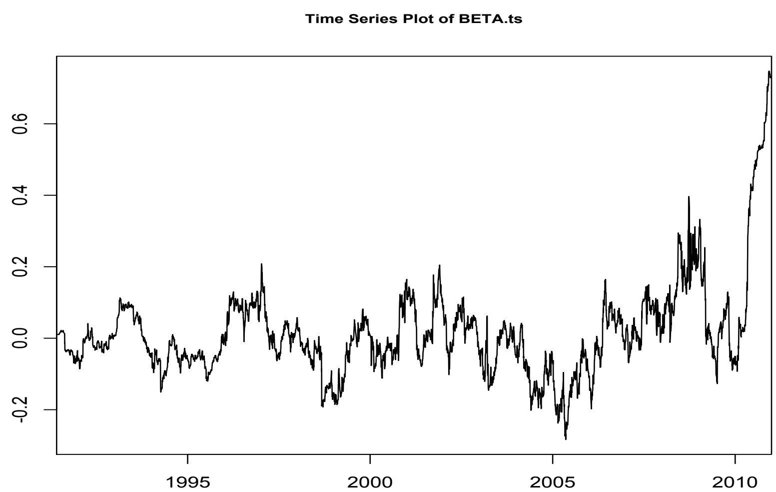

FIGURE 2. Instantaneous Dependence (β) of the Goldman Sachs Total Return Commodity Index upon the S&P 500 returns

3.3. **High Frequency Trading.** In the last few years, the notion of one price, publicly known, and at which transactions can happen in arbitrary sizes has been challenged. The existence and the importance of liquidity frictions and price impact due to the size and frequency of trades are recognized as the source of many of the most spectacular failures (LTCM, Amaranth, Lehman, ...) prompting new research in applications of stochastic optimization to optimal execution and predatory trading for example.

Another important driver in the change of direction in quantitative finance research is the growing role of algorithmic and high frequency trading. Indeed, it is commonly accepted that between sixty and seventy percent of trading is electronic nowadays. Market makers and brokers are now mostly electronic, and while they are claimed to be liquidity providers – the jury is still out on that one – occurrences like the flash crash of May 6, 2010, and the computer glitches like those which took down Knight Capital, have raised serious concerns, and research on the development of Limit Order Book (LOB) models and their impact on trading is clearly one of the emerging topics in quantitative finance research. See for example the recent book [9].

3.4. **Back to Basics: Stochastic Equilibrium and Stochastic Games.** More recently these tools have been adapted to problems involving multiple 'agents' optimizing for themselves but interacting with each other through a market. These involve analyzing and computing an equilibrium, which may come from a market clearing condition, for example, or in enforcing the strong competition of a Nash equilibrium. There has been much recent progress in stochastic differential games. We mention, for example, recent works of Lasry and Lions who consider Mean Field Games in which there are a large number of players and competition is felt only through an average of one's competitors, with each player's impact on the average being negligible.

Problems of price impact, stability, liquidity and formation of bubbles remain of vital interest and have produced very interesting mathematics along the way. Not long after the 1987 crash, there was much concern to what extent the large drop was caused or exacerbated by program traders whose computers were automatically hedging options positions causing them to sell mechanically when prices went down, pushing them down further. In a continuous time framework, this type of feedback model of price impact was initiated in in the late 1990s, and since then, there have been many influential studies of situations where an investor in not simply a "price taker" and how large stock positions are sold off in pieces to avoid large price depressing trades, the problem of optimal execution. A recent survey of work on asset price bubbles (and subsequent crashes) can be found in [11].

FURTHER READING

[1] R. Carmona and M. Coulon. A survey of commodity markets and structural approaches to modeling electricity. In F. Benth, editor, *Energy Markets, Proceedings of the WPI Special Year*, pages 1– 42. Springer Verlag, New York, NY, 2013.
[2] R. Carmona and M. Tehranchi. *Interest Rate Models: an Infinite Dimensional Stochastic Analysis Perspective*. Springer Finance. Springer, 2006.
[3] D. Filipovic. *Term Strucutre Models: a Graduate Course*. Springer-Verlag, 2009.
[4] J.-P. Fouque and J. Langsam., editors. *Handbook of Risk*. Cambridge University Press, 2012.
[5] J.-P. Fouque, G. Papanicolaou, R. Sircar, and K. Sølna. *Multiscale Stochastic Volatility for Equity, Interest-Rate and Credit Derivatives*. Cambridge University Press, 2011.
[6] J. Gatheral. *The Volatility Surface: a Practitioner's Guide*. John Wiley and Sons, Inc., 2006.
[7] P. Glasserman. *Monte Carlo Methods in Financial Engineering*. Applications of Mathematics 53. Springer, 2003.
[8] P. Guasoni and J. Muhle-Karbe. Portfolio choice with transaction costs: a user's guide. In V. Henderson and R. Sircar, editors, *Paris-Princeton Lectures on Mathematical Finance 2013*. Springer, 2013.
[9] C.A. Lehalle and S. Laruelle. *Market Microstructure in Practice*. World Scientific, 2013.
[10] R.C. Merton. *Continuous-Time Finance*. Blackwell, 1992.
[11] P. Protter. A mathematical theory of financial bubbles. In V. Henderson and R. Sircar, editors, *Paris-Princeton Lectures on Mathematical Finance 2013*. Springer, 2013.
[12] G. Swindle. *Valuation and Risk Management in Energy Markets*. Cambridge University Press, 2014.