Energy Markets II: Spread Options & Asset Valuation

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European Call on the difference between two indexes

Calendar Spread Options

- Single Commodity at two different times

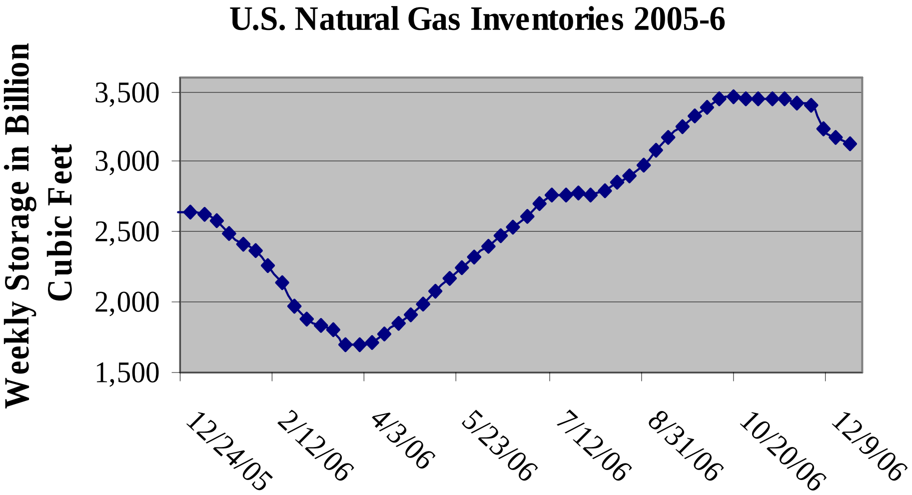
$$\mathbb{E}\{(I(T_2) - I(T_1) - K)^+\}$$

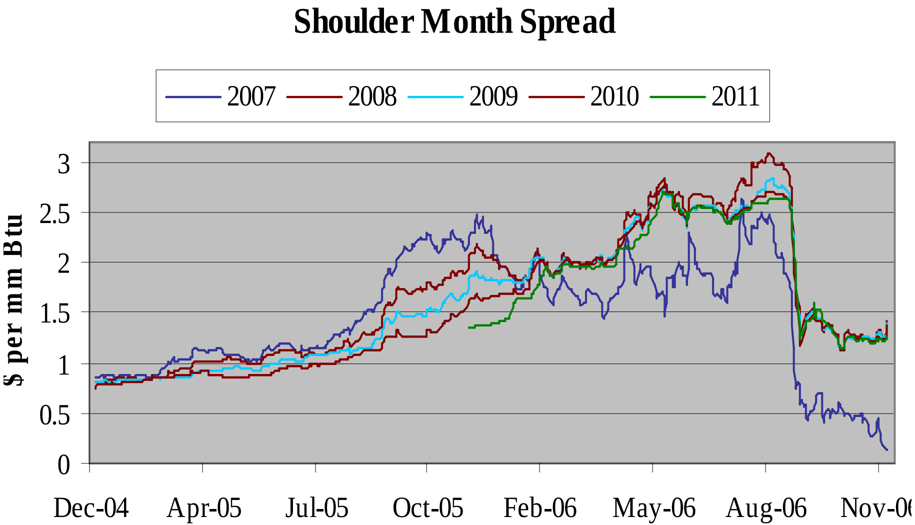
- Mathematically easier (only one underlier)

Amaranth largest (and **fatal**) positions

- Shoulder Natural Gas Spread (play on inventories)
- **Long** March Gas / **Short** April Gas
  - Depletion stops in March / injection starts in April
  - Can be fatal: ***widow maker spread***

Seasonality of Gas Inventory




Shoulder Month Spread
2007
2008
2009
2010
2011
$ per mm Btu
3
2.5
2
1.5
1
0.5
0
Dec-04
Apr-05
Jul-05
Oct-05
Feb-06
May-06
Aug-06
Nov-06

More Spread Options

- **Cross Commodity**
  - Crush Spread: between Soybean and soybean products (meal & oil)
  - Crack Spread:
    - gasoline crack spread between Crude and Unleaded
    - heating oil crack spread between Crude and HO
  - **Spark spread**

$$S_t = F_E(t) - H_{eff} F_G(t)$$

H_{eff} **Heat Rate**

Synthetic Generation

Present value of profits for future power generation (case of one fuel)

$$\mathbb{E}\{\int_0^T D(0,t)(\tilde{F}_P(t,\tau) - H * \tilde{F}_G(t,\tau) - K)^+ \, dt\}$$

where

- $\tau > 0$ fixed (small)
- $D(0,t)$ **discount factor** to compute present values
- $\tilde{F}_P(t,\tau)$ (resp. $\tilde{F}_G(t,\tau)$) price at time t of a power (resp. gas) contract with delivery $t+\tau$
- H **Heat Rate**
- K **Operation and Maintenance** cost (sometimes denoted $O\&M$)

Basket of Spread Options

Deterministic discounting (with constant interest rate)

$$D(t, T) = e^{-r(T-t)}$$

Interchange **expectation** and **integral**

$$\int_0^T e^{-rt}\mathbb{E}\{(\tilde{F}_P(t,\tau) - H * \tilde{F}_G(t,\tau) - K)^+\}\, dt$$

Continuous **stream of spread options**

In Practice

- **Discretize time**, say daily

$$\sum_{t=0}^{T} e^{-rt}\mathbb{E}\{(\tilde{F}_P(t,\tau) - H * \tilde{F}_G(t,\tau) - K)^+$$

- **Bin** Daily Production in **Buckets** B_k's (e.g. 5×16, 2×16, 7×8, settlement locations, ...).

$$\sum_{t=0}^{T} e^{-r(T-t)} \sum_{k} \mathbb{E}\{(\tilde{F}_P^{(k)}(t,\tau) - H^{(k)} * \tilde{F}_G^{(k)}(t,\tau) - K^{(k)})^+\}$$

Basket of Spark Spread Options

Spread Mathematical Challenge

$$p = e^{-rT}\mathbb{E}\{(I_2(T) - I_1(T) - K)^+\}$$

- Underlying indexes are spot prices
  - Geometric Brownian Motions ($K = 0$ Margrabe)
  - Geometric Ornstein-Uhlembeck (OK for Gas)
  - Geometric Ornstein-Uhlembeck with jumps (OK for Power)
- Underlying indexes are forward/futures prices
  - HJM-type models with deterministic coefficients

Problem

finding closed form formula and/or fast/sharp approximation for

$$\mathbb{E}\{(\alpha e^{\gamma X_1} - \beta e^{\delta X_2} - \kappa)^+\}$$

for a Gaussian vector (X_1, X_2) of $N(0,1)$ random variables with correlation ρ.

Sensitivities?

Easy Case : Exchange Option & Margrabe Formula

$$p = e^{-rT}\mathbb{E}\{(S_2(T) - S_1(T))^+\}$$

- $S_1(T)$ and $S_2(T)$ **log-normal**
- p given by a formula *à la Black-Scholes*

$$p = x_2 N(d_1) - x_1 N(d_0)$$

with

$$d_1 = \frac{\ln(x_2/x_1)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T} \qquad d_0 = \frac{\ln(x_2/x_1)}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T}$$

and:

$$x_1 = S_1(0), \quad x_2 = S_2(0), \quad \sigma^2 = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2$$

- Deltas are also given by ”closed form formulae”.

Proof of Margrabe Formula

$$p = e^{-rT}\mathbb{E}_{\mathbb{Q}}\{(S_2(T) - S_1(T))^+\} = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left\{\left(\frac{S_2(T)}{S_1(T)} - 1\right)^+ S_1(T)\right\}$$

- $\mathbb{Q}$ risk-neutral probability measure
- Define (Girsanov) $\mathbb{P}$ by:

$$\left.\frac{d\mathbb{P}}{d\mathbb{Q}}\right|_{\mathcal{F}_T} = S_1(T) = \exp\left(-\frac{1}{2}\sigma_1^2 T + \sigma_1 \hat{W}_1(T)\right)$$

- Under $\mathbb{P}$,
  - $\hat{W}_1(t) - \sigma_1 t$ and $\hat{W}_2(t)$
  - S_2/S_1 is geometric Brownian motion under $\mathbb{P}$ with volatility

$$\sigma^2 = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2$$

$$p = S_1(0)\mathbb{E}_{\mathbb{P}}\left\{\left(\frac{S_2(T)}{S_1(T)} - 1\right)^+\right\}$$

Black-Scholes formula with $K = 1$, σ as above.

Pricing Calendar Spreads in Forward Models

Model

$$dF(t,T) = F(t,T)[\mu(t,T)dt + \sum_{k=1}^{n} \sigma_k(t,T)dW_k(t)]$$

$\mu(t,T)$ and $\sigma_k(t,T)$ deterministic so

forward prices are log-normal

Calendar Spread involves prices of two forward contracts with different maturities

$$S_1(t) = F(t,T_1) \qquad \text{and} \qquad S_2(t) = F(t,T_2),$$

Price at time t of a calendar spread option with maturity T and strike K

$$\mathbb{E}\{(F(T,T_2) - F(T,T_1) - K)^*\}$$

Use formula for

$$\mathbb{E}\{(\alpha e^{\gamma X_1} - \beta e^{\delta X_2} - \kappa)^+\}$$

with

$$\alpha = e^{-r[T-t]} F(t, T_2), \quad \beta = \sqrt{\sum_{k=1}^{n} \int_t^T \sigma_k(s, T_2)^2 ds},$$

$$\gamma = e^{-r[T-t]} F(t, T_1), \quad \text{and} \qquad \delta = \sqrt{\sum_{k=1}^{n} \int_t^T \sigma_k(s, T_1)^2 ds}$$

and $\kappa = e^{-r(T-t)}$ ($\mu \equiv 0$ per risk-neutral dynamics)

$$\rho = \frac{1}{\beta\delta} \sum_{k=1}^{n} \int_t^T \sigma_k(s, T_1) \sigma_k(s, T_2)\, ds$$

Pricing Spark Spreads in Forward Models

Cross-commodity

- subscript **e** for forward prices, times-to-maturity, volatility functions, . . . relative to electric power
- subscript **g** for quantities pertaining to natural gas.

Pay-off

$$\left(F_e(T, T_e) - H * F_g(T, T_g) - K\right)^+ .$$

- $T < \min\{T_e, T_g\}$
- Heat rate H
- Strike K given by O& M costs

Natural

- **Buyer** owner of a power plant that transforms gas into electricity,
- **Protection** against low electricity prices and/or high gas prices.

Joint Dynamics of the Commodities

$$\begin{cases} dF_e(t,T_e) &= F_e(t,T_e)[\mu_e(t,T_e)dt + \sum_{k=1}^n \sigma_{e,k}(t,T_e)dW_k(t)] \\ dF_g(t,T_g) &= F_g(t,T_g)[\mu_g(t,T_g)dt + \sum_{k=1}^n \sigma_{g,k}(t,T_g)dW_k(t)] \end{cases}$$

- Each commodity has its own volatility factors
- between The two dynamics share the **same** driving Brownian motion processes W_k, hence **correlation.**

Fitting Join Cross-Commodity Models

- on any given day t we have
  - electricity forward contract prices for $N^{(e)}$ times-to-maturity $\tau_1^{(e)} < \tau_2^{(e)}, \ldots < \tau_{N^{(e)}}^{(e)}$
  - natural gas forward contract prices for $N^{(g)}$ times-to-maturity $\tau_1^{(g)} < \tau_2^{(g)}, \ldots < \tau_{N^{(g)}}^{(g)}$

Typically $N^{(e)} = 12$ and $N^{(g)} = 36$ (possibly more).

- Estimate instantaneous vols $\sigma^{(e)}(t)$ & $\sigma^{(g)}(t)$ 30 days rolling window
- For each day t, the $N = N^{(e)} + N^{(g)}$ dimensional random vector $\mathbf{X}(t)$

$$\mathbf{X}(t) = \begin{bmatrix} \left(\frac{\log \tilde{F}_e(t+1,\tau_j^{(e)}) - \log \tilde{F}_e(t,\tau_j^{(e)})}{\sigma^{(e)}(t)} \right)_{j=1,\ldots,N^{(e)}} \\ \left(\frac{\log \tilde{F}_g(t+1,\tau_j^{(g)}) - \log \tilde{F}_g(t,\tau_j^{(g)})}{\sigma^{(g)}(t)} \right)_{j=1,\ldots,N^{(g)}} \end{bmatrix}$$

- Run PCA on historical samples of $\mathbf{X}(t)$
- Choose small number n of factors
- for $k = 1, \ldots, n$,
  - first $N^{(e)}$ coordinates give the electricity volatilities $\tau \hookrightarrow \sigma_k^{(e)}(\tau)$ for $k = 1, \ldots, n$
  - remaining $N^{(g)}$ coordinates give the gas volatilities $\tau \hookrightarrow \sigma_k^{(g)}(\tau)$.

Skip gory details

Pricing a Spark Spread Option

Price at time t

$$p_t = e^{-r(T-t)}\mathbb{E}_t\left\{(F_e(T,T_e) - H * F_g(T,T_g) - K)^+\right\}$$

$F_e(T,T_e)$ and $F_g(T,T_g)$ are log-normal under the pricing measure calibrated by PCA

$$F_e(T,T_e) = F_e(t,T_e)\exp\left[-\frac{1}{2}\sum_{k=1}^{n}\int_t^T \sigma_{e,k}(s,T_e)^2 ds + \sum_{k=1}^{n}\int_t^T \sigma_{e,k}(s,T_e)dW_k(s)\right]$$

and:

$$F_g(T,T_g) = F_g(t,T_g)\exp\left[-\frac{1}{2}\sum_{k=1}^{n}\int_t^T \sigma_{g,k}(s,T_g)^2 ds + \sum_{k=1}^{n}\int_t^T \sigma_{g,k}(s,T_g)dW_k(s)\right]$$

Set

$$S_1(t) = H * F_g(t,T_g) \qquad \text{and} \qquad S_2(t) = F_e(t,T_e)$$

Pricing a Spark Spread Option

Use the constants

$$\alpha = e^{-r(T-t)} F_e(t, T_e), \qquad \text{and} \qquad \beta = \sqrt{\sum_{k=1}^{n} \int_t^T \sigma_{e,k}(s, T_e)^2\, ds}$$

for the first log-normal distribution,

$$\gamma = He^{-r(T-t)} F_g(t, T_g), \qquad \text{and} \qquad \delta = \sqrt{\sum_{k=1}^{n} \int_t^T \sigma_{g,k}(s, T_g)^2\, ds}$$

for the second one, $\kappa = e^{-r(T-t)} K$ and

$$\rho = \frac{1}{\beta\delta} \int_t^T \sum_{k=1}^{n} \sigma_{e,k}(s, T_e) \sigma_{g,k}(s, T_g) ds$$

for the correlation coefficient.

Approximations

- Fourier Approximations (Madan, Carr, Dempster, . . .)
- Bachelier approximation
- Zero-strike approximation
- Kirk approximation
- Upper and Lower Bounds

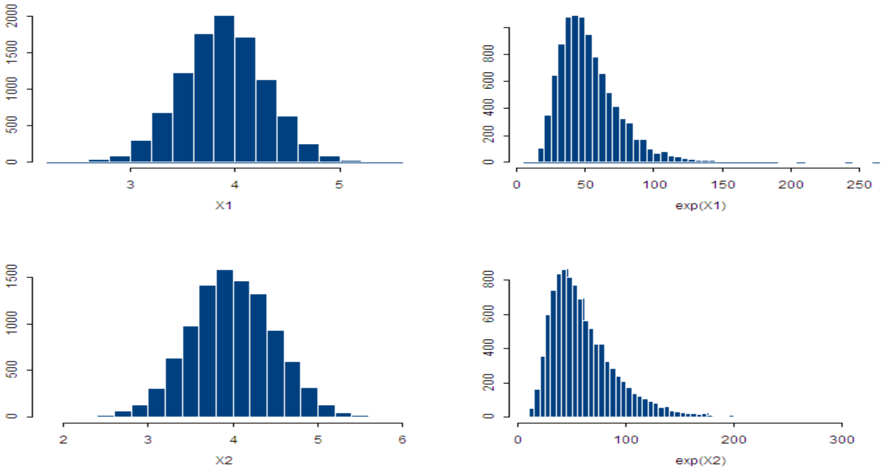
Can we also approximate the **Greeks** ?

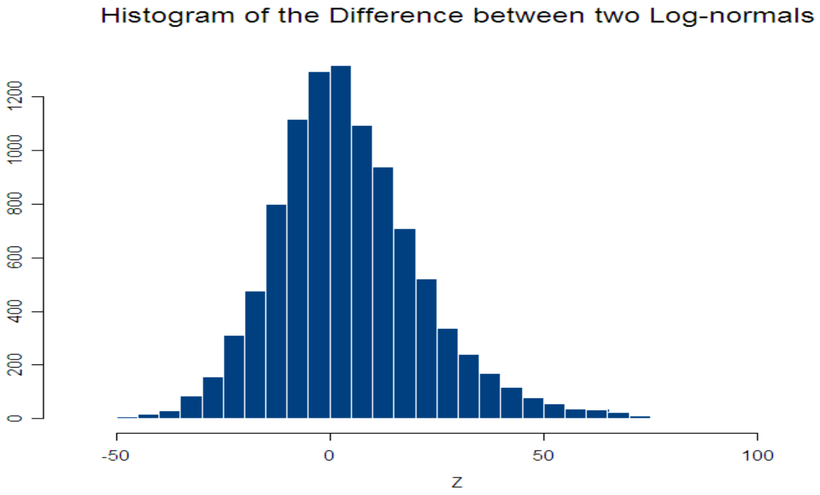
Bachelier Approximation

- Generate $x_1^{(1)}, x_2^{(1)}, \cdots, x_N^{(1)}$ from $N(\mu_1, \sigma_1^2)$
- Generate $x_1^{(2)}, x_2^{(2)}, \cdots, x_N^{(2)}$ from $N(\mu_1, \sigma_1^2)$
- Correlation ρ
- Look at the distribution of

$$e^{x_1^{(2)}} - e^{x_1^{(1)}}, e^{x_2^{(2)}} - e^{x_2^{(1)}}, \cdots, e^{x_N^{(2)}} - e^{x_N^{(1)}}$$

Log-Normal Samples




Histogram of the Difference between two Log-normals
0
200
400
600
800
1000
1200
-50
0
50
100
Z

Bachelier Approximation

- Assume $(S_2(T) - S_1(T)$ is Gaussian
- Match the first two moments

$$p = \left(m(T) - Ke^{-rT}\right)\Phi\left(\frac{m(T) - Ke^{-rT}}{s(T)}\right) + s(T)\varphi\left(\frac{m(T) - Ke^{-rT}}{s(T)}\right)$$

with:

$$\begin{aligned} m(T) &= (x_2 - x_1)e^{(\mu - r)T} \\ s^2(T) &= e^{2(\mu - r)T}\left[x_1^2\left(e^{\sigma_1^2 T} - 1\right) - 2x_1x_2\left(e^{\rho\sigma_1\sigma_2 T} - 1\right) + x_2^2\left(e^{\sigma_2^2 T} - 1\right)\right] \end{aligned}$$

Easy to compute the Greeks !

Zero-Strike Approximation

$$p = e^{-rT}\mathbb{E}\{(S_2(T) - S_1(T) - K)^+\}$$

- Assume $S_2(T) = F_E(T)$ is **log-normal**
- Replace $S_1(T) = H * F_G(T)$ by $\tilde{S}_1(T) = S_1(T) + K$
- Assume $S_2(T)$ and $S_1(T)$ are **jointly log-normal**
- Use **Margrabe** formula for $p = e^{-rT}\mathbb{E}\{(S_2(T) - \tilde{S}_1(T))^+\}$

Use the Greeks from Margrabe formula !

Kirk Approximation

$$\hat{p}^K = x_2 \Phi \left(\frac{\ln\left(\frac{x_2}{x_1 + Ke^{-rT}}\right)}{\sigma^K} + \frac{\sigma^K}{2} \right) - (x_1 + Ke^{-rT}) \Phi \left(\frac{\ln\left(\frac{x_2}{x_1 + Ke^{-rT}}\right)}{\sigma^K} - \frac{\sigma^K}{2} \right)$$

where

$$\sigma^K = \sqrt{\sigma_2^2 - 2\rho\sigma_1\sigma_2 \frac{x_1}{x_1 + Ke^{-rT}} + \sigma_1^2 \left(\frac{x_1}{x_1 + Ke^{-rT}} \right)^2}.$$

Exactly what we called "Zero Strike Approximation"!!!

Upper and Lower Bounds

$$\Pi(\alpha,\beta,\gamma,\delta,\kappa,\rho) = \mathbb{E}\left\{\left(\alpha e^{\beta X_1-\beta^2/2} - \gamma e^{\delta X_2-\delta^2/2} - \kappa\right)^+\right\}$$

where

- α, β, γ, δ and κ real constants
- X_1 and X_2 are jointly Gaussian $N(0,1)$
- correlation ρ

$\alpha = x_2 e^{-q_2 T}$ $\beta = \sigma_2\sqrt{T}$ $\gamma = x_1 e^{-q_1 T}$ $\delta = \sigma_1\sqrt{T}$ and $\kappa = Ke^{-rT}$.

Strategy for a Lower Bound

$$\mathbb{E}\{X^+\} = \sup_{0 \le Y \le 1} \mathbb{E}\{XY\}$$

So in particular

$$\mathbb{E}\{X^+\} \ge \sup_{u_1, u_2, d \in \mathbb{R}} \mathbb{E}\{X \mathbf{1}_{\{u_1 X_1 + u_2 X_2 \le d\}}\}$$

and we apply this to

$$X = \alpha e^{\beta X_1 - \beta^2/2} - \gamma e^{\delta X_2 - \delta^2/2} - \kappa$$

so everything can be computed!

A Precise Lower Bound

$$\hat{p} = x_2 e^{-q_2 T}\Phi\left(d^* + \sigma_2\cos(\theta^* + \phi)\sqrt{T}\right) - x_1 e^{-q_1 T}\Phi\left(d^* + \sigma_1\sin\theta^*\sqrt{T}\right) - Ke^{-rT}\Phi(d^*)$$

where

- θ^* is the solution of

$$\frac{1}{\delta\cos\theta}\ln\left(-\frac{\beta\kappa\sin(\theta+\phi)}{\gamma[\beta\sin(\theta+\phi)-\delta\sin\theta]}\right) - \frac{\delta\cos\theta}{2}$$
$$= \frac{1}{\beta\cos(\theta+\phi)}\ln\left(-\frac{\delta\kappa\sin\theta}{\alpha[\beta\sin(\theta+\phi)-\delta\sin\theta]}\right) - \frac{\beta\cos(\theta+\phi)}{2}$$

- the angle ϕ is defined by setting $\rho = \cos\phi$
- d^* is defined by

$$d^* = \frac{1}{\sigma\cos(\theta^* - \psi)\sqrt{T}}\ln\left(\frac{x_2 e^{-q_2 T}\sigma_2\sin(\theta^* + \phi)}{x_1 e^{-q_1 T}\sigma_1\sin\theta^*}\right) - \frac{1}{2}(\sigma_2\cos(\theta^* + \phi) + \sigma_1\cos\theta^*)\sqrt{T}$$

- the angles ϕ and ψ are chosen in $[0, \pi]$ such that:

$$\cos\phi = \rho \qquad \text{and} \qquad \cos\psi = \frac{\sigma_1 - \rho\sigma_2}{\sigma},$$

Remarks on this Lower Bound

- $\hat{p}$ is equal to the true price p when
  - $K = 0$
  - $x_1 = 0$
  - $x_2 = 0$
  - $\rho = -1$
  - $\rho = +1$
- Margrabe formula when $K = 0$ because

$$\theta^* = \pi + \psi = \pi + \arccos\left(\frac{\sigma_1 - \rho\sigma_2}{\sigma}\right).$$

with:

$$\sigma = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$$

Delta Hedging

The portfolio comprising at each time $t \leq T$

$$\Delta_1 = -e^{-q_1 T}\Phi\left(d^* + \sigma_1 \cos\theta^* \sqrt{T}\right)$$

and

$$\Delta_2 = e^{-q_2 T}\Phi\left(d^* + \sigma_2 \cos(\theta^* + \phi)\sqrt{T}\right)$$

units of each of the underlying assets is a **sub-hedge**

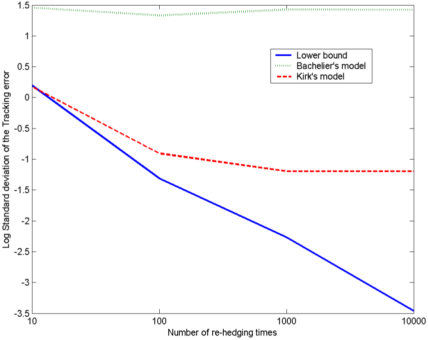
*its value at maturity is a.s. a **lower bound** for the pay-off*

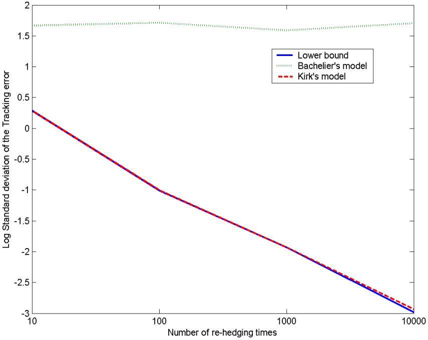
The Other Greeks

- ϑ_1 and ϑ_2 sensitivities w.r.t. volatilities σ_1 and σ_2
- χ sensitivity w.r.t. correlation ρ
- κ sensitivity w.r.t. strike price K
- Θ sensitivity w.r.t. maturity time T

$$
\begin{aligned}
\vartheta_1 &= x_1 e^{-q_1 T} \varphi\left(d^* + \sigma_1 \cos\theta^* \sqrt{T}\right) \cos\theta^* \sqrt{T} \\
\vartheta_2 &= -x_2 e^{-q_2 T} \varphi\left(d^* + \sigma_2 \cos(\theta^* + \phi) \sqrt{T}\right) \cos(\theta^* + \phi) \sqrt{T} \\
\chi &= -x_1 e^{-q_1 T} \varphi\left(d^* + \sigma_1 \cos\theta^* \sqrt{T}\right) \sigma_1 \frac{\sin\theta^*}{\sin\phi} \sqrt{T} \\
\kappa &= -\Phi\left(d^*\right) e^{-rT} \\
\Theta &= \frac{\sigma_1 \vartheta_1 + \sigma_2 \vartheta_2}{2T} - q_1 x_1 \Delta_1 - q_2 x_2 \Delta_2 - rK\kappa
\end{aligned}
$$

Comparisons





Behavior of the tracking error as the number of re-hedging times increases. The model data are $x_1 = 100$, $x_2 = 110$, $\sigma_1 = 10\%$, $\sigma_2 = 15\%$ and $T = 1$. $\rho = 0.9$, $K = 30$ (left) and $\rho = 0.6$, $K = 20$ (right).

Tolling Agreements?

Stylized Version

- **Leasing an Energy Asset**
  - Fossil Fuel Power Plant
  - Oil Refinery
  - Pipeline
- **Owner of the Agreement**
  - Decides **when** and **how** to use the asset (e.g. run the power plant)
  - Has someone else do the leg work

The Classical (Real Option) Approach

- Lifetime of the plant $[T_1, T_2]$
- C **capacity** of the plant (in MWh)
- H **heat rate** of the plant (in MMBtu/MWh)
- P_t price of **power** on day t
- G_t price of **fuel** (gas) on day t
- K fixed **Operating Costs**
- **Value of the Plant (ORACLE)**

$$C \sum_{t=T_1}^{T_2} e^{-rt} \mathbb{E}\{(P_t - HG_t - K)^+\}$$

String of Spark Spread Options

Plant Operation Model: the Finite Mode Case

- Markov process (state of the world) $X_t = (X_t^{(1)}, X_t^{(2)}, \cdots)$ (e.g. $X_t^{(1)} = P_t, \quad X_t^{(2)} = G_t, \quad X_t^{(3)} = O_t$ for a dual plant)
- Plant characteristics
  - $\mathbb{Z}_M \triangleq \{0, \cdots, M-1\}$ modes of operation of the plant
  - $H_0, H_1 \cdots, H_{M-1}$ **heat rates**
  - $\{C(i,j)\}_{(i,j)\in\mathbb{Z}_M}$ regime **switching costs** $(C(i,j) = C(i,\ell) + C(\ell,j))$
  - $\psi_i(t,x)$ **reward** at time t when world in state x, plant in mode i
- Operation of the plant (control) $u = (\xi, \mathcal{T})$ where
  - $\xi_k \in \mathbb{Z}_M \triangleq \{0, \cdots, M-1\}$ successive modes
  - $0 \leqslant \tau_{k-1} \leqslant \tau_k \leqslant T$ switching times
- T (horizon) length of the tolling agreement
- Total **reward**

$$H(x, i, [0,T]; u)(\omega) \triangleq \int_0^T \neg\psi_{u_s}(s, X_s)\, ds - \sum_{\tau_k < T} C(u_{\tau_k -}, u_{\tau_k})$$

Stochastic Control Problem

- $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t^X), \mathbb{P})$ (risk neutral) stochastic basis
- $\mathcal{U}(t))$ *acceptable* controls on $[t, T]$
  - adapted càdlàg $\mathbb{Z}_M$-valued processes u of a.s. finite variation on $[t, T]$

Optimal Switching Problem

$$J(t,x,i) = \sup_{u \in \mathcal{U}(t)} J(t,x,i;u),$$

where

$$\begin{aligned} J(t,x,i;u) &= \mathbb{E}\big[H(x,i,[t,T];u) \big| X_t = x, u_t = i\big] \\ &= \mathbb{E}\big[\int_0^T \neg\psi_{u_s}(s,X_s)\,ds - \sum_{\tau_k < T} C(u_{\tau_k -}, u_{\tau_k}) \big| X_t = x, u_t = i\big] \end{aligned}$$

Iterative Optimal Stopping

$$\mathcal{U}^k(t) \triangleq \{(\xi, \mathcal{T}) \in \mathcal{U}(t) \colon \tau_\ell = T \text{ for } \ell \geqslant k+1\}$$

Admissible strategies on $[t, T]$ with at most k switches

$$J^k(t,x,i) \triangleq \text{esssup}_{u \in \mathcal{U}^k(t)} \mathbb{E}\Big[\int_t^T \neg\psi_{u_s}(s, X_s)\, ds - \sum_{t \leqslant \tau_k < T} C(u_{\tau_k -}, u_{\tau_k}) \Big| X_t = x, u_t = i\Big].$$

Alternative **recursive construction**

$$J^0(t,x,i) \triangleq \mathbb{E}\Big[\int_t^T \neg\psi_i(s, X_s)\, ds \Big| X_t = x\Big],$$

$$J^k(t,x,i) \triangleq \sup_{\tau \in \mathcal{S}_t} \mathbb{E}\Big[\int_t^\tau \neg\psi_i(s, X_s)\, ds + \mathcal{M}^{k,i}(\tau, X_\tau) \Big| X_t = x\Big].$$

Intervention operator $\mathcal{M}$

$$\mathcal{M}^{k,i}(t,x) \triangleq \max_{j \neq i} \Big\{ -C_{i,j} + J^{k-1}(t,x,j) \Big\}.$$

Studied mathematically by **Hamadène - Jeanblanc** ($M = 2$).

Alternative Formulations

- Variational Formulation and Viscosity Solutions of PDEs
- System of Reflected Backward Stochastic Differential Equations (BSDEs)

- Time Step $\Delta t = T/M^\sharp$
- Time grid $\mathcal{S}^\Delta = \{m\Delta t,\ m = 0, 1, \dots, M^\sharp\}$
- Switches are allowed in $\mathcal{S}^\Delta$

DPP

For $t_1 = m\Delta t, t_2 = (m+1)\Delta t$ consecutive times

$$J^k(t_1, X_{t_1}, i) = \max\Big(\mathbb{E}\Big[\int_{t_1}^{t_2} \neg\psi_i(s, X_s)\, ds + J^k(t_2, X_{t_2}, i) | \mathcal{F}_{t_1}\Big],\ \mathcal{M}^{k,i}(t_1, X_{t_1})\Big)$$
$$\simeq \Big(\psi_i(t_1, X_{t_1})\,\Delta t + \mathbb{E}\big[J^k(t_2, X_{t_2}, i) | \mathcal{F}_{t_1}\big]\Big) \vee \Big(\max_{j \neq i}\{-C_{i,j} + J^{k-1}(t_1, X_{t_1}, j)\}\Big). \tag{1}$$

Tsitsiklis - van Roy

Longstaff-Schwartz Version

Recall

$$J^k(m\Delta t, x, i) = \mathbb{E}\Big[\sum_{j=m}^{\tau^k} \psi_i(j\Delta t, X_{j\Delta t})\,\Delta t + \mathcal{M}^{k,i}(\tau^k \Delta t, X_{\tau^k \Delta t}) \Big|\, X_{m\Delta t} = x\Big].$$

Analogue for τ^k:

$$\tau^k(m\Delta t, x^\ell_{m\Delta t}, i) = \begin{cases} \tau^k((m+1)\Delta t, x^\ell_{(m+1)\Delta t}, i), & \text{no switch;} \\ m, & \text{switch,} \end{cases} \tag{2}$$

and the set of paths on which we switch is given by $\{\ell \colon \hat{\jmath}^\ell(m\Delta t; i) \neq i\}$ with

$$\hat{\jmath}^\ell(t_1; i) = \arg\max_j \Big(-C_{i,j} + J^{k-1}(t_1, x^\ell_{t_1}, j),\ \psi_i(t_1, x^\ell_{t_1})\Delta t + \hat{E}_{t_1}\left[J^k(t_2, \cdot, i)\right](x^\ell_{t_1}) \Big). \tag{3}$$

The full recursive *pathwise* construction for J^k is

$$J^k(m\Delta t, x^\ell_{m\Delta t}, i) = \begin{cases} \psi_i(m\Delta t, x^\ell_{m\Delta t})\,\Delta t\ + J^k((m+1)\Delta t, x^\ell_{(m+1)\Delta t}, i), & \text{no switch;} \\ -C_{i,j} + J^{k-1}(m\Delta t, x^\ell_{m\Delta t}, j), & \text{switch to } j. \end{cases} \tag{4}$$

Remarks

- Regression used solely to update the optimal stopping times τ^k
- Regressed values never stored
- Helps to eliminate potential biases from the regression step.

Algorithm

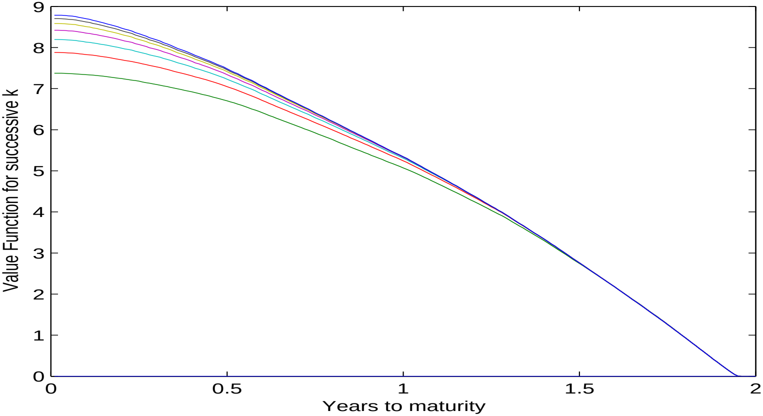
1. Select a set of basis functions (B_j) and algorithm parameters $\Delta t, M^\sharp, N^p, \bar{K}, \delta$.
2. Generate N^p paths of the driving process: $\{x^\ell_{m\Delta t}, m = 0, 1, \ldots, M^\sharp, \ell = 1, 2, \ldots, N^p\}$ with fixed initial condition $x^\ell_0 = x_0$.
3. Initialize the value functions and switching times $J^k(T, x^\ell_T, i) = 0$, $\tau^k(T, x^\ell_T, i) = M^\sharp \ \forall i, k$.
4. Moving backward in time with $t = m\Delta t$, $m = M^\sharp, \ldots, 0$ repeat the Loop:
   - Compute inductively the layers $k = 0, 1, \ldots, \bar{K}$ (evaluate $\mathbb{E}\left[J^k(m\Delta t + \Delta t, \cdot, i) | \mathcal{F}_{m\Delta t}\right]$ by linear regression of $\{J^k(m\Delta t + \Delta t, x^\ell_{m\Delta t+\Delta t}, i)\}$ against $\{B_j(x^\ell_{m\Delta t})\}_{j=1}^{N^B}$, then add the reward $\psi_i(m\Delta t, x^\ell_{m\Delta t}) \cdot \Delta t$)
   - Update the switching times and value functions
5. end Loop.
6. Check whether $\bar{K}$ switches are enough by comparing $J^{\bar{K}}$ and $J^{\bar{K}-1}$ (they should be equal).

Observe that during the main loop we only need to store the buffer $J(t, \cdot), \ldots, J(t + \delta, \cdot)$; and $\tau(t, \cdot), \ldots, \tau(t + \delta, \cdot)$.

Example 1

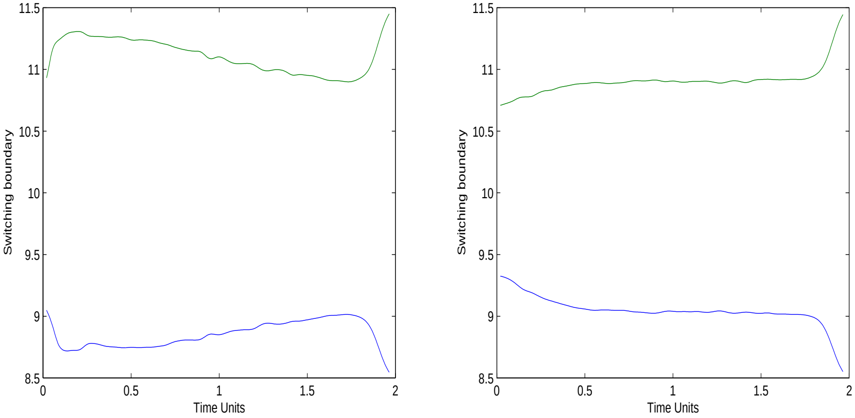
$$dX_t = 2(10 - X_t)\,dt + 2\,dW_t, \qquad X_0 = 10,$$

- Horizon $T = 2$,
- Switch separation $\delta = 0.02$.
- Two regimes
- Reward rates $\psi_0(X_t) = 0$ and $\psi_1(X_t) = 10(X_t - 10)$
- Switching cost $C = 0.3$.



$J^k(t, x, 0)$ as a function of t

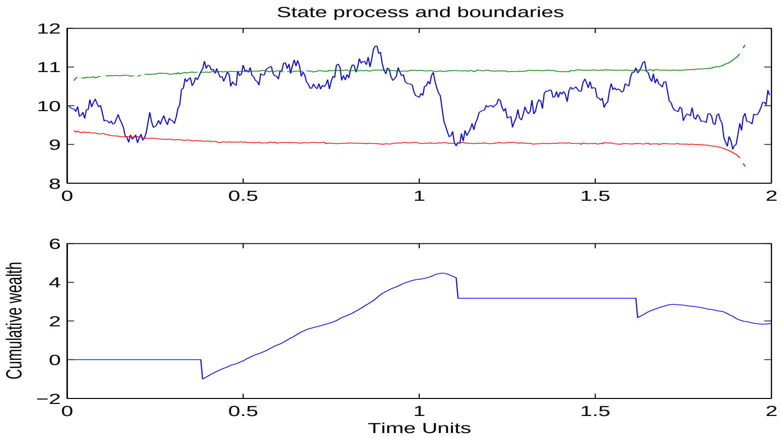
Exercise Boundaries



$k = 2$ (left) $k = 7$ (right)

NB: Decreasing boundary around $t = 0$ is an artifact of the Monte Carlo.

One Sample


State process and boundaries
12
11
10
9
8
0
0.5
1
1.5
2
Cumulative wealth
6
4
2
0
−2
0
0.5
1
1.5
2
Time Units

Example 2: Comparisons

Spark spread $X_t = (P_t, G_t)$

$$\begin{cases} \log(P_t) \sim OU(\kappa = 2, \theta = \log(10), \sigma = 0.8) \\ \log(G_t) \sim OU(\kappa = 1, \theta = \log(10), \sigma = 0.4) \end{cases}$$

- $P_0 = 10$, $G_0 = 10$, $\rho = 0.7$
- Agreement Duration $[0, 0.5]$
- Reward functions

$$\begin{aligned} \psi_0(X_t) &= 0 \\ \psi_1(X_t) &= 10(P_t - G_t) \\ \psi_2(X_t) &= 20(P_t - 1.1\,G_t) \end{aligned}$$

- **Switching costs**

$$C_{i,j} = 0.25|i - j|$$

Numerical Comparison

Method	Mean	Std. Dev	Time (m)
Explicit FD	5.931	–	25
LS Regression	5.903	0.165	1.46
TvR Regression	5.276	0.096	1.45
Kernel	5.916	0.074	3.8
Quantization	5.658	0.013	400^*

Table: Benchmark results for Example 2.

Example 3: Dual Plant & Delay

$$\begin{cases} \log(P_t) \sim OU(\kappa = 2, \theta = \log(10), \sigma = 0.8), \\ \log(G_t) \sim OU(\kappa = 1, \theta = \log(10), \sigma = 0.4), \\ \log(O_t) \sim OU(\kappa = 1, \theta = \log(10), \sigma = 0.4), . \end{cases}$$

- $P_0 = G_0 = O_0 = 10$, $\rho_{pg} = 0.5$,$\rho_{po} = 0.3$, $\rho_{go} = 0$
- Agreement Duration $T = 1$
- Reward functions

$$\begin{aligned} \psi_0(X_t) &\equiv 0 \\ \psi_1(X_t) &= 5 \cdot (P_t - G_t) \\ \psi_2(X_t) &= 5 \cdot (P_t - O_t), \\ \psi_3(X_t) &= 5 \cdot (3P_t - 4G_t) \\ \psi_4(X_t) &= 5 \cdot (3P_t - 4O_t). \end{aligned}$$

- Switching costs $C_{i,j} \equiv 0.5$
- Delay $\delta = 0, 0.01, 0.03$ (up to ten days)

Numerical Results

Setting	No Delay	$\delta = 0.01$	$\delta = 0.03$
Base Case	13.22	12.03	10.87
Jumps in P_t	23.33	22.00	20.06
Regimes 0-3 only	11.04	10.63	10.42
Regimes 0-2 only	9.21	9.16	9.14
Gas only: $0, 1, 3$	9.53	7.83	7.24

Table: LS scheme with 400 steps and 16000 paths.

Remarks

- High δ lowers profitability by over 20%.
- Removal of regimes: without regimes 3 and 4 expected profit drops from 13.28 to 9.21.

Example 4: Exhaustible Resources

Include I_t current level of resources left (I_t non-increasing process).

$$J(t,x,c,i) = \sup_{\tau,j} \mathbb{E}\Big[\int_t^\tau \neg\psi_i(s,X_s)\,ds + J(\tau,X_\tau,I_\tau,j) - C_{i,j} \,|\, X_t = x, I_t = c\Big]. \qquad (5)$$

◇ Resource depletion (boundary condition) $J(t,x,0,i) \equiv 0$.
◇ Not really a control problem I_t can be computed ***on the fly***

Mining example of Brennan and Schwartz varying the initial copper price X_0

Method/ X_0	0.3	0.4	0.5	0.6	0.7	0.8
BS '85	1.45	4.35	8.11	12.49	17.38	22.68
PDE FD	1.42	4.21	8.04	12.43	17.21	22.62
RMC	1.33	4.41	8.15	12.44	17.52	22.41

Extensions

- Extension to **Gas Storage** valuation
- Extension to **Hydro** valuation
- Improve the theoretical results
  - Need to improve delays
  - Need **convergence analysis**
  - Need better analysis of **exercise boundaries**
  - Need to implement duality upper bounds
    - we have approximate value functions
    - we have approximate exercise boundaries
    - so we have lower bounds

Extending the Analysis Adding Access to a Financial Market

Porchet-Touzi

- Same (Markov) factor process $X_t = (X_t^{(1)}, X_t^{(2)}, \cdots)$ as before
- Same plant characteristics as before
- Same operation control $u = (\xi, \mathcal{T})$ as before
- Same maturity T (end of tolling agreement) as before
- **Reward** for operating the plant

$$H(x, i, T; u)(\omega) \triangleq \int_0^T \neg\psi_{u_s}(s, X_s)\, ds - \sum_{\tau_k < T} C(u_{\tau_k -}, u_{\tau_k})$$

Hedging/Investing in Financial Market

Access to a financial market (possibly incomplete)

- y initial wealth
- π_t investment portfolio
- $Y_T^{y,\pi}$ corresponding terminal wealth from investment
- **Utility function** $U(y) = -e^{-\gamma y}$
- Maximum expected utility

$$v(y) = \sup_{\pi} \mathbb{E}\{U(Y_T^{y,\pi})\}$$

- With the power plant (tolling contract)

$$V(x, i, y) = \sup_{u,\pi} \mathbb{E}\{U(Y_T^{y,\pi} + H(x, i, T; u))\}$$

INDIFFERENCE PRICING

$$\overline{p} = p(x, i, y) = \sup\{p \geq 0; V(x, i, y) \geq v(y)\}$$

Analysis of

- BSDE formulation
- PDE formulation

- **Spread Options, Swings, and Asset Valuation**
  - R.C. & V. Durrleman: Pricing and Hedging Spread Options, *SIAM Review* **45** (2004) 627 - 685
  - R.C. & V. Durrleman: Pricing and Hedging Multivariate Contingent Claims, The Journal of Computation Finance **9**(2) (2005) 1-25.
  - R.C. & N. Touzi: Optimal Multiple Stopping and Valuation of Swing Options, *Mathematical Finance* **18** (2008) 239-268.
  - R.C. & S. Dayanik: Optimal Multiple Stopping of Linear Diffusions, *Mathematics of Operations Research* **33** (2) (2008) 446–460.
  - R.C. & M. Ludkovski: Pricing Asset Scheduling Flexibility using Optimal Switching. (2007) *Applied Mathematical Finance* **15** (5), (2008) 405–447.
  - R.C. & M. Ludkovski: Valuation of Energy Storage: an Optimal Switching Approach. *Quantitative Finance* (2009)