Optimal Execution: I. Limit Order Book & Price Impact Models

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Standard Assumptions in Finance

Black-Scholes theory

- Price given by a **single** number
- **infinite** liquidity
  - one **can** buy or sell **any** quantity **at** this price
  - with **NO IMPACT** on the asset price
- Fixes to account for liquidity frictions
  - Transaction Costs (Constantinides, Davis, Paras, Zariphopoulou, Shreve, Soner, ...)
  - liquidity $\sim$ transaction cost (Cetin-Jarrow-Protter)

Not satisfactory for

- Large trades (over short periods)
- High Frequency Trading

Need **Market Microstructure**

- e.g. understand how are buy and sell orders executed?

New Markets

- **Quote Driven Markets**
  - **Market Maker** or **Dealer** centralizes **buy** and **sell** orders and **provides liquidity** by setting bid and ask quotes.
    Ex: **NYSE** specialist system
- **Order Driven Markets**
  - electronic platforms aggregate all available orders in a

    Limit Order Book (LOB)

    Ex: NYSE, NASDAQ, LSE

High Frequency Trading

Speculative figures – **Sound plausible**

- HFT accounts for 60% of all share volume.
- 10% of that is **predatory** $\approx$ 600 million shares per day
- At \$0.01-\$0.02 per share, predatory HFT is profiting \$6-\$12 million a day or \$1.5-\$3 billion e year

Algorithmic Trading – **Source of concern**

- Moving computing facilities **closer** to trading platform (**latency**)
- Relying on / competing with **Benchmark Tracking** execution algorithms

Limit Order Book (LOB)

List of all the waiting **buy** and **sell** orders

- Prices are multiple of the tick size
- For a given price, orders are arranged in a **First-In-First-Out** (FIFO) stack
- At each time t
  - The **bid** price B_t is the price of the highest **waiting buy order**
  - The **ask** price A_t is the price of the lowest **waiting sell order**
- The state of the order book is modified by **order book events**:
  - **limit orders**
  - **market orders**
  - **cancelations**
- **consolidated order book**: If the stock is traded in several venues, one aggregates over all (**visible**) trading venues. We shall **not** talk about **pools** in these lectures.

The Role of a LOB

- Crucial in high frequency finance: explains **transaction costs**.
- ***Liquidity providers*** post trading intentions: Bids and Offers.
- ***Liquidity takers*** execute certain orders: adverse selection.

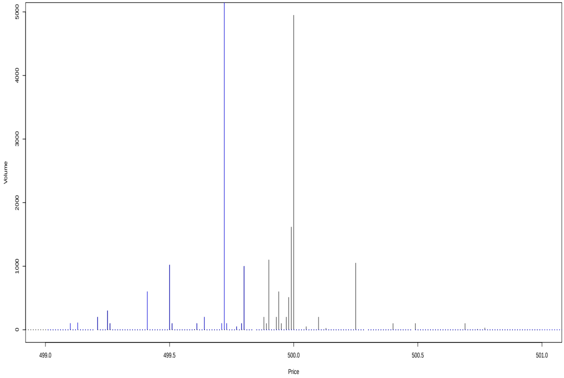
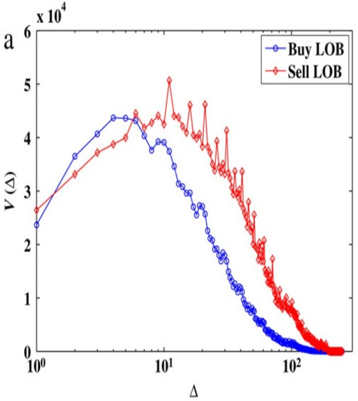
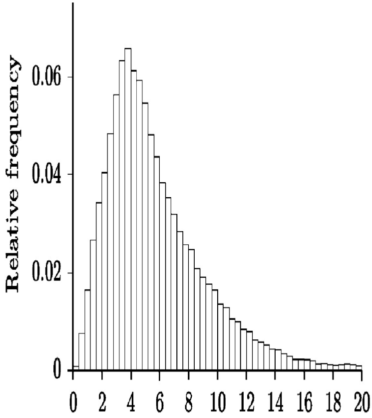


Figure: Apple snapshot order book at 8:43 on NASDAQ.

Empirical literature



Gu - Chen - Zhou(2008)



Naes - Skjeltorp(2006)

Biais(1995), Maslov - Mills(2001), Bouchaud - Potters(2003,2004), Weber - Rosenow(2005), Naes - Skjeltorp(2006), Gu - Chen - Zhou(2008), Chakrborti - Toke - Patriarca - Abergel(2011)

Limit Orders

A **limit order** sits in the order book until it is

- either executed against a **matching** market order
- or it is canceled

A **limit order**

- may be executed very quickly if it corresponds to a price near the bid and the ask
- may take a long time if
  - the market price moves away from the requested price
  - the requested price is too far from the bid/ask.
- can be canceled at any time

Typically, a **limit order** waits for a **match**

- transaction **cost** is **known**
- execution **time** is **uncertain**

Market Orders

A **market order** is an order to buy/sell a certain quantity of the asset at the **best available price** in the book.

- Agents can put a **market order** that, for a buy (resp. sell) order,
  - the first share(s) will be traded at the ask (resp. bid) price
  - the remaining one(s) will be traded some ticks upper (resp. lower)

  in order to fill the order size.
- The ask (resp. bid) price is then modified accordingly.
- When either the bid or ask queue is **depleted** by
  - market orders
  - cancelations

  the price is **updated** up or down to the next level of the order book.

Typically a **market order** consumes the cheapest limit orders

- **immediate execution** (if the book is filled enough)
- **price** per share instead **uncertain** (depends upon the order size)

Cancellations

- Agents can put a **cancellation** of x orders in a given queue reduces the queue size by x
- When either the bid or ask **queue is depleted** by market orders and cancelations, the **price moves** up or down to the next level of the order book.

LOB Dynamics

- Actual trades come in two forms
- Agents can put a **limit order** and wait that this order matches another one
  - transaction cost is known
  - execution time is uncertain
- Agents can put a **market order** that consumes the cheapest limit orders in the book
  - immediate execution (if the book is filled enough)
  - price per share instead depends on the order size

  For a buy (resp. sell) order, the first share will be traded at the ask (resp. bid) price while the last one will be traded some ticks upper (resp. lower) in order to fill the order size. The ask (resp. bid) price is then modified accordingly.
- Agents can put a **cancellation** of x orders in a given queue reduces the queue size by x
- When either the bid or ask queue is depleted by market orders and cancelations, the price moves up or down to the next level of the order book.

Order Book Modeling Objectives

Offer a **framework** to investigate order impact on execution prices

- Optimal multi-period liquidation strategies against a limit order book
- Detailed but tractable stochastic model of spread and transaction costs
- Benchmark tracking slippage
- Opportunity costs of delayed trading

Existing **Literature** (very partial list, only relevant to these lectures)

- **Equilibrium models**: **Parlour (1998), Foucault et al. (2005), Rosu (2009)**
- **Empirical studies**: **Bouchaud et al. (2002), Farmer et al. (2004), Hollifield et al. (2004)**
- **Reduced form models**
  - Stochastic dynamic models: **Bouchaud et al. (2008), Smith et al. (2003), Bovier et al. (2006), Luckock (2003), Maslov and Mills (2001)**
  - Queuing theory based models: **Cont et al. (2010)**

Order Book Models

Roughly speaking, LOB is a set of two histograms (**Bids** and **Asks**)
Reduced form model: Markov process $(O_t)_t$ on a large state space of order books $\mathcal{O}$.

- **Smith-Farmer-Guillemot-Krishnamurthy (SFGK) Model**
- **Market orders** (**buys** and **sells**) arrive according to a **Poisson process** with rate $\mu/2$
- **Cancellation** of existing limit orders: outstanding limit orders die at a rate ν

Another Model Capturing Stylized Facts

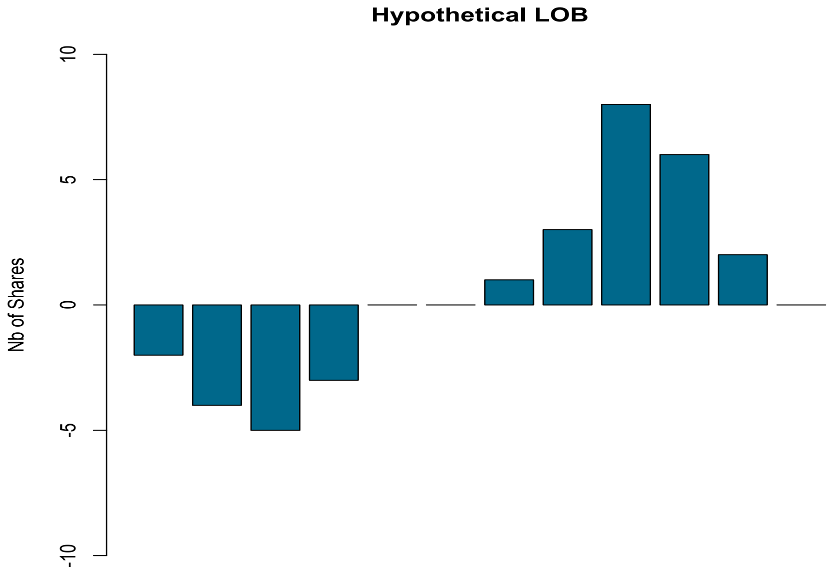
Cont-Stoikov-Talreja

- $\mathcal{P} = \{1, 2, \cdots, n\}$ **price grid** in multiples of price tick
- LOB at time t $O(t) = (O_1(t), O_2(t), \cdots, O_n(t))$
  - $|O_p(t)|$ is the number of outstanding limit orders at price p
  - There are $-O_p(t)$ bid orders at price p if $O_p(t) < 0$
  - There are $O_p(t)$ ask orders at price p if $O_p(t) > 0$
- Admissible **state space**

$$\mathcal{O} = \Big\{ O \in \mathbb{Z}^n;\ \exists 1 \leq k \leq \ell \leq n,\ O_p < 0 \text{ for } p \leq k,$$
$$O_p = 0 \text{ for } k < p < \ell,\ O_p > 0 \text{ for } \ell \leq p \Big\}$$

- **Ask price** at time t:
  $P_A(t) := (n+1) \wedge \inf\{p;\ 1 \leq p \leq n,\ O_p(t) > 0\}$
- **Bid price** at time t:
  $P_B(t) := 0 \vee \sup\{p;\ 1 \leq p \leq n,\ O_p(t) < 0\}$
- **Mid-price** $\tilde{P}(t) = \frac{1}{2}[P_A(t) + P_B(t)]$
- **Bid-Ask spread** $\tilde{S}(t) = P_A(t) - P_B(t)$

A Typical State of the LOB



LOB Dynamics

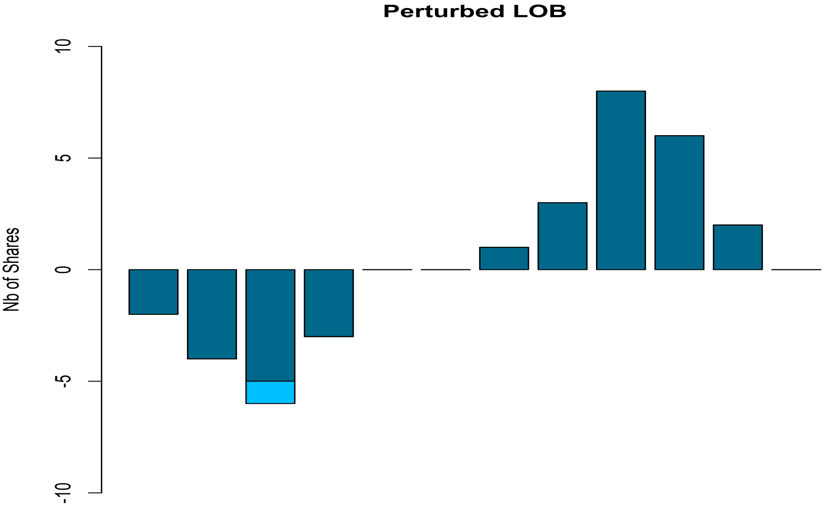
- For the sake of **simplicity**, we assume that the changes to the LOB happen

  one share at a time!

- We review the **events** causing the LOB state **transitions**
- Convenient Notation $O^{p\pm 1}$ as a transition from O

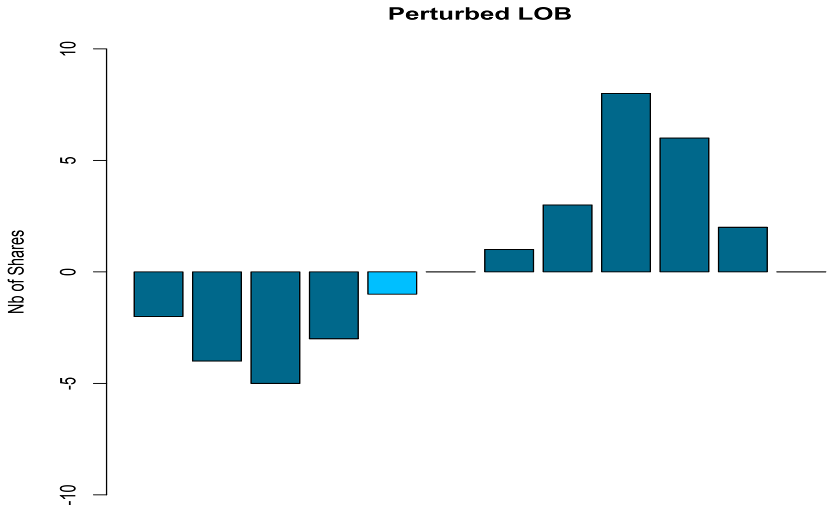
$$O_i^{p\pm 1} = \begin{cases} O_i & \text{if } i \neq p \\ O_i \pm 1 & \text{if } i = p \end{cases}$$

Limit buy order at price level $p < P_A(t)$



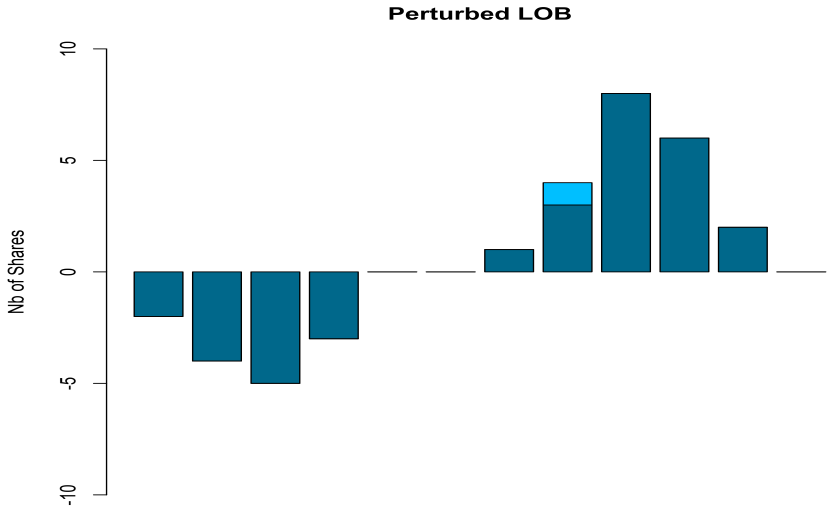
Increases the quantity at level p: $O(t) \hookrightarrow O(t)^{p-1}$

Limit buy order at price level $p < P_A(t)$



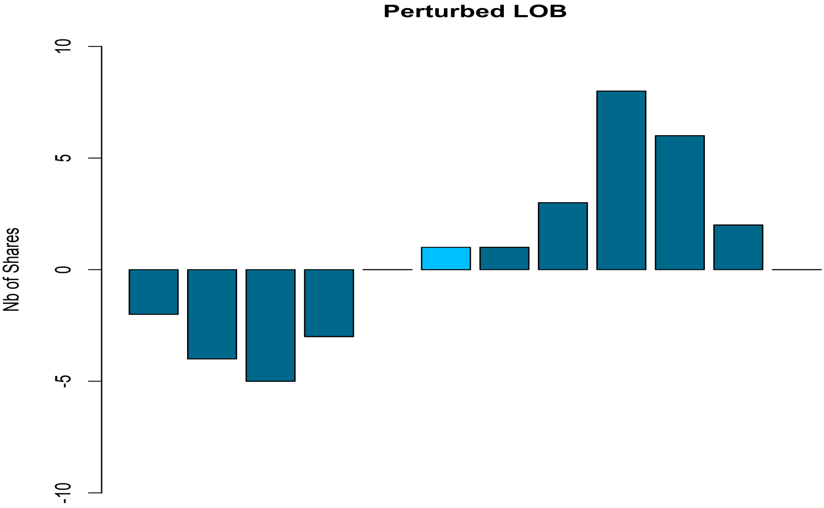
Increases the quantity at level p: $O(t) \hookrightarrow O(t)^{p-1}$

Limit sell order at price level $p > P_B(t)$



Increases the quantity at level p: $O(t) \hookrightarrow O(t)^{p+1}$

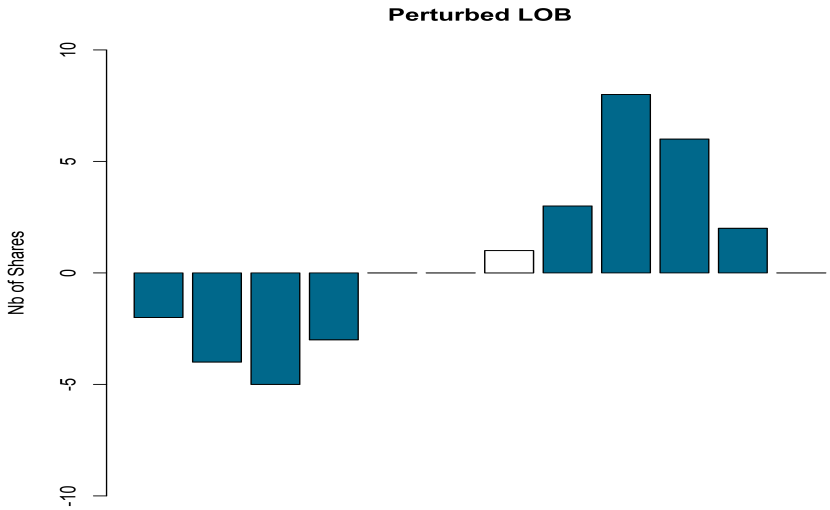
Limit sell order at price level $p > P_B(t)$



Increases the quantity at level p: $O(t) \hookrightarrow O(t)^{p+1}$

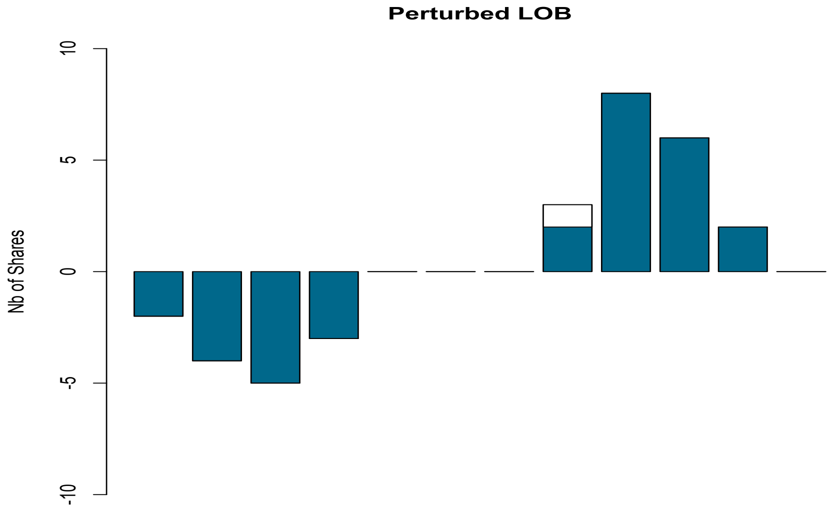
Market buy order

Decreases the quantity at the ask price: $O(t) \hookrightarrow O(t)^{P_A(t)-1}$



Followed by another Market buy order

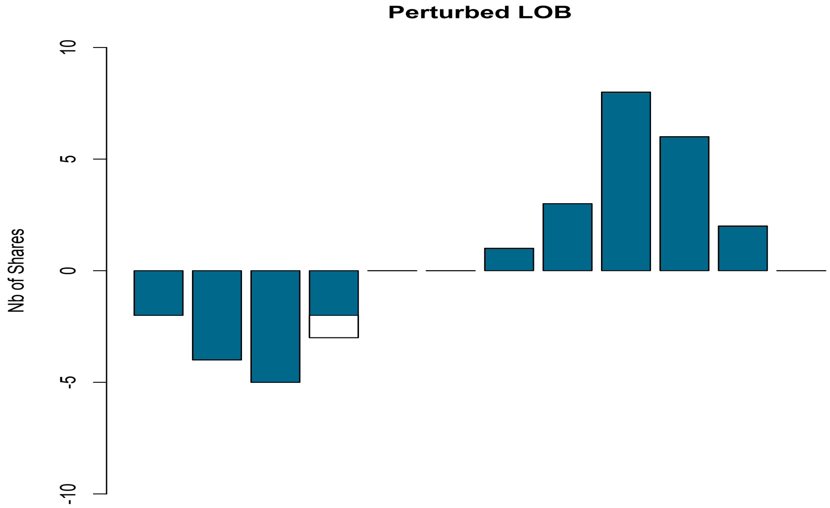
Decreases the quantity at the ask price: $O(t) \hookrightarrow O(t)^{P_A(t)-1}$



Now the Ask price $P_A(O)$ changes

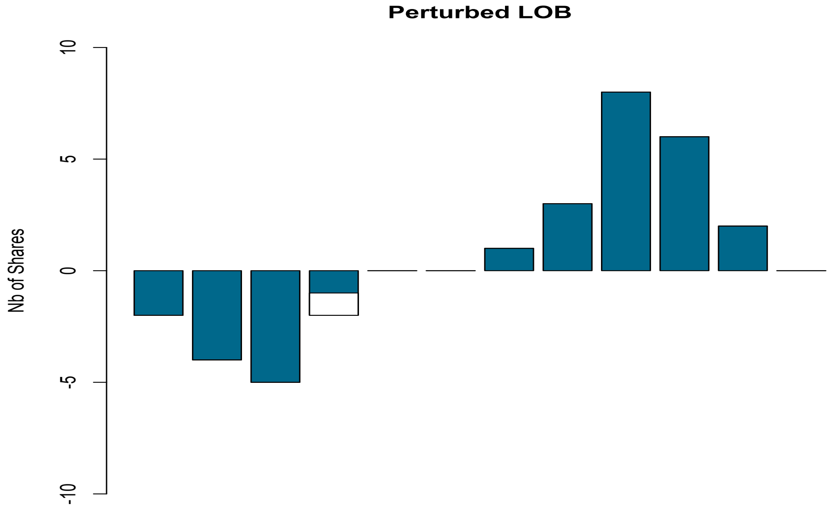
Market sell order

Decreases the quantity at the bid price: $O(t) \hookrightarrow O(t)^{P_B(t)+1}$



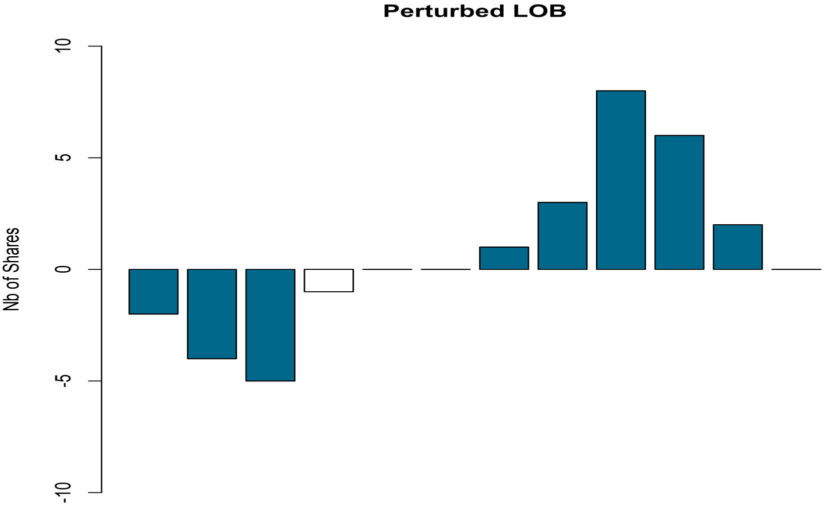
Followed by Another Market sell order

Decreases the quantity at the bid price: $O(t) \hookrightarrow O(t)^{P_B(t)+1}$



Followed by Still Another Market sell order

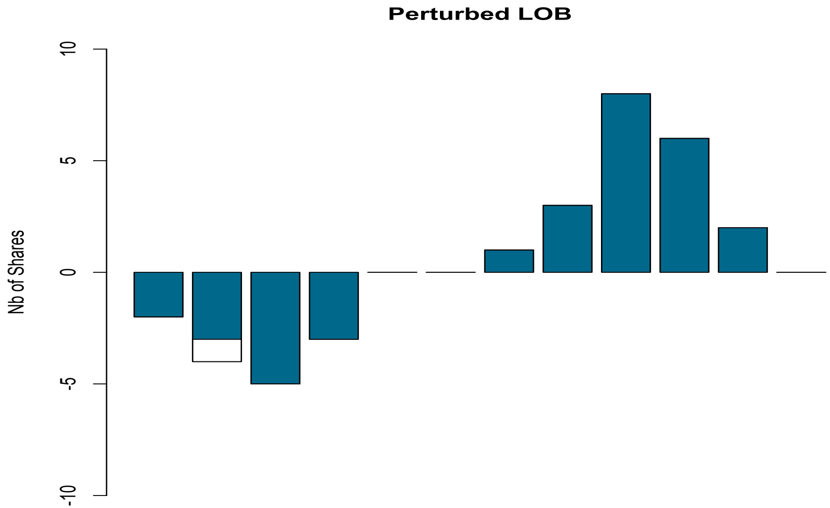
Decreases the quantity at the bid price: $O(t) \hookrightarrow O(t)^{P_B(t)+1}$



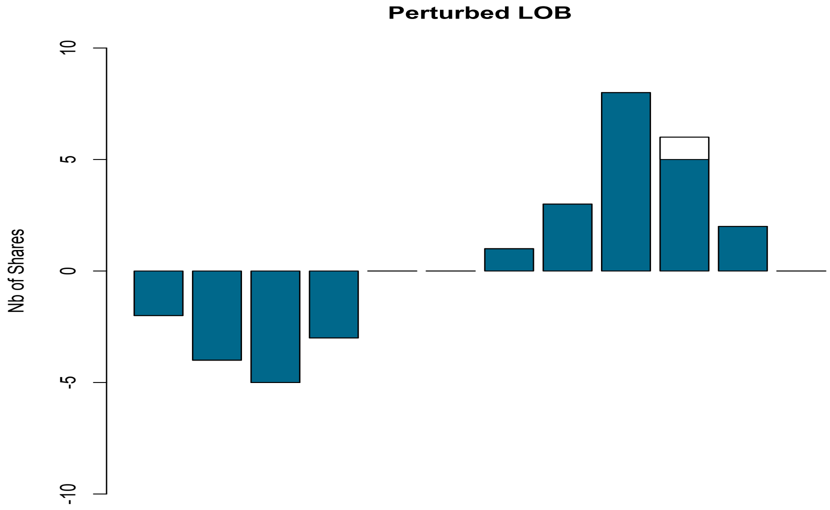
Now the Bid price $P_B(O)$ changes

Cancellation of an outstanding limit buy order at price level $p < P_B(t)$

Decreases the quantity at level p: $O(t) \hookrightarrow O(t)^{p+1}$



Cancellation of an outstanding limit sell order at price level $p > P_A(t)$



Practical Assumptions

- Limit buy (respectively sell) orders arrive at a distance of i ticks from the opposite best quote at independent, exponential times with rate $\lambda(i) = Ki^{-\beta}$ for some $K > 0$ and $\beta > 0$
- Market buy (respectively sell) orders arrive at independent, exponential times with constant rate μ
- Cancellations of limit orders at a distance of i ticks from the opposite best quote occur at a rate proportional to the number of outstanding orders: If the number of outstanding orders at that level is x, then the cancellation rate is $\theta(i)x$.
- The above events are mutually independent.

Summary

Under these assumptions, $\underline{O} = [O(t)]_{t\geq 0}$ is a continuous-time Markov chain with state space $\mathcal{O}$ and transition rates:

- $O \hookrightarrow O^{p-1}$ with rate $\lambda(P_A(t) - p)$ for $p < P_A(t)$
- $O \hookrightarrow O^{p-1}$ with rate $\theta(p - P_B(t))|O_p|$ for $p > P_B(t)$
- $O \hookrightarrow O^{p+1}$ with rate $\lambda(p - P_B(t))$ for $p > P_B(t)$
- $O \hookrightarrow O^{p+1}$ with rate $\theta(P_A(t) - p)|O_p|$ for $p < P_A(t)$
- $O \hookrightarrow O^{P_B(t)+1}$ with rate μ
- $O \hookrightarrow O^{P_A(t)-1}$ with rate μ

This chain remains in $\mathcal{O}$ if it starts from there, i.e.

$$P_B(t) \leq P_A(t), \qquad \text{far all } t > 0$$

if it is true at time $t = 0$.

Cont-Stoikov-Talreja Model

- Descriptive Analysis
- Use ideas from textbfqueuing theory
  - first passage times of Birth-and-Death processes
  - Laplace transform techniques
- Compute / Estimate Probabilities of Conditional Events
- Not sufficient for optimal execution strategies

Optimization Problems

Goal of a LOB model is to

- **Understand the costs of transactions**
- **Develop efficient** (if not **optimal**) trading procedures

Typical challenge

- **Sell** x_0 units of an asset and maximize the sales revenues, using a limited number of **market orders** **only**

$$\sup_{\tau_1 < \cdots < \tau_n < T} \mathbb{E}(U(\sum_{i=1}^{n} P_B(\tau_i)))$$

where U is a utility function and $\mathbb{E}$ is the expectation over a model for the dynamics of the LOB O_t

Searching for optimal ***strategies*** / ***market timing rules*** is a **stochastic control problem** in **prohibitively** **high** dimension

Alternative ”Amlgren-Chriss Price Impact” Model

(originally proposed in discrete time)

- **Permanent price impact** given by a function g of trading speed

$$dP_t^{mid} = g(v(t))dt + \sigma dW_t$$

- **Temporary price impact** given by function h of trading speed

$$P_t^{trans} = P_t^{mid} + h(v(t))$$

- **Problem:** find deterministic continuous *transaction path* to maximize **mean-variance** reward.
  - Closed form solution when **permanent** and **instantaneous price impact** functions g and h are **linear**
  - **Efficient frontier**: Speed of trading and hence risk/return controlled by risk aversion parameter

Widely used in industry

Criticisms

- Mid-price P_t^{mid} **arithmetic Brownian motion** + drift
  - Can become negative
  - Reasonable only for short times
- Possible issues with **rate of trading** in continuous time?
- **Price impact** more complex than instantaneous + permanent
- What is the link between **Price Impact** and **LOB** dynamics?
  - e.g. can we combine elegant description of risk-return trade-off in Almgren / Chriss with detail of Smith-Farmer type models?
- **Empirical evidence** that instantaneous price impact is stochastic in many markets