Limit Order Books

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Standard Assumptions in Finance

Black-Scholes theory

- Price given by a **single** number
- **infinite** liquidity
  - one **can** buy or sell **any** quantity **at** this price
  - with **NO IMPACT** on the asset price
- Fixes to account for liquidity frictions
  - Transaction Costs (Constantinides, Davis, Paras, Zariphopoulou, Shreve, Soner, ...)
  - liquidity $\sim$ transaction cost (Cetin-Jarrow-Protter)

Not satisfactory for

- Large trades (over short periods)
- High Frequency Trading

Need **Market Microstructure**

- e.g. understand how are buy and sell orders executed?

New Markets

- **Quote Driven Markets**
  - **Market Maker** or **Dealer** centralizes **buy** and **sell** orders and **provides liquidity** by setting bid and ask quotes.
    Ex: **NYSE** specialist system

- **Order Driven Markets**
  - electronic platforms aggregate all available orders in a

    Limit Order Book (LOB)

    Ex: NYSE, NASDAQ, LSE
- Same stock traded on **several venues**
- **Price discovery** made difficult as most instruments can be traded off market without printing the trade to a publicly accessible data source
  - **Competition** between markets leads to **lower fees** and **smaller tick sizes**
- Creation of **Dark Pools**
- Increase in updating **frequency** of **order books**

High Frequency Trading

Speculative figures – **Sound plausible**

- HFT accounts for 60 – 75% of all share volume.
- 10% of that is **predatory** $\approx$ 600 million shares per day
- At \$0.01-\$0.02 per share, predatory HFT is profiting \$6-\$12 million a day or \$1.5-\$3 billion e year

Algorithmic Trading – **Source of concern**

- Moving computing facilities **closer** to trading platform (**latency**)
- Relying on / competing with **Benchmark Tracking** execution algorithms

Pros & Cons

Pros

- Smaller tick size;
- HF traders provide extra liquidity
- Dark pools reduce trade execution costs from price impact
- Markets are more efficient

Cons

- Expensive technological **arms race**
- **Dark** trading incentivizes **price manipulation**, **fishing** and **predatory trading**
- Little or no oversight possible by humans (e.g. **flash crash**) & increased **systemic risk**
- HF trading algorithms do not use **economic fundaments** (e.g. value & profitability of a firm)

Some Highly Publicized Mishaps

Flash Crash of May 6, 2010

- Dow Jones IA plunged about 1000 points (recovered in minutes)
- Biggest one-day point decline (998.5 points)
  - At 2:32 pm a mutual fund program started to sell 75,000 E-Mini S&P 500 contracts ($\approx$ 4.1 billion USD) at an execution rate of 9%
  - HT trading programs were among the buyers: **quickly bought and resold** contracts to each other
  - **hot-potato** volume effect, combined sales drove ”the E-mini price down 3% in just 4 minutes

Other Notable Crashes

- Associated Press' Twitter account **hack**
  - **White House** bombed
  - **President Obama** injured
  - **DJIA lost 140 points** and recovered in minutes
- Several **mini flash crashes** on NASDAQ in 2012

Limit Order Book (LOB)

List of all the waiting **buy** and **sell** orders

- Prices are multiple of the tick size
- For a given price, orders are arranged in a **First-In-First-Out** (FIFO) stack
- At each time t
  - The **bid** price B_t is the price of the highest **waiting buy order**
  - The **ask** price A_t is the price of the lowest **waiting sell order**
- The state of the order book is modified by **order book events**:
  - **limit orders**
  - **market orders**
  - **cancelations**
- **consolidated order book**: If the stock is traded in several venues, one aggregates over all (**visible**) trading venues.
- Here, **little** or **no** discussion of **pools**

The Role of a LOB

- Crucial in high frequency finance: explains **transaction costs**.
- ***Liquidity providers*** post trading intentions: Bids and Offers.
- ***Liquidity takers*** execute certain orders: adverse selection.

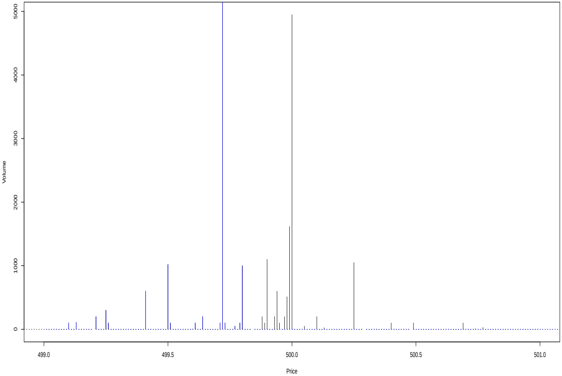
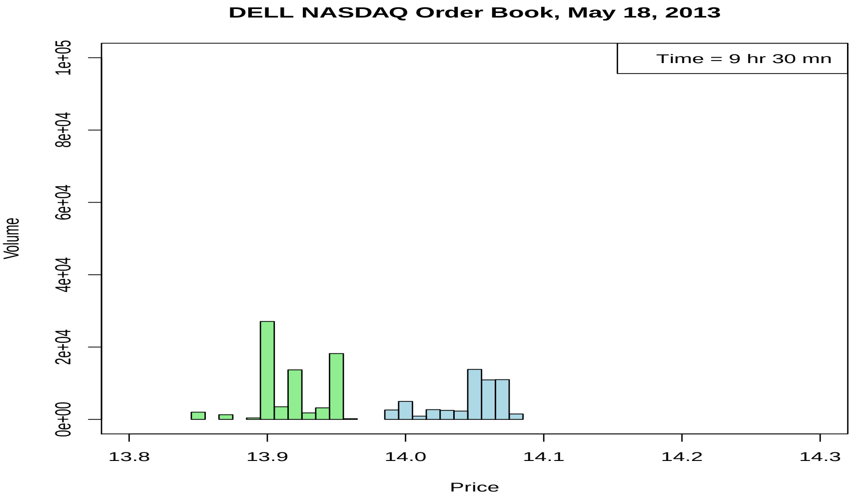


Figure : Snapshot of Apple order book at 8:43 (NASDAQ)

DELL Limit Order Book on May 18, 2013



Limit Orders

A limit order sits in the order book until it is

- either executed against a **matching** market order
- or it is **canceled**

A limit order

- may be executed very quickly if it corresponds to a price near the bid and the ask
- may take a long time if
  - the market price moves away from the requested price
  - the requested price is too far from the bid/ask.
- can be canceled at any time

Typically, a **limit order** waits for a **match**

- transaction **cost** is **known**
- execution **time** is **uncertain**

Market Orders

A **market order** is an order to buy/sell a certain quantity of the asset at the **best available price** in the book.

- Agents can put a **market order** that, for a buy (resp. sell) order,
  - the first share(s) will be traded at the ask (resp. bid) price
  - the remaining one(s) will be traded some ticks upper (resp. lower)

  in order to fill the order size.
- The ask (resp. bid) price is then modified accordingly.
- When either the bid or ask queue is **depleted** by
  - market orders
  - cancelations

  the price is **updated** up or down to the next level of the order book.

Typically a **market order** consumes the cheapest limit orders

- **immediate execution** (if the book is filled enough)
- **price** per share instead **uncertain** (depends upon the order size)

Cancellations

- Agents can put a **cancellation** of x orders in a given queue reduces the queue size by x
- When either the bid or ask **queue is depleted** by market orders and cancelations, the **price moves** up or down to the next level of the order book.

LOB Dynamics Summary

- Actual trades come in two forms
- Agents can put a **limit order** and wait that this order matches another one
  - transaction cost is known
  - execution time is uncertain
- Agents can put a **market order** that consumes the cheapest limit orders in the book
  - immediate execution (if the book is filled enough)
  - price per share instead depends on the order size

  For a buy (resp. sell) order, the first share will be traded at the ask (resp. bid) price while the last one will be traded some ticks upper (resp. lower) in order to fill the order size. The ask (resp. bid) price is then modified accordingly.
- Agents can put a **cancellation** of x orders in a given queue reduces the queue size by x
- When either the bid or ask queue is depleted by market orders and cancelations, the price moves up or down to the next level of the order book.

Market Impact of Large Fills

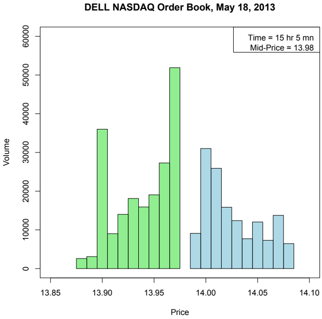
- Current **mid-price** $p_{mid} = (p_{Bid} + p_{Ask})/2 = 13.98$
- Fill size $N = 76015$ (e.g. buy)
- n_1 shares available at best bid p_1, n_2 shares at price $p_2 > p_1$, $\ldots$
- n_k shares at price $p_k > p_{k-1}$ $N = n_1 + n_2 + n_3 + \cdots + n_k$
- **colorblueTransaction cost**

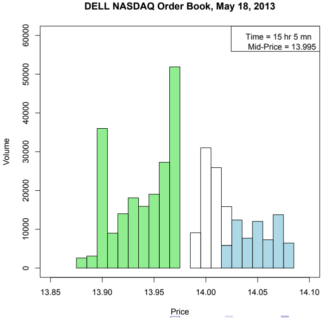
$$n_1 p_1 + n_2 p_2 + \cdots + n_k p_k = 1064578$$

- **Effective price**

$$p_{eff} = \frac{1}{N}(n_1 p_1 + n_2 p_2 + \cdots + n_k p_k) = 14.00484$$

- New **mid-price** $p_{mid} = 13.995$





A LOB Idiosyncrasy: Hidden Liquidity

- Some exchanges (e.g. NASDAQ & NYSE) allow **Hidden Orders**
- Made **visible** to the broader market **after being executed**
- **Controversial**
  - barrier to the implementation of a fully transparent market
  - impediment to price discovery and information dissemination

Results of First Empirical Analyzes

- Encourage **fishing**
- After it is **revealed** that a hidden order was executed
  - rash increase of order placement inside the bid-ask after
- HF Traders **divided** in two groups
  - Traders try to take advantage of the remaining hidden liquidity
  - Traders try to steal execution priority from the fully hidden orders

”Partially Hidden” Orders: Iceberg Orders

- **Dark liquidity** posted inside the LOB
- Two components: the **shown quantity** and the **hidden remainder**
- Order queued with the **lit** part of the LOB, only the shown quantity is visible
- When the order reaches the front of the queue, **only the display quantity is filled**
- Trade (price & quantity filled) **revealed**
- **hidden part** put at the back of the queue
- Sometimes **extra execution fee** charged by the exchange

Dark Pools / Crossing Networks

- **Electronic engine** that matches buy and sell orders without routing them to lit exchanges
- **Raison d'être:** move large amounts without impacting the price (no need for iceberg orders)
- **Run by private brokerages**
  - Ex: Liquidnet, Pipeline, ITG's Posit, Goldman's SIGMA X.
  - Participants submit (wish) lists of orders to a matching engine
  - Matched orders are executed at the midpoint of the bid-ask spread.
  - **Pros:** trade at mid-point can be better than on a lit market
  - **Cons:** May have to wait a long time for a match to occur
- Regulated by SEC (in the US) as **Alternative Trading Systems**
  - Little or no public disclosure
  - Not much has been done to increase *transparency*
- Trading on dark pools $\approx$ 32% of trades in 2012 (!)

Order Book Modeling Objectives

Offer a **framework** to investigate order impact on execution prices

- Optimal multi-period liquidation strategies against a limit order book
- Detailed but tractable stochastic model of spread and transaction costs
- Benchmark tracking slippage
- Opportunity costs of delayed trading

Existing **Literature** (very partial list, only relevant to these lectures)

- **Equilibrium models**: **Parlour (1998), Foucault et al. (2005), Rosu (2009)**
- **Empirical studies**: **Bouchaud et al. (2002), Farmer et al. (2004), Hollifield et al. (2004)**
- **Reduced form models**
  - Stochastic dynamic models: **Bouchaud et al. (2008), Smith et al. (2003), Bovier et al. (2006), Luckock (2003), Maslov and Mills (2001)**
  - Queuing theory based models: **Cont et al. (2010)**

Order Book Models

Roughly speaking, LOB is a set of two histograms (**Bids** and **Asks**)
Reduced form model: Markov process $(O_t)_t$ on a large state space of order books $\mathcal{O}$.

- **Smith-Farmer-Guillemot-Krishnamurthy (SFGK) Model**
- **Market orders** (**buys** and **sells**) arrive according to a **Poisson process** with rate $\mu/2$
- **Cancellation** of existing limit orders: outstanding limit orders **"die"** at a rate ν

Another Model Capturing Stylized Facts

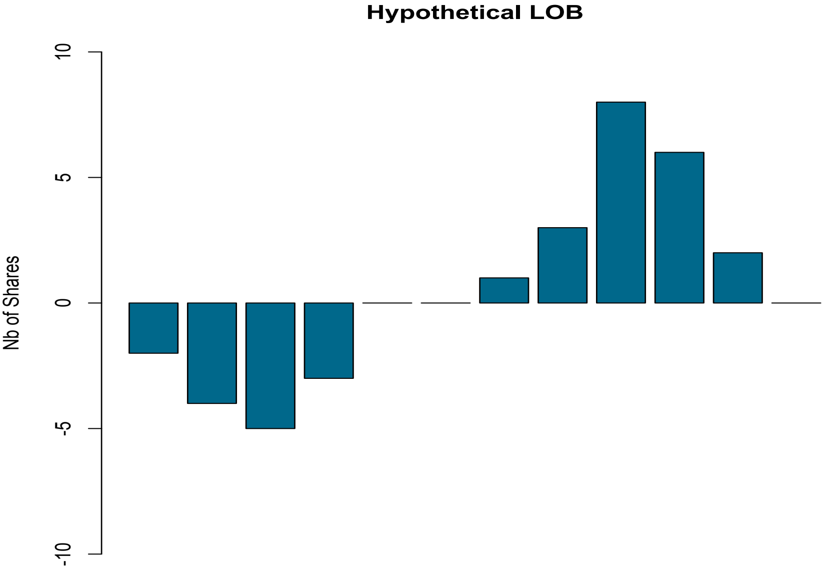
Cont-Stoikov-Talreja

- $\mathcal{P} = \{1, 2, \cdots, n\}$ **price grid** in multiples of price tick
- LOB at time t $O(t) = (O_1(t), O_2(t), \cdots, O_n(t))$
  - $|O_p(t)|$ is the number of outstanding limit orders at price p
  - There are $-O_p(t)$ bid orders at price p if $O_p(t) < 0$
  - There are $O_p(t)$ ask orders at price p if $O_p(t) > 0$
- Admissible **state space**

$$\mathcal{O} = \Big\{ O \in \mathbb{Z}^n;\ \exists 1 \leq k \leq \ell \leq n,\ O_p < 0 \text{ for } p \leq k, \\ O_p = 0 \text{ for } k < p < \ell,\ O_p > 0 \text{ for } \ell \leq p \Big\}$$

- **Ask price** at time t:
  $P_A(t) := (n+1) \wedge \inf\{p;\ 1 \leq p \leq n,\ O_p(t) > 0\}$
- **Bid price** at time t:
  $P_B(t) := 0 \vee \sup\{p;\ 1 \leq p \leq n,\ O_p(t) < 0\}$
- **Mid-price** $\tilde{P}(t) = \frac{1}{2}[P_A(t) + P_B(t)]$
- **Bid-Ask spread** $\tilde{S}(t) = P_A(t) - P_B(t)$

A Typical State of the LOB



LOB Dynamics

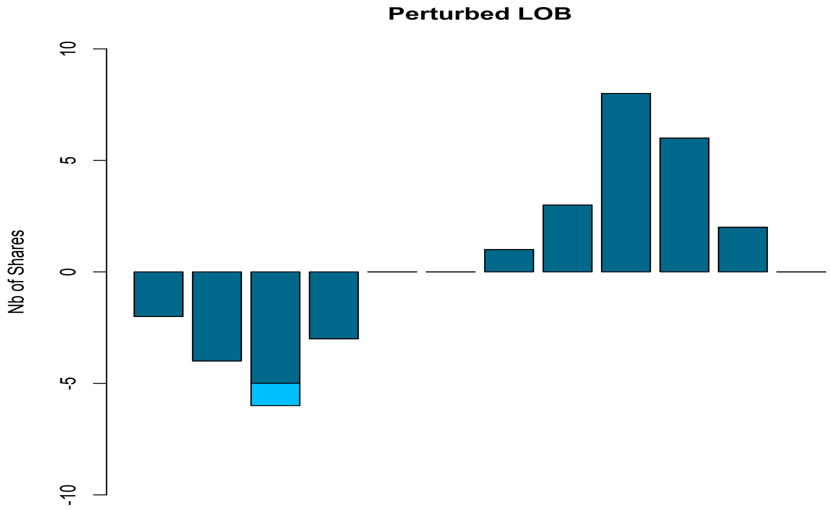
- For the sake of **simplicity**, we assume that the changes to the LOB happen

  one share at a time!

- We review the **events** causing the LOB state **transitions**
- Convenient Notation $O^{p\pm1}$ as a transition from O

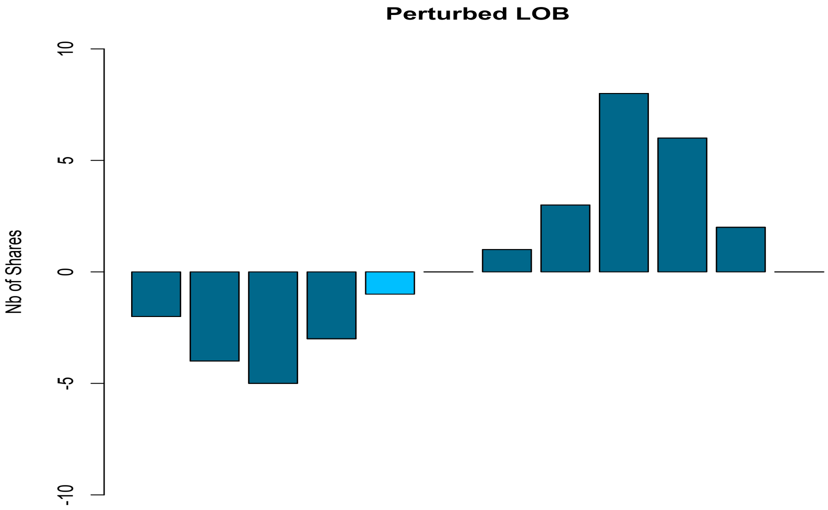
$$O_i^{p\pm1} = \begin{cases} O_i & \text{if } i \neq p \\ O_i \pm 1 & \text{if } i = p \end{cases}$$

Limit buy order at price level $p < P_A(t)$



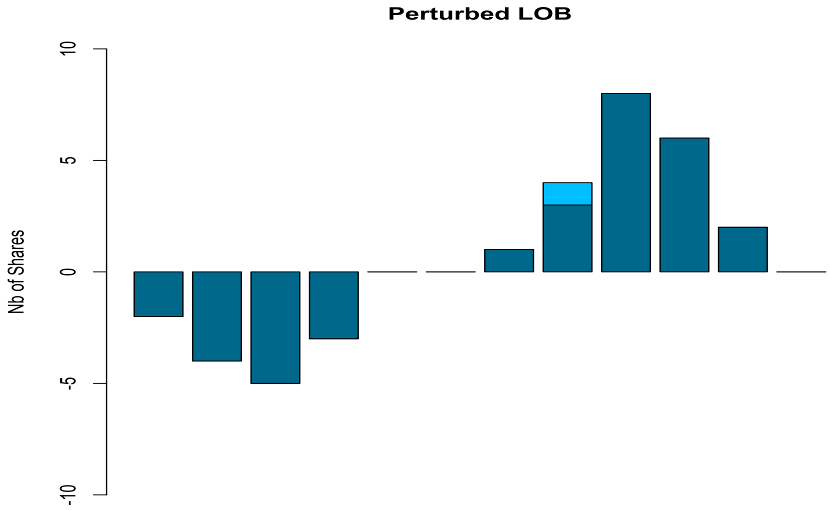
Increases the quantity at level p: $O(t) \hookrightarrow O(t)^{p-1}$

Limit buy order at price level $p < P_A(t)$



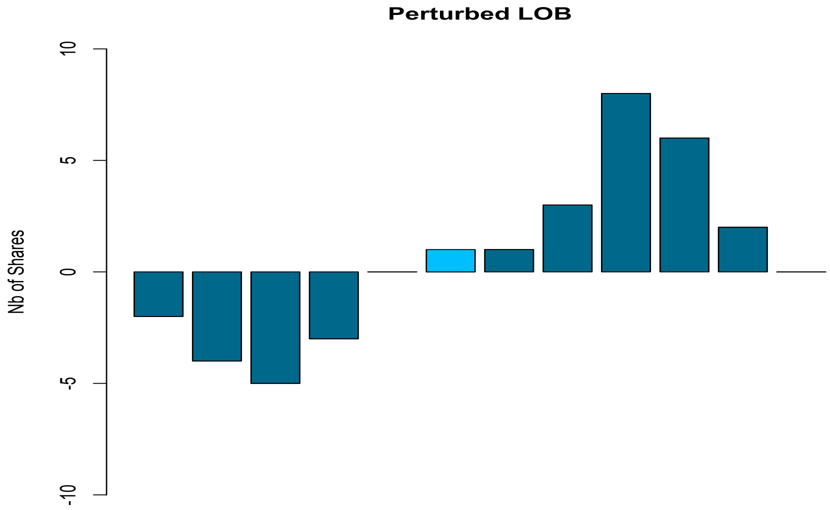
Increases the quantity at level p: $O(t) \hookrightarrow O(t)^{p-1}$

Limit sell order at price level $p > P_B(t)$



Increases the quantity at level p: $O(t) \hookrightarrow O(t)^{p+1}$

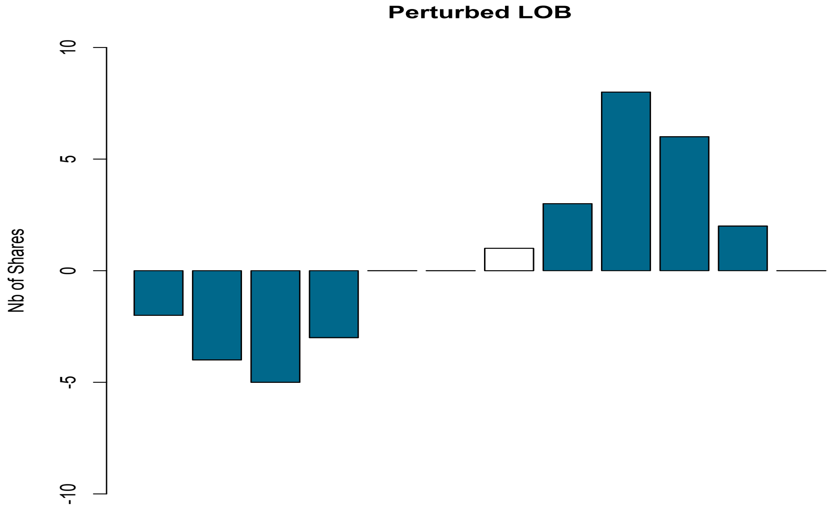
Limit sell order at price level $p > P_B(t)$



Increases the quantity at level p: $O(t) \hookrightarrow O(t)^{p+1}$

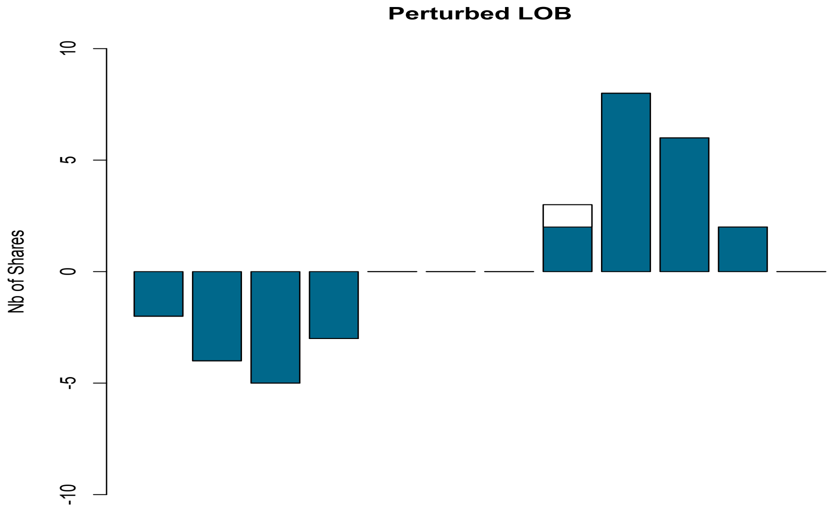
Market buy order

Decreases the quantity at the ask price: $O(t) \hookrightarrow O(t)^{P_A(t)-1}$



Followed by another Market buy order

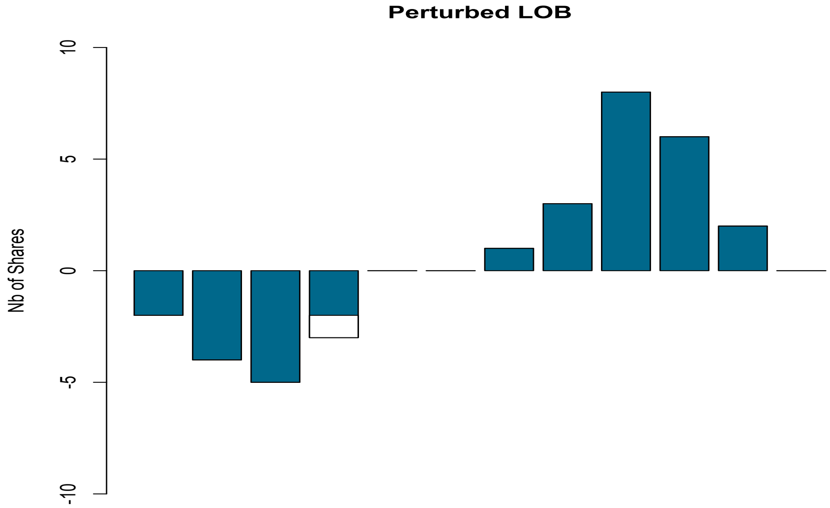
Decreases the quantity at the ask price: $O(t) \hookrightarrow O(t)^{P_A(t)-1}$



Now the Ask price $P_A(O)$ changes

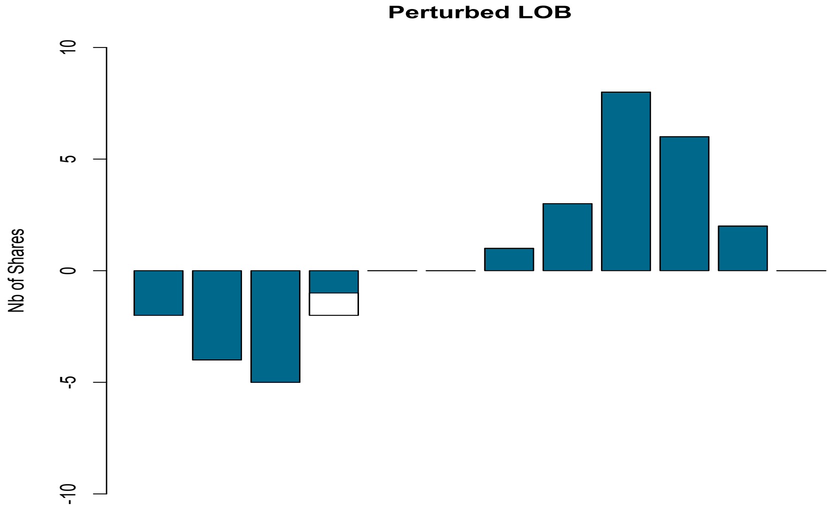
Market sell order

Decreases the quantity at the bid price: $O(t) \hookrightarrow O(t)^{P_B(t)+1}$



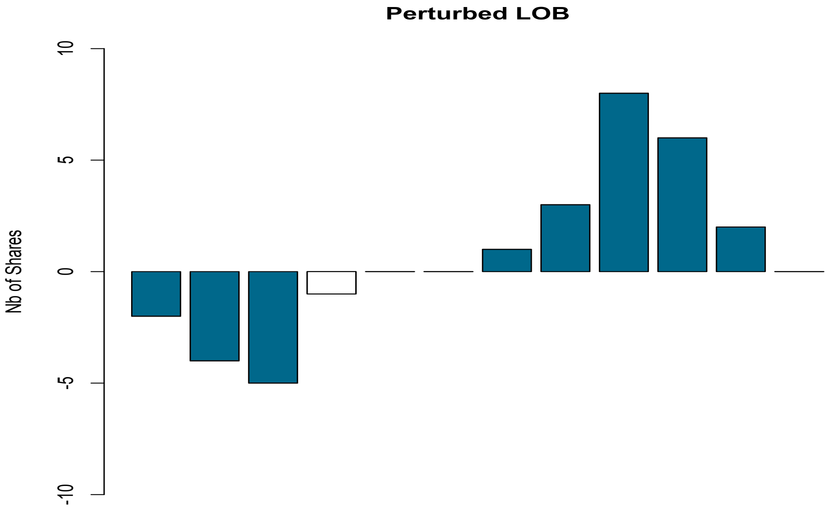
Followed by Another Market sell order

Decreases the quantity at the bid price: $O(t) \hookrightarrow O(t)^{P_B(t)+1}$



Followed by Still Another Market sell order

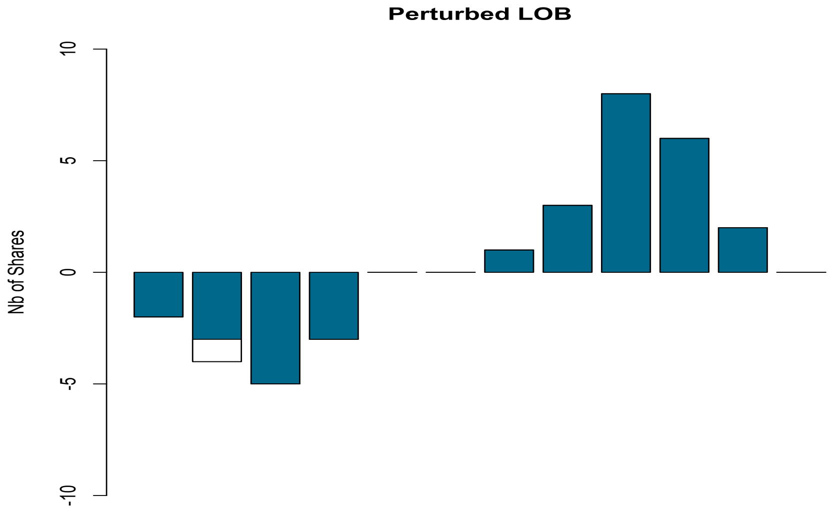
Decreases the quantity at the bid price: $O(t) \hookrightarrow O(t)^{P_B(t)+1}$



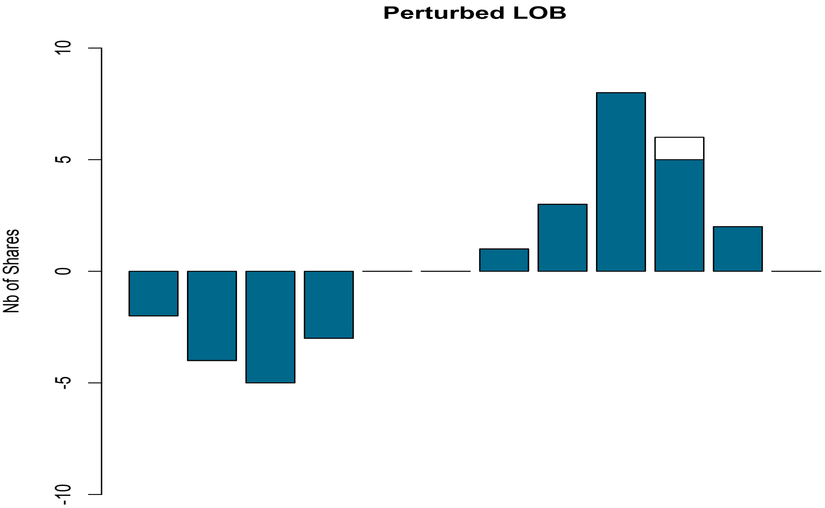
Now the Bid price $P_B(O)$ changes

Cancellation of an outstanding limit buy order at price level $p < P_B(t)$

Decreases the quantity at level p: $O(t) \hookrightarrow O(t)^{p+1}$



Cancellation of an outstanding limit sell order at price level $p > P_A(t)$



Practical Assumptions

- Limit buy (respectively sell) orders arrive at a distance of i ticks from the opposite best quote at independent, exponential times with rate $\lambda(i) = Ki^{-\beta}$ for some $K > 0$ and $\beta > 0$
- Market buy (respectively sell) orders arrive at independent, exponential times with constant rate μ
- Cancellations of limit orders at a distance of i ticks from the opposite best quote occur at a rate proportional to the number of outstanding orders: If the number of outstanding orders at that level is x, then the cancellation rate is $\theta(i)x$.
- The above events are mutually independent.

Summary

Under these assumptions, $\underline{O} = [O(t)]_{t\geq 0}$ is a continuous-time Markov chain with state space $\mathcal{O}$ and transition rates:

- $O \hookrightarrow O^{p-1}$ with rate $\lambda(P_A(t) - p)$ for $p < P_A(t)$
- $O \hookrightarrow O^{p-1}$ with rate $\theta(p - P_B(t))|O_p|$ for $p > P_B(t)$
- $O \hookrightarrow O^{p+1}$ with rate $\lambda(p - P_B(t))$ for $p > P_B(t)$
- $O \hookrightarrow O^{p+1}$ with rate $\theta(P_A(t) - p)|O_p|$ for $p < P_A(t)$
- $O \hookrightarrow O^{P_B(t)+1}$ with rate μ
- $O \hookrightarrow O^{P_A(t)-1}$ with rate μ

This chain remains in $\mathcal{O}$ if it starts from there, i.e.

$$P_B(t) \leq P_A(t), \qquad \text{far all } t > 0$$

if it is true at time $t = 0$.

Cont-Stoikov-Talreja Model

- Descriptive Analysis
- Use ideas from **queuing theory**
  - first passage times of Birth-and-Death processes
  - Laplace transform techniques
- Compute / Estimate Probabilities of Conditional Events
- Not sufficient for optimal execution strategies

Extensions

Incorporate the most important stylised facts of order flow.

- Limit order arrival rates conditional on distance from e.g. best price on same side [MF05]
- Existing limit orders cancelled and immediately resubmitted [Tse06]
- Aggressiveness of orders depends on depth [LS05]
- Fewer market orders when the spread is large [BJP03]
- More limit orders inside spread when depth at best is large [EHJJ03]
- (Long-range) autocorrelation of signs of consecutive market orders [BGPW03] Patrick Hewlett

Optimization Problems

Goal of a LOB model is to

- **Understand the costs of transactions**
- **Develop efficient** (if not **optimal**) trading procedures

Typical challenge

- **Sell** x_0 units of an asset and maximize the sales revenues, using a limited number of **market orders** **only**

$$\sup_{\tau_1 < \cdots < \tau_n < T} \mathbb{E}(U(\sum_{i=1}^{n} P_B(\tau_i)))$$

where U is a utility function and $\mathbb{E}$ is the expectation over a model for the dynamics of the LOB O_t

Searching for optimal ***strategies*** / ***market timing rules*** is a **stochastic control problem** in **prohibitively** **high** dimension