BSDEs with polynomial growth generators

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Abstract

In this paper, we give existence and uniqueness results for backward stochastic differential equations when the generator has polynomial growth in the state variable. We deal with the case of fixed terminal time as well as the case of random terminal time. The need for this type of extension of the classical existence and uniqueness results comes from the desire to provide a probabilistic representation of the solutions of semilinear partial differential equations in the spirit of a nonlinear Feynman-Kac formula. Indeed in many applications of interest, the nonlinearity is polynomial, see e.g. the Allen-Cahn equation or the standard nonlinear heat and Schrödinger equations.

1 Introduction

It is by now well-known that there exists a unique, adapted and square integrable, solution to a backward stochastic differential equation (BSDE for short) of type

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \qquad 0 \le t \le T,$$

provided that the generator is Lipschitz in both the variables y and z. We refer to the original work of E. PARDOUX and S. PENG [13, 14] for the general theory and to N. EL KAROUI, S. PENG AND M.-C. QUENEZ [6] for a survey of the applications of this theory in finance. Since the first existence and uniqueness result established by E. PARDOUX and S. PENG in 1990, a lot of works, including R. W. R. DARLING, E. PARDOUX [5], S. HAMADENE [8], M. KOBYLANSKI [9], J.-P. LEPELTIER, J. SAN MARTIN [10, 11], see also the references therein, have tried to weaken the Lipschitz assumption on the generator. Most of these works deal only with real-valued BSDEs [8, 9, 10, 11] because of their dependence on the use of the comparison theorem for BSDEs (see e.g. N. EL KAROUI, S. PENG, M.-C. QUENEZ [6, Theorem 2.2]). Furthermore, except in [11], the generator is always assumed to be at most linear in the state variable. Let us mention nevertheless an exception: in [11], J.-P. LEPELTIER and J. SAN MARTIN accomodate a growth of the generator of the following type: $C\big(1 + |x|\big|\log|x|\big|\big)$, $C\big(1 + |x|\big|\log\big|\log|x|\big|\big|\big)$ ...

On the other hand, one of the most promising field of application for the theory of BSDEs is the analysis of elliptic and parabolic partial differential equations (PDEs for short) and we refer to E. PARDOUX [12] for a survey of their relationships. Indeed, as it was revealed by S. PENG [17] and by E. PARDOUX, S. PENG [14] (see also the contributions of G. BARLES, R. BUCKDAHN,

E. Pardoux [1], Ph. Briand [3], E. Pardoux, F. Pradeilles, Z. Rao [15], E. Pardoux, S. Zhang [16] among others), BSDEs provide a probabilistic representation of solutions (viscosity solutions in the most general case) of semilinear PDEs. This provides a generalization to the nonlinear case of the well known Feynman-Kac formula. In many examples of semilinear PDEs, the nonlinearity is not of linear growth (as implied by a global Lipschitz condition) but instead, it is of polynomial growth, see e.g. the nonlinear heat equation analyzed by M. Escobedo, O. Kavian and H. Matano in [7]) or the Allen-Cahn equation (G. Barles, H. M. Soner, P. E. Souganidis [2]). If one attempts to study these semilinear PDEs by means of the nonlinear version of the Feynman-Kac formula, alluded to above, one has to deal with BSDEs whose generators with nonlinear (though polynomial) growth. Unfortunately, existence and uniqueness results for the solutions of BSDE's of this type were not available when we first started this investigation, and filling this gap in the literature was at the origin of this paper.

In order to overcome the difficulties introduced by the polynomial growth of the generator, we assume that the generator satisfies a kind of monotonicity condition in the state variable. This condition is very useful in the study of BSDEs with random terminal time. See the works of S. Peng [17], R. W. R. Darling, E. Pardoux [5], Ph. Briand, Y. Hu [4] for attempts in the spirit of our investigation. Even though it looks rather technical at first, it is especially natural in our context: indeed, it is plain to check that it is satisfied in all the examples of semilinear PDEs quoted above.

The rest of the paper is organized as follows. In the next section, we fix some notation, we stae our main assumptions and we prove a technical proposition which will be needed in the sequel. In section 3, we deal with the case of BSDEs with fixed terminal time: we prove an existence and uniqueness result and we establish some a priori estimates for the solutions of BSDEs in this context. In section 4, we consider the case of BSDEs with random terminal times. BSDEs with random terminal times play a crucial role in the analysis of the solutions of elliptic semilinear PDEs. They were first introduced by S. Peng [17] and then studied in a more general framework by R. W. R. Darling, E. Pardoux [5]. These equations are also considered in [12].

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2 Preliminaries

2.1 Notation and Assumptions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space carrying a d-dimensional Brownian motion $(W_t)_{t\geq 0}$, and $\left(\mathcal{F}_t\right)_{t\geq 0}$ be the filtration generated by $(W_t)_{t\geq 0}$. As usual we assume that each σ-field $\mathcal{F}_t$ has been augmented with the $\mathbb{P}$-null sets to make sure that $\left(\mathcal{F}_t\right)_{t\geq 0}$ is right continuous and complete. For $y \in \mathbb{R}^k$, we denote by $|y|$ its Euclidean norm and if z belongs to $\mathbb{R}^{k\times d}$, $\|z\|$ denotes $\left\{\mathrm{tr}(zz^*)\right\}^{1/2}$. For $q > 1$, we define the following spaces of processes:

- $\mathcal{S}_q = \left\{ \psi \text{ progressively measurable;} \quad \psi_t \in \mathbb{R}^k; \quad \|\psi\|_{\mathcal{S}_q}^q := \mathbb{E}\left[\sup_{0\leq t\leq T} |\psi_t|^q\right] < \infty \right\}$,

- $\mathcal{H}_q = \left\{ \psi \text{ progressively measurable;} \quad \psi_t \in \mathbb{R}^{k\times d}; \quad \|\psi\|_q^q := \mathbb{E}\left[\left(\int_0^T \|\psi_t\|^2 dt\right)^{q/2}\right] < \infty \right\}$, and we consider the Banach space $\mathcal{B}_q = \mathcal{S}_q \times \mathcal{H}_q$ endowed with the norm:

$$\|(Y,Z)\|_q^q = \mathbb{E}\Big[\sup_{0\le t\le T}|Y_t|^q\Big] + \mathbb{E}\Big[\Big(\int_0^T \|Z_t\|^2 dt\Big)^{q/2}\Big].$$

We now introduce the generator of our BSDEs. We assume that f is a function defined on $\Omega \times [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d}$, with values in $\mathbb{R}^k$ in such a way that the process $\big(f(t,y,z)\big)_{t\in[0,T]}$ is progressively measurable for each (y,z) in $\mathbb{R}^k \times \mathbb{R}^{k\times d}$. Furthermore we make the following assumption.

(A 1). There exist constants $\gamma \ge 0$, $\mu \in \mathbb{R}$, $C \ge 0$ and $p > 1$ such that $\mathbb{P} - a.s.$, we have:

1. $\forall t,\ \forall y,\ \forall (z,z'),\quad \big|f(t,y,z) - f(t,y,z')\big| \le \gamma\|z-z'\|$;
2. $\forall t,\ \forall z,\ \forall (y,y'),\quad (y-y')\cdot\big(f(t,y,z) - f(t,y',z)\big) \le -\mu|y-y'|^2$;
3. $\forall t,\ \forall y,\ \forall z,\quad \big|f(t,y,z)\big| \le \big|f(t,0,z)\big| + C\big(1+|y|^p\big)$;
4. $\forall t,\ \forall z,\quad y \longmapsto f(t,y,z)$ is continuous.

We refer to the condition (A 1).2 as a monotonicity condition. Our goal is to study the BSDE

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds - \int_t^T Z_s dW_s, \qquad 0 \le t \le T, \tag{1}$$

when the generator f satisfies the above assumption. In the classical case $p = 1$, the terminal condition ξ and the process $\big(f(t,0,0)\big)_{t\in[0,T]}$ are assumed to be square integrable. In the nonlinear case $p > 1$, we need stronger integrability conditions on both ξ and $\big(f(t,0,0)\big)_{t\in[0,T]}$.We suppose that:

(A 2). ξ is an $\mathcal{F}_T$-measurable random variable with values in $\mathbb{R}^k$ such that

$$\mathbb{E}\Big[|\xi|^{2p}\Big] + \mathbb{E}\Big[\Big(\int_0^T \big|f(s,0,0)\big|^2 ds\Big)^p\Big] < \infty.$$

Remark. We consider here only the case $p > 1$ since the case $p = 1$ is treated in the works of R. W. R. Darling, E. Pardoux [5] and E. Pardoux [12].

2.2 A First a priori Estimate

We end these preliminaries by establishing an a priori estimate for BSDEs in the case where ξ and $f(t,0,0)$ are bounded. The following proposition is a mere generalization of a result of S. Peng [18, Theorem 2.2] who proved the same result under a stronger assumption on f namely,

$$\forall t,y,z, \qquad \big|f(t,y,z)\big| \le \alpha + \nu|y| + \kappa\|z\|.$$

Our contribution is merely to remark that his proof requires only an estimate of $y\cdot f(t,y,z)$ and thus that the result should still true in our context. We include a proof for the sake of completeness.

Proposition 2.1 *Let $\big((Y_t,Z_t)\big)_{t\in[0,T]} \in \mathcal{B}_2$ be a solution of the BSDE* (1)*. Let us assume moreover that for each t, y, z,*

$$y\cdot f(t,y,z) \le \alpha|y| + \nu|y|^2 + \kappa|y|\cdot\|z\|, \ \ and,\ \|\xi\|_\infty \le \delta.$$

Then, for each $\varepsilon > 0$, we have, setting $\beta = \varepsilon + 2\nu + \kappa^2$,

$$\sup_{0\le t\le T}|Y_t|^2 \le \delta^2 e^{\beta T} + \frac{\alpha^2}{\varepsilon\beta}\big(e^{\beta T} - 1\big).$$

Proof. Let us fix $t \in [0,T]$, β will be chosen later in the proof. Applying Itô's formula to $e^{\beta(s-t)}|Y_s|^2$ between t and T, we obtain:

$$|Y_t|^2 + \int_t^T e^{\beta(s-t)}\big(\beta|Y_s|^2 + \|Z_s\|^2\big)ds = |\xi|^2 e^{\beta(T-t)} + 2\int_t^T e^{\beta(s-t)} Y_s \cdot f(s,Y_s,Z_s)ds - M_t,$$

provided we write M_t for $2\displaystyle\int_t^T e^{\beta(s-t)} Y_s \cdot Z_s dW_s$. Using the assumption on (ξ,f) it follows that:

$$|Y_t|^2 + \int_t^T e^{\beta(s-t)}\big(\beta|Y_s|^2 + \|Z_s\|^2\big)ds \le \delta^2 e^{\beta T} + 2\int_t^T e^{\beta(s-t)}\big\{\alpha|Y_s| + \nu|Y_s|^2 + \kappa|Y_s|\cdot\|Z_s\|\big\}ds - M_t.$$

Using the inequality $2ab \le \frac{a^2}{\eta} + \eta b^2$, we obtain, for any $\varepsilon > 0$,

$$\begin{aligned}|Y_t|^2 + \int_t^T e^{\beta(s-t)}\big(\beta|Y_s|^2 + \|Z_s\|^2\big)ds \quad &\le \quad \delta^2 e^{\beta T} + \int_t^T e^{\beta(s-t)}\Big\{\frac{\alpha^2}{\varepsilon} + (\varepsilon + 2\nu + \kappa^2)|Y_s|^2\Big\}ds \\ &\quad + \int_t^T e^{\beta(s-t)}\|Z_s\|^2 ds - 2\int_t^T e^{\beta(s-t)} Y_s \cdot Z_s dW_s,\end{aligned}$$

and choosing $\beta = \varepsilon + 2\nu + \kappa^2$ yields the inequality

$$|Y_t|^2 \le \delta^2 e^{\beta T} + \frac{\alpha^2}{\varepsilon\beta}\big(e^{\beta T} - 1\big) - 2\int_t^T e^{\beta(s-t)} Y_s \cdot Z_s dW_s.$$

Taking the conditional expectation with respect to $\mathcal{F}_t$ of both sides, we get immediately that:

$$\forall t \in [0,T], \qquad |Y_t|^2 \le \delta^2 e^{\beta T} + \frac{\alpha^2}{\varepsilon\beta}\big(e^{\beta T} - 1\big),$$

which completes the proof. □

3 BSDEs with Fixed Terminal Times

The goal of this section is to study the BSDE (1) for fixed (deterministic) terminal time T under the assumption (A 1) and (A 2). We first prove uniqueness, then we prove an a priori estimate and finally we turn to existence.

3.1 Uniqueness and a priori Estimates

This subsection is devoted to the proof of uniqueness and to the study of the integrability properties of the solutions of the BSDE (1).

Theorem 3.1 *If (A 1).1–2 hold, the BSDE* (1) *has at most one solution in the space* $\mathcal{B}_2$.

Proof. Suppose that we have two solutions in the space $\mathcal{B}_2$, say (Y^1,Z^1) and (Y^2,Z^2). Setting $\delta Y \equiv Y^1 - Y^2$ and $\delta Z \equiv Z^1 - Z^2$ for notational convenience, for each real number α and for each $t \in [0,T]$, taking expectations in Itô's formula gives:

$$\mathbb{E}\Big[e^{\alpha t}|\delta Y_t|^2 + \int_t^T e^{\alpha s}\|\delta Z_s\|^2 ds\Big] = \mathbb{E}\Big[\int_t^T e^{\alpha s}\big\{2\delta Y_s \cdot \big(f(s,Y_s^1,Z_s^1) - f(s,Y_s^2,Z_s^2)\big) - \alpha|\delta Y_s|^2\big\}ds\Big].$$

The vanishing of the expectation of the stochastic integral is easily justified in view of Burkholder's inequality. Using the monotonicity of f and the Lipschitz assumption, we get:

$$\mathbb{E}\Big[e^{\alpha t}|\delta Y_t|^2+\int_t^T e^{\alpha s}\|\delta Z_s\|^2 ds\Big]\le \mathbb{E}\Big[2\gamma\int_t^T e^{\alpha s}|\delta Y_s|\|\delta Z_s\|ds-(\alpha+2\mu)\int_t^T e^{\alpha s}|\delta Y_s|^2 ds\Big].$$

Hence, we see that

$$\mathbb{E}\Big[e^{\alpha t}|\delta Y_t|^2+\int_t^T e^{\alpha s}\|\delta Z_s\|^2 ds\Big]\le (2\gamma^2-2\mu-\alpha)\mathbb{E}\Big[\int_t^T e^{\alpha s}|\delta Y_s|^2 ds\Big]+\frac{1}{2}\mathbb{E}\Big[\int_t^T e^{\alpha s}\|\delta Z_s\|^2 ds\Big].$$

We conclude the proof of uniqueness by choosing $\alpha=2\gamma^2-2\mu+1$. $\square$

We close this section with the derivation of some a priori estimates in the space $\mathcal{B}_{2p}$. These estimates give short proofs of existence and uniqueness in the Lipschitz context. They were introduced in a "L^p framework" by N. El Karoui, S. Peng, M.-C. Quenez [6] to treat the case of Lipschitz generators.

Proposition 3.2 *For $i=1,2$ we let $(Y^i,Z^i)\in\mathcal{B}_{2p}$ be a solution of the BSDE*

$$Y_t^i=\xi^i+\int_t^T f^i(s,Y_s^i,Z_s^i)ds-\int_t^T Z_s^i dW_s,\qquad 0\le t\le T,$$

where (ξ^i,f^i) satisfies the assumptions (A 1) and (A 2) with constants γ_i, μ_i and C_i. Let ε such that $0<\varepsilon<1$ and $\alpha\ge(\gamma_1)^2/\varepsilon-2\mu_1$. Then there exists a constant K_p^ε which depends only on p and on ε such that:

$$\mathbb{E}\Big[\sup_{0\le t\le T}e^{p\alpha t}|\delta Y_t|^{2p}+\Big(\int_0^T e^{\alpha t}\|\delta Z_t\|^2 dt\Big)^p\Big]\le K_p^\varepsilon\mathbb{E}\Big[e^{\alpha pT}|\delta\xi|^{2p}+\Big(\int_0^T e^{\frac{\alpha}{2}s}|\delta f_s|ds\Big)^{2p}\Big],$$

where $\delta\xi=\xi^1-\xi^2$, $\delta Y\equiv Y^1-Y^2$, $\delta Z\equiv Z^1-Z^2$ and $\delta f_\cdot\equiv f^1(\cdot,Y_\cdot^2,Z_\cdot^2)-f^2(\cdot,Y_\cdot^2,Z_\cdot^2)$. Moreover, if $\alpha>(\gamma_1)^2/\varepsilon-2\mu_1$, we have also, setting $\nu=\alpha-(\gamma_1)^2/\varepsilon+2\mu_1$,

$$\mathbb{E}\Big[\Big(\int_0^T e^{\alpha t}|\delta Y_t|^2 dt\Big)^p\Big]\le\frac{K_p^\varepsilon}{\nu^p}\mathbb{E}\Big[e^{\alpha pT}|\delta\xi|^{2p}+\Big(\int_0^T e^{\frac{\alpha}{2}s}|\delta f_s|ds\Big)^{2p}\Big].$$

Proof. As usual we start with Itô's formula to see that:

$$\begin{aligned} e^{\alpha t}|\delta Y_t|^2+\int_t^T e^{\alpha s}\|\delta Z_s\|^2 ds &= e^{\alpha T}|\delta\xi|^2+2\int_t^T e^{\alpha s}\delta Y_s\cdot\big(f^1(s,Y_s^1,Z_s^1)-f^2(s,Y_s^2,Z_s^2)\big)ds\\ &\quad -\int_t^T \alpha e^{\alpha s}|\delta Y_s|^2 ds-M_t\end{aligned}$$

where we set $M_t=2\displaystyle\int_t^T e^{\alpha s}\delta Y_s\cdot\delta Z_s dW_s$ for each $t\in[0,T]$. In order to use the monotonicity of f^1 and the Lipschitz assumption on f^1, we split one term into three parts, precisely we write:

$$\begin{aligned}\delta Y_s\cdot\big(f^1(s,Y_s^1,Z_s^1)-f^2(s,Y_s^2,Z_s^2)\big) &= \delta Y_s\cdot\big(f^1(s,Y_s^1,Z_s^1)-f^1(s,Y_s^2,Z_s^1)\big)\\ &\quad+\delta Y_s\cdot\big(f^1(s,Y_s^2,Z_s^1)-f^1(s,Y_s^2,Z_s^2)\big)\\ &\quad+\delta Y_s\cdot\big(f^1(s,Y_s^2,Z_s^2)-f^2(s,Y_s^2,Z_s^2)\big),\end{aligned}$$

and the inequality $2\gamma_1|Y_s|\cdot\|Z_s\| \le \big((\gamma_1)^2/\varepsilon\big)|Y_s|^2+\varepsilon\|Z_s\|^2$ implies that:

$$\begin{aligned} e^{\alpha t}|\delta Y_t|^2+(1-\varepsilon)\int_t^T e^{\alpha s}\|\delta Z_s\|^2ds &\le e^{\alpha T}|\delta\xi|^2+\int_t^T e^{\alpha s}\Big\{-\alpha-2\mu_1+\frac{(\gamma_1)^2}{\varepsilon}\Big\}|\delta Y_s|^2ds \\ &\quad +2\int_t^T e^{\alpha s}|\delta Y_s|\cdot|\delta f_s|ds-M_t. \end{aligned}$$

Setting $\nu=\alpha+2\mu_1-(\gamma_1)^2/\varepsilon$, the previous inequality can be rewritten in the following way

$$\begin{aligned} e^{\alpha t}|\delta Y_t|^2+(1-\varepsilon)\int_t^T e^{\alpha s}\|\delta Z_s\|^2ds+\nu\int_t^T e^{\alpha s}|\delta Y_s|^2ds &\le e^{\alpha T}|\delta\xi|^2-M_t \\ &\quad +2\int_t^T e^{\alpha s}|\delta Y_s|\cdot|\delta f_s|ds. \end{aligned} \tag{2}$$

Taking the conditional expectation with respect to $\mathcal{F}_t$ of the previous inequality, we deduce since the conditional expectation of M_t vanishes,

$$e^{\alpha t}|\delta Y_t|^2 \le \mathbb{E}\Big\{e^{\alpha T}|\delta\xi|^2+2\int_0^T e^{\alpha s}|\delta Y_s|\cdot|\delta f_s|ds\,\Big|\,\mathcal{F}_t\Big\},$$

and since $p>1$, Doob's maximal inequality implies:

$$\begin{aligned} \mathbb{E}\Big[\sup_{0\le t\le T}e^{p\alpha t}|\delta Y_t|^{2p}\Big] &\le K_p\mathbb{E}\Big[e^{p\alpha T}|\delta\xi|^{2p}+\Big(\int_0^T e^{\alpha s}|\delta Y_s|\cdot|\delta f_s|ds\Big)^p\Big] \\ &\le K_p\mathbb{E}\Big[e^{p\alpha T}|\delta\xi|^{2p}+\sup_{0\le t\le T}\big\{e^{(p\alpha/2)t}|\delta Y_t|^p\big\}\Big(\int_0^T e^{(\alpha/2)s}|\delta f_s|ds\Big)^p\Big]. \end{aligned}$$

where we use the notation K_p for a constant depending only on p and whose value could be changing from line to line. Thanks to the inequality $ab\le a^2/2+b^2/2$, we get

$$\mathbb{E}\Big[\sup_{0\le t\le T}e^{p\alpha t}|\delta Y_t|^{2p}\Big]\le K_p\mathbb{E}\Big[e^{\alpha pT}|\delta\xi|^{2p}+\Big(\int_0^T e^{(\alpha/2)s}|\delta f_s|ds\Big)^{2p}\Big]+\frac{1}{2}\mathbb{E}\Big[\sup_{0\le t\le T}e^{p\alpha t}|\delta Y_t|^{2p}\Big],$$

which gives

$$\mathbb{E}\Big[\sup_{0\le t\le T}e^{p\alpha t}|\delta Y_t|^{2p}\Big]\le K_p\mathbb{E}\Big[e^{\alpha pT}|\delta\xi|^{2p}+\Big(\int_0^T e^{(\alpha/2)s}|\delta f_s|ds\Big)^{2p}\Big]. \tag{3}$$

Now coming back to the inequality (2), we have since $\varepsilon<1$,

$$\int_0^T e^{\alpha s}\|\delta Z_s\|^2ds\le\frac{1}{1-\varepsilon}\Big(e^{\alpha T}|\delta\xi|^2+2\int_0^T e^{\alpha s}|\delta Y_s|\cdot|\delta f_s|ds-2\int_0^T e^{\alpha s}\delta Y_s\cdot\delta Z_s dW_s\Big),$$

and by Burkholder-Davis-Gundy's inequality we obtain

$$\begin{aligned} \mathbb{E}\Big[\Big(\int_0^T e^{\alpha s}\|\delta Z_s\|^2ds\Big)^p\Big] &\le K_p^\varepsilon\mathbb{E}\Big[e^{\alpha pT}|\delta\xi|^{2p}+\Big(\int_0^T e^{\alpha s}|\delta Y_s|\cdot|\delta f_s|ds\Big)^p\Big] \\ &\quad +K_p^\varepsilon\mathbb{E}\Big[\Big(\int_0^T e^{2\alpha s}|\delta Y_s|^2\|\delta Z_s\|^2ds\Big)^{p/2}\Big], \end{aligned}$$

and thus it follows easily that:

$$\begin{aligned}\mathbb{E}\Big[\Big(\int_0^T e^{\alpha s}\|\delta Z_s\|^2 ds\Big)^p\Big] &\le K_p^\varepsilon \mathbb{E}\Big[e^{\alpha pT}|\delta\xi|^{2p} + \sup_{0\le t\le T}\{e^{(p\alpha/2)t}|\delta Y_t|^p\}\Big(\int_0^T e^{(\alpha/2)s}|\delta f_s|ds\Big)^p\Big] \\ &+K_p^\varepsilon \mathbb{E}\Big[\sup_{0\le t\le T}\{e^{(p\alpha/2)t}|\delta Y_t|^p\}\Big(\int_0^T e^{\alpha s}\|\delta Z_s\|^2 ds\Big)^{p/2}\Big],\end{aligned}$$

which yields the inequality, using one more time the inequality $ab \le a^2/2 + b^2/2$,

$$\begin{aligned}\mathbb{E}\Big[\Big(\int_0^T e^{\alpha s}\|\delta Z_s\|^2 ds\Big)^p\Big] &\le K_p^\varepsilon \mathbb{E}\Big[e^{\alpha pT}|\delta\xi|^{2p} + \sup_{0\le t\le T} e^{p\alpha t}|\delta Y_t|^{2p} + \Big(\int_0^T e^{(\alpha/2)s}|\delta f_s|ds\Big)^{2p}\Big] \\ &+\frac{1}{2}\mathbb{E}\Big[\Big(\int_0^T e^{\alpha s}\|\delta Z_s\|^2 ds\Big)^p\Big].\end{aligned}$$

Taking into account the upper bound found for $\mathbb{E}\Big[\sup_{0\le t\le T} e^{p\alpha t}|\delta Y_t|^{2p}\Big]$ given in (3), we derive from the above inequality,

$$\mathbb{E}\Big[\Big(\int_0^T e^{\alpha s}\|\delta Z_s\|^2 ds\Big)^p\Big] \le K_p^\varepsilon \mathbb{E}\Big[e^{\alpha pT}|\delta\xi|^{2p} + \Big(\int_0^T e^{(\alpha/2)s}|\delta f_s|ds\Big)^{2p}\Big],$$

which concludes the first part of this proposition. For the second assertion we simply remark that (2) gives

$$\nu\int_0^T e^{\alpha s}|\delta Y_s|^2 ds \le \Big(e^{\alpha T}|\delta\xi|^2 + 2\int_0^T e^{\alpha s}|\delta Y_s|\cdot|\delta f_s|ds - 2\int_0^T e^{\alpha s}\delta Y_s\cdot\delta Z_s dW_s\Big),$$

A similar computation gives:

$$\begin{aligned}\nu^p\mathbb{E}\Big[\Big(\int_0^T e^{\alpha s}|\delta Y_s|^2 ds\Big)^p\Big] &\le K_p^\varepsilon \mathbb{E}\Big[e^{\alpha pT}|\delta\xi|^{2p} + \sup_{0\le t\le T} e^{p\alpha t}|\delta Y_t|^{2p} + \Big(\int_0^T e^{(\alpha/2)s}|\delta f_s|ds\Big)^{2p}\Big] \\ &+\frac{1}{2}\mathbb{E}\Big[\Big(\int_0^T e^{\alpha s}\|\delta Z_s\|^2 ds\Big)^p\Big],\end{aligned}$$

which completes the proof using the first part of the proposition already shown and keeping in mind that if $\alpha > (\gamma_1)^2/\varepsilon - 2\mu_1$ then $\nu > 0$. □

Corollary 3.3 *Under the assumptions and with the notation of the previous proposition, there exists a constant K, depending only on p, T, μ_1 and γ_1 such that:*

$$\mathbb{E}\Big[\sup_{0\le t\le T}|\delta Y_t|^{2p} + \Big(\int_0^T \|\delta Z_t\|^2 dt\Big)^p\Big] \le K\mathbb{E}\Big[|\delta\xi|^{2p} + \Big(\int_0^T |\delta f_s|ds\Big)^{2p}\Big].$$

Proof. From the previous proposition, we have (taking $\varepsilon = 1/2$):

$$\mathbb{E}\Big[\sup_{0\le t\le T} e^{p\alpha t}|\delta Y_t|^{2p} + \Big(\int_0^T e^{\alpha t}\|\delta Z_t\|^2 dt\Big)^p\Big] \le K_p\mathbb{E}\Big[e^{\alpha pT}|\delta\xi|^{2p} + \Big(\int_0^T e^{\frac{\alpha}{2}s}|\delta f_s|ds\Big)^{2p}\Big],$$

and thus

$$e^{-pT\alpha^-}\mathbb{E}\Big[\sup_{0\le t\le T}|\delta Y_t|^{2p} + \Big(\int_0^T \|\delta Z_t\|^2 dt\Big)^p\Big] \le K_p e^{pT\alpha^+}\mathbb{E}\Big[|\delta\xi|^{2p} + \Big(\int_0^T |\delta f_s|ds\Big)^{2p}\Big].$$

It is enough to set $K = e^{p|\alpha|T} K_p$ to conclude the proof. □

Remark. It is plain to check that the assumptions (A1).3–4 are not needed in the above proofs of the results of Proposition 3.2 and its corollary.

Corollary 3.4 *Let $\big((Y_t, Z_t)\big)_{0\le t\le T} \in \mathcal{B}_{2p}$ be a solution of the BSDE* (1) *and let us assume that $\xi \in \mathrm{L}^{2p}$ and assume also that there exists a process $(f_t)_{0\le t\le T} \in \mathcal{H}_{2p}(\mathbb{R}^k)$ such that*

$$\forall (s,y,z) \in [0,T]\times\mathbb{R}^k\times\mathbb{R}^{k\times d}, \qquad y\cdot f(s,y,z) \le |y|\cdot|f_s| - \mu|y|^2 + \gamma|y|\cdot\|z\|.$$

Then, if $0<\varepsilon<1$ and $\alpha \ge \gamma^2/\varepsilon - 2\mu$, there exists a constant K_p^ε which depends only on p and on ε such that:

$$\mathbb{E}\Big[\sup_{0\le t\le T} e^{p\alpha t}|Y_t|^{2p} + \Big(\int_0^T e^{\alpha t}\|Z_t\|^2 dt\Big)^p\Big] \le K_p^\varepsilon \mathbb{E}\Big[e^{\alpha p T}|\xi|^{2p} + \Big(\int_0^T e^{\frac{\alpha}{2}s}|f_s|ds\Big)^{2p}\Big],$$

Proof. As usual we start with Itô's formula to see that

$$: e^{\alpha t}|Y_t|^2 + \int_t^T e^{\alpha s}\|Z_s\|^2 ds = e^{\alpha T}|\xi|^2 + 2\int_t^T e^{\alpha s} Y_s\cdot f(s,Y_s,Z_s)ds - \int_t^T \alpha e^{\alpha s}|Y_s|^2 ds - M_t,$$

provided we set $M_t = 2\int_t^T e^{\alpha s} Y_s\cdot Z_s dW_s$ for each $t\in[0,T]$. Using the assumption on $y\cdot f(s,y,z)$ and then the inequality $2\gamma|Y_s|\cdot\|Z_s\| \le \big(\gamma^2/\varepsilon\big)|Y_s|^2 + \varepsilon\|Z_s\|^2$, we deduce that

$$\begin{aligned} e^{\alpha t}|Y_t|^2 + (1-\varepsilon)\int_t^T e^{\alpha s}\|Z_s\|^2 ds \quad &\le \quad e^{\alpha T}|\xi|^2 + \int_t^T e^{\alpha s}\Big\{-\alpha - 2\mu + \frac{\gamma^2}{\varepsilon}\Big\}|Y_s|ds \\ &\quad +2\int_t^T e^{\alpha s}|Y_s|\cdot|f_s|ds - M_t. \end{aligned}$$

Since $\alpha \ge 2\mu - \gamma^2/\varepsilon$, the previous inequality implies

$$e^{\alpha t}|Y_t|^2 + (1-\varepsilon)\int_t^T e^{\alpha s}\|Z_s\|^2 ds \le e^{\alpha T}|\xi|^2 + 2\int_t^T e^{\alpha s}|Y_s|\cdot|f_s|ds - M_t.$$

This inequality is exactly the same as the inequality (2). As a consequence we can complete the proof of this as in the proof of Proposition 3.2. □

3.2 Existence

In this subsection, we study the existence of solutions for the BSDE (1) under the assumptions (A 1) and (A 2). We shall prove that the BSDE (1) has a solution in the space $\mathcal{B}_{2p}$. We may assume, without lost of generality, that the constant μ is equal to 0. Indeed, $(Y_t, Z_t)_{t\in[0,T]}$ solves the BSDE (1) in $\mathcal{B}_{2p}$ if and only if, setting for each $t\in[0,T]$,

$$\overline{Y}_t = e^{-\mu t}Y_t, \text{ and } \overline{Z}_t = e^{-\mu t}Z_t,$$

the process $\big(\overline{Y}, \overline{Z}\big)$ solves in $\mathcal{B}_{2p}$ the following BSDE:

$$\overline{Y}_t = \overline{\xi} + \int_t^T \overline{f}(s,\overline{Y}_s,\overline{Z}_s)ds - \int_t^T \overline{Z}_s dW_s, \qquad 0\le t\le T,$$

where $\overline{\xi} = e^{-\mu T}\xi$ and $\overline{f}(t,y,z) = e^{-\mu t} f(t, e^{\mu t} y, e^{\mu t} z) + \mu y$. Since $(\overline{\xi}, \overline{f})$ satisfies the assumption (A 1) and (A 2) with $\overline{\gamma} = \gamma$, $\overline{\mu} = 0$ and $\overline{C} = C \exp\left(T\{(p-1)\mu^+ + \mu^-\}\right) + |\mu|$, we shall assume that $\mu = 0$ in the remaining of this section.

Our proof is based on the following strategy: first, we solve the problem when the function f does not depend on the variable z and then we use a fix point argument using the a priori estimate given in subsection 3.1, Proposition 3.2 and Corollary 3.3. The following proposition gives the first step.

Proposition 3.5 *Let the assumptions (A 1) and (A 2) hold. Given a process $(V_t)_{0\le t\le T}$ in the space $\mathcal{H}_{2p}$, there exists a unique solution $\left((Y_t, Z_t)\right)_{t\in[0,T]}$ in the space $\mathcal{B}_{2p}$ to the BSDE*

$$Y_t = \xi + \int_t^T f(s, Y_s, V_s)ds - \int_t^T Z_s dW_s, \qquad 0 \le t \le T. \tag{4}$$

Proof. We shall write in the sequel $h(s,y)$ in place of $f(s,y,V_s)$. Of course h satisfies the assumption (A 1) with the same constants as f and $\left(h(\cdot,0)\right)$ belongs to $\mathcal{H}_{2p}$ since f is Lipschitz with respect to z and the process V belongs to $\mathcal{H}_{2p}$. What we would like to do is to construct a sequence of Lipschitz (globally in y uniformly with respect to (ω, s)) functions h_n which approximate h and which are monotone. However, we only manage to construct a sequence for which each h_n is monotone in a given ball (the radius depends on n). As we will see later in the proof, this "local" monotonicity is sufficient to obtain the result. This is mainly due to Proposition 2.1 whose key idea can be traced back to a work of S. Peng [18, Theorem 2.2].

We shall use an approximate identity. Let $\rho : \mathbb{R}^k \longrightarrow \mathbb{R}_+$ be a nonnegative $\mathcal{C}^\infty$ function with the unit ball for support and such that $\int \rho(u)du = 1$ and define for each integer $n \ge 1$, $\rho_n(u) = n\rho(nu)$. We denote also, for each integer n, by Θ_n a $\mathcal{C}^\infty$ function from $\mathbb{R}^k$ to $\mathbb{R}_+$ such that $0 \le \Theta_n \le 1$, $\Theta_n(u) = 1$ for $|u| \le n$ and $\Theta_n(u) = 0$ as soon as $|u| \ge n+1$. We set moreover

$$\xi_n = \begin{cases} \xi & \text{if } |\xi| \le n, \\ n\dfrac{\xi}{|\xi|} & \text{otherwise,} \end{cases} \qquad \text{and,} \quad \tilde{h}_n(s,y) = \begin{cases} h(s,y) & \text{if } |h(s,0)| \le n, \\ \dfrac{n}{|h(s,0)|} h(s,y) & \text{otherwise.} \end{cases}$$

Such an $\tilde{h}_n$ satisfies the assumption (A 1) and moreover we have $|\xi_n| \le n$ and $|\tilde{h}_n(s,0)| \le n$. Finally we set $q(n) = \left[e^{1/2}(n+2C)\sqrt{1+T^2}\right] + 1$ where $[r]$ stands as usual for the integer part of r and we define

$$h_n(s,\cdot) = \rho_n * \left(\Theta_{q(n)+1}\tilde{h}_n(s,\cdot)\right) \qquad s \in [0,T].$$

We first remark that $h_n(s,y) = 0$ whenever $|y| \ge q(n)+3$ and that $h_n(s,\cdot)$ is globally Lipschitz with respect to y uniformly in (ω, s). Indeed, $h_n(s,\cdot)$ is a smooth function with compact support and thus we have $\sup_{y\in\mathbb{R}^k} \left|\nabla h_n(s,y)\right| = \sup_{|y|\le q(n)+3} \left|\nabla h_n(s,y)\right|$ and, from the growth assumption on f (A 1).3, it is not hard to check that $|\tilde{h}_n(s,y)| \le n \wedge |h(s,0)| + C\left(1+|y|^p\right)$ which implies that

$$\left|\nabla h_n(s,y)\right| \le \left(n\left\{n + C(1+2^{p-1}|y|^p)\right\} + C2^{p-1}\right) \int \left|\nabla\rho(u)\right| du.$$

As an immediate consequence, the function h_n is globally Lipschitz with respect to y uniformly in (ω, s). In addition $|\xi_n| \le n$ and $|h_n(s,0)| \le n \wedge |h(s,0)| + 2C$ and thus Theorem 5.1 in [6] provides a solution (Y^n, Z^n) to the BSDE

$$Y_t^n = \xi_n + \int_t^T h_n(s, Y_s^n)ds - \int_t^T Z_s^n dW_s, \qquad 0 \le t \le T, \tag{5}$$

which belongs actually to $\mathcal{B}_q$ for each $q > 1$. In order to apply Proposition 2.1 we observe that, for each y,

$$\begin{aligned} y \cdot h_n(s,y) &= \int \rho_n(u)\Theta_{q(n)+1}(y-u)y \cdot \tilde{h}_n(s,y-u)du \\ &= \int \rho_n(u)\Theta_{q(n)+1}(y-u)y \cdot \{\tilde{h}_n(s,y-u) - \tilde{h}_n(s,-u)\}du \\ &\quad + \int \rho_n(u)\Theta_{q(n)+1}(y-u)y \cdot \tilde{h}_n(s,-u)du. \end{aligned}$$

Hence, we deduce that, since the function $\tilde{h}_n(s,\cdot)$ is monotone (recall that $\mu = 0$) in this section) and in view of the growth assumption on f we have:

$$\forall (s,y) \in \Omega \times [0,T], \qquad y \cdot h_n(s,y) \le \big(n \wedge |h(s,0)| + 2C\big)|y|. \tag{6}$$

This estimate will turn out to be very useful in the sequel. Indeed, we can apply Proposition 2.1 to the BSDE (5) to show that, for each n, choosing $\varepsilon = 1/T$,

$$\sup_{0\le t\le T} |Y_t^n| \le (n+2C)e^{1/2}\sqrt{1+T^2}. \tag{7}$$

On the other hand, the inequality (6) allows one to use Corollary 3.4, to obtain, for a constant K_p depending only on p:

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\sup_{0\le t\le T} |Y_t^n|^{2p} + \left(\int_0^T \|Z_t^n\|^2 dt\right)^p\right] \le K_p \mathbb{E}\left[|\xi|^{2p} + \left(\int_0^T \big\{|h(s,0)| + 2C\big\}ds\right)^{2p}\right]. \tag{8}$$

It is worth noting that, thanks to $|h(s,0)| \le |f(s,0,0)| + \gamma\|V_s\|$, the right hand side of the previous inequality is finite. We want to prove that the sequence $\big((Y^n,Z^n)\big)_{\mathbb{N}}$ converges towards the solution of the BSDE (4) and in order to do that we first show that the sequence $\big((Y^n,Z^n)\big)_{\mathbb{N}}$ is a Cauchy sequence in the space $\mathcal{B}_2$. This fact relies mainly on the following property: h_n satisfies the monotonicity condition in the ball of radius $q(n)$. Indeed, fix $n \in \mathbb{N}$ and let us pick y, y' such that $|y| \le q(n)$ and $|y'| \le q(n)$. We have:

$$\begin{aligned} (y-y')\cdot(h_n(s,y) - h_n(s,y') &= (y-y')\cdot \int \rho_n(u)\Theta_{q(n)+1}(y-u)\tilde{h}_n(s,y-u)du \\ &\quad -(y-y')\cdot \int \rho_n(u)\Theta_{q(n)+1}(y'-u)\tilde{h}_n(s,y'-u)du. \end{aligned}$$

But, since $|y|, |y'| \le q(n)$ and since the support of ρ_n is included in the unit ball, we get from the fact that $\Theta_{q(n)+1}(x) = 1$ as soon as $|x| \le q(n)+1$,

$$(y-y')\cdot(h_n(s,y) - h_n(s,y') = \int \rho_n(u)(y-y')\cdot\big(\tilde{h}_n(s,y-u) - \tilde{h}_n(s,y'-u)\big)du.$$

Hence, by the monotonicity of $\tilde{h}_n$, we get

$$\forall y, y' \in \overline{\mathrm{B}(0,q(n))}, \qquad (y-y')\cdot(h_n(s,y) - h_n(s,y') \le 0. \tag{9}$$

We now turn to the convergence of $\big((Y^n,Z^n)\big)_{\mathbb{N}}$. Let us fix two integers m and n such that $m \ge n$. Itô's formula gives, for each $t \in [0,T]$,

$$|\delta Y_t|^2 + \int_t^T \|\delta Z_s\|^2 ds = |\delta\xi|^2 + 2\int_t^T \delta Y_s \cdot \big(h_m(s,Y_s^m) - h_n(s,Y_s^n)\big)ds - 2\int_t^T \delta Y_s \cdot \delta Z_s dW_s,$$

where we have set $\delta\xi = \xi_m - \xi_n$, $\delta Y \equiv Y^m - Y^n$ and $\delta Z \equiv Z^m - Z^n$. We split one term of the previous inequality into two parts, precisely we write:

$$\delta Y_s \cdot \big(h_m(s,Y_s^m) - h_n(s,Y_s^n)\big) = \delta Y_s \cdot \big(h_m(s,Y_s^m) - h_m(s,Y_s^n)\big) + \delta Y_s \cdot \big(h_m(s,Y_s^n) - h_n(s,Y_s^n)\big).$$

But in view of the estimate (7), we have $|Y_s^m| \le q(m)$ and $|Y_s^n| \le q(n) \le q(m)$. Thus, using the property (9), the first part of the right hand side of the previous inequality is non-positive and it follows that

$$|\delta Y_t|^2 + \int_t^T \|\delta Z_s\|^2 ds \le |\delta\xi|^2 + 2\int_t^T |\delta Y_s| \cdot \big|h_m(s,Y_s^n) - h_n(s,Y_s^n)\big| ds - 2\int_t^T \delta Y_s \cdot \delta Z_s dW_s. \quad (10)$$

In particular, we have

$$\mathbb{E}\Big[\int_0^T \|\delta Z_s\|^2 ds\Big] \le 2\mathbb{E}\Big[|\delta\xi|^2 + \int_0^T |\delta Y_s| \cdot \big|h_m(s,Y_s^n) - h_n(s,Y_s^n)\big| ds\Big],$$

and coming back to (10), Burkholder's inequality implies

$$\mathbb{E}\Big[\sup_{0\le t\le T} |\delta Y_t|^2\Big] \le K\mathbb{E}\Big[|\delta\xi|^2 + \int_0^T |\delta Y_s| \cdot \big|h_m(s,Y_s^n) - h_n(s,Y_s^n)\big| ds + \Big(\int_0^T |\delta Y_s|^2 \|\delta Z_s\|^2 ds\Big)^{1/2}\Big],$$

and then using the inequality $ab \le a^2/2 + b^2/2$ we obtain the following inequality:

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\le t\le T} |\delta Y_t|^2\Big] \quad &\le \quad K\mathbb{E}\Big[|\delta\xi|^2 + \int_0^T |\delta Y_s| \cdot \big|h_m(s,Y_s^n) - h_n(s,Y_s^n)\big| ds\Big] \\ &\quad + \frac{1}{2}\mathbb{E}\Big[\sup_{0\le t\le T} |\delta Y_t|^2\Big] + \frac{K^2}{2}\mathbb{E}\Big[\int_0^T \|Z_s\|^2 ds\Big],\end{aligned}$$

from which we get, for another constant still denoted by K,

$$\mathbb{E}\Big[\sup_{0\le t\le T} |\delta Y_t|^2 + \int_0^T \|\delta Z_s\|^2 ds\Big] \le K\mathbb{E}\Big[|\delta\xi|^2 + \int_0^T |\delta Y_s| \cdot \big|h_m(s,Y_s^n) - h_n(s,Y_s^n)\big| ds\Big].$$

Obviously, since $\xi \in \mathrm{L}^{2p}$, $\delta\xi$ tends to 0 in L^2 as $n, m \to \infty$ with $m \ge n$. So, we have only to prove that

$$\mathbb{E}\Big[\int_0^T |\delta Y_s| \cdot \big|h_m(s,Y_s^n) - h_n(s,Y_s^n)\big| ds\Big] \longrightarrow 0, \qquad \text{as} \quad n \to \infty.$$

For any nonnegative number k, we write

$$\begin{aligned} S_n^m &= \mathbb{E}\Big[\int_0^T \mathbb{1}_{|Y_s^n|+|Y_s^m|\le k} |\delta Y_s| \cdot \big|h_m(s,Y_s^n) - h_n(s,Y_s^n)\big| ds\Big], \\ R_n^m &= \mathbb{E}\Big[\int_0^T \mathbb{1}_{|Y_s^n|+|Y_s^m|> k} |\delta Y_s| \cdot \big|h_m(s,Y_s^n) - h_n(s,Y_s^n)\big| ds\Big], \end{aligned}$$

and with these notations we have

$$\mathbb{E}\Big[\int_0^T |\delta Y_s| \cdot \big|h_m(s,Y_s^n) - h_n(s,Y_s^n)\big| ds\Big] = S_n^m + R_n^m$$

and hence, the following inequality:

$$\mathbb{E}\Big[\int_0^T |\delta Y_s| \cdot \big|h_m(s,Y_s^n) - h_n(s,Y_s^n)\big| ds\Big] \le k\mathbb{E}\Big[\int_0^T \sup_{|y|\le k} \big|h_m(s,y) - h_n(s,y)\big| ds\Big] + R_n^m. \quad (11)$$

First we deal with R_n^m and using Hölder's inequality we get the following upper bound:

$$R_n^m \leq \left\{\mathbb{E}\Big[\int_0^T \mathbb{1}_{|Y_s^n|+|Y_s^m|\geq k} ds\Big]\right\}^{\frac{p-1}{2p}} \left\{\mathbb{E}\Big[\int_0^T |\delta Y_s|^{\frac{2p}{p+1}} \big|h_m(s,Y_s^n) - h_n(s,Y_s^n)\big|^{\frac{2p}{p+1}} ds\Big]\right\}^{\frac{p+1}{2p}}.$$

Setting $A_n^m = \mathbb{E}\Big[\int_0^T |\delta Y_s|^{\frac{2p}{p+1}} \big|h_m(s,Y_s^n) - h_n(s,Y_s^n)\big|^{\frac{2p}{p+1}} ds\Big]$ for notational convenience, we have

$$R_n^m \leq \left\{\int_0^T \mathbb{P}\big(|Y_s^n| + |Y_s^m| \geq k\big) ds\right\}^{\frac{p-1}{2p}} A_n^{m\,\frac{p+1}{2p}},$$

and Chebyshev's inequality yields:

$$\begin{aligned} R_n^m &\leq k^{1-p}\left\{\int_0^T \mathbb{E}\Big[\big(|Y_s^n| + |Y_s^m|\big)^{2p}\Big] ds\right\}^{\frac{p-1}{2p}} A_n^{m\,\frac{p+1}{2p}} \\ &\leq 2^p T^{\frac{p-1}{2p}}\left\{\sup_{n\in\mathbb{N}} \mathbb{E}\Big[\sup_{0\leq t\leq T} |Y_t^n|^{2p}\Big]\right\}^{\frac{p-1}{2p}} k^{1-p} A_n^{m\,\frac{p+1}{2p}}. \end{aligned} \tag{12}$$

We have already seen that $\sup_{n\in\mathbb{N}} \mathbb{E}\Big[\sup_{0\leq t\leq T} |Y_t^n|^{2p}\Big]$ is finite (cf. (8)) and we shall prove that A_n^m remains bounded as n, m vary. To do this, let us recall that

$$A_n^m = \mathbb{E}\Big[\int_0^T |\delta Y_s|^{\frac{2p}{p+1}} \big|h_m(s,Y_s^n) - h_n(s,Y_s^n)\big|^{\frac{2p}{p+1}} ds\Big],$$

and using Young's inequality ($ab \leq \frac{1}{r}a^r + \frac{1}{r^*}b^{r^*}$ whenever $\frac{1}{r} + \frac{1}{r^*} = 1$) with $r = p+1$ and $r^* = \frac{p+1}{p}$, we deduce that

$$A_n^m \leq \frac{1}{p+1}\mathbb{E}\Big[\int_0^T |\delta Y_s|^{2p} ds\Big] + \frac{p}{p+1}\mathbb{E}\Big[\int_0^T \big|h_m(s,Y_s^n) - h_n(s,Y_s^n)\big|^2 ds\Big].$$

The first part of the last upper bound remains bounded as n, m vary since from (8) we know that $\sup_{n\in\mathbb{N}} \mathbb{E}\Big[\sup_{0\leq t\leq T} |Y_t^n|^{2p}\Big]$ is finite. Moreover, we derive easily from the assumption (A 1) that $\big|h_n(s,y)\big| \leq n \wedge \big|h(s,0)\big| + 2^p C\big(1 + |y|^p\big)$, and then,

$$\big|h_m(s,Y_s^n) - h_n(s,Y_s^n)\big| \leq 2\big|h(s,0)\big| + 2^{p+1}C\big(1 + |Y_s^n|^p\big),$$

which yields the inequality, taking into account the assumption (A 1).1,

$$\mathbb{E}\Big[\int_0^T \big|h_m(s,Y_s^n) - h_n(s,Y_s^n)\big|^2 ds\Big] \leq K_p \mathbb{E}\Big[\int_0^T \big\{|f(s,0,0)|^2 + \|V_s\|^2 + 1 + |Y_s^n|^{2p}\big\} ds\Big].$$

Taking into account (8) and the integrability assumption on both V and $f(\cdot,0,0)$, we have proved that $\sup_{n\leq m} A_n^m < \infty$.

Coming back to the inequality (12), we get, for a constant κ, $R_n^m \leq \kappa k^{1-p}$, and since $p > 1$, R_n^m can be made arbitrary small by choosing k large enough. Thus, in view of the estimate (11), it remains only to check that, for each fixed $k > 0$,

$$\mathbb{E}\Big[\int_0^T \sup_{|y|\leq k} \big|h_m(s,y) - h_n(s,y)\big| ds\Big]$$

goes to 0 as n tends to infinity uniformly with respect to m to get the convergence of $\big((Y^n,Z^n)\big)_{\mathbb{N}}$ in the space $\mathcal{B}_2$. But, since $h(s,\cdot)$ is continuous ($\mathbb{P}-a.s.$, $\forall s$), $h_n(s,\cdot)$ converges towards $h(s,\cdot)$ uniformly on compact sets. Taking into account that $\sup_{|y|\leq k}\big|h_n(s,y)\big| \leq \big|h(s,0)\big| + 2^pC\big(1+k^p\big)$ Lebesgue's convergence theorem gives the result.

Thus, the sequence $\big((Y^n,Z^n)\big)_{\mathbb{N}}$ converges towards a progressively measurable process (Y,Z) in the space $\mathcal{B}_2$. Moreover, since $\big((Y^n,Z^n)\big)_{\mathbb{N}}$ is bounded in $\mathcal{B}_{2p}$ (see (8)), Fatou's lemma implies that (Y,Z) belongs also to the space $\mathcal{B}_{2p}$.

It remains to check that (Y,Z) solves the BSDE (4) which is nothing but

$$Y_t = \xi + \int_t^T h(s,Y_s)ds - \int_t^T Z_s dW_s, \qquad 0\leq t\leq T.$$

Of course, we want to pass to the limit in the BSDE (5). Let us first remark that $\xi_n \longrightarrow \xi$ in L^{2p} and that for each $t\in[0,T]$, $\displaystyle\int_t^T Z_s^n dW_s \longrightarrow \int_t^T Z_s dW_s$ since Z^n converges to Z in the space $\mathcal{H}_2(\mathbb{R}^{k\times d})$. Actually, we only need to prove that for $t\in[0,T]$,

$$\int_t^T h_n(s,Y_s^n)ds \longrightarrow \int_t^T h(s,Y_s)ds, \qquad \text{as}\, n\to\infty.$$

For this, we shall see that $h_n(\cdot,Y_\cdot^n)$ tends to $h(\cdot,Y_\cdot)$ in the space $\mathrm{L}^1(\Omega\times[0,T])$. Indeed,

$$\mathbb{E}\Big[\int_0^T \big|h_n(s,Y_s^n)-h(s,Y_s)\big|ds\Big] \leq \mathbb{E}\Big[\int_0^T \big|h_n(s,Y_s^n)-h(s,Y_s^n)\big|ds\Big] + \mathbb{E}\Big[\int_0^T \big|h(s,Y_s^n)-h(s,Y_s)\big|ds\Big].$$

The first term of the right hand side of the previous inequality tends to 0 as n goes to ∞ by the same argument we use earlier in the proof to see that $\mathbb{E}\big[\int_0^T |\delta Y_s|\cdot|h_m(s,Y_s^n)-h_n(s,Y_s^n)|ds\big]$ goes to 0. For the second term, we shall firstly prove that there exists a converging subsequence. Indeed, since Y^n converges to Y is the space $\mathcal{S}_2$, there exists a subsequence (Y^{n_j}) such that $\mathbb{P}$–a.s.,

$$\forall t\in[0,T], \qquad Y_t^{n_j} \longrightarrow Y_t.$$

Since $h(t,\cdot)$ is continuous ($\mathbb{P}$–a.s., $\forall t$), $\mathbb{P}$–a.s. $\big(\forall t,\quad h(t,Y_t^{n_j}) \longrightarrow h(t,Y_t)\big)$. Moreover, since $Y\in\mathcal{S}_{2p}$ and $(Y_n)_{\mathbb{N}}$ is bounded in $\mathcal{S}_{2p}$ ((8)), it is not hard to check from the growth assumption on f that

$$\sup_{j\in\mathbb{N}} \mathbb{E}\Big[\int_0^T \big|h(s,Y_s^{n_j})-h(s,Y_s)\big|^2 ds\Big] < \infty,$$

and then the result follows by uniform integrability of the sequence. Actually, the convergence hold for the whole sequence since each subsequence has a converging subsequence. Finally, we can pass to the limit in the BSDE (5) and the proof is complete. □

With the help of this proposition, we can now construct a solution (Y,Z) to the BSDE (1). We claim the following result:

Theorem 3.6 *Under the assumptions (A 1) and (A 2), the BSDE* (1) *has a unique solution* (Y,Z) *in the space* $\mathcal{B}_{2p}$.

Proof. The uniqueness part of this statement is already proved in Theorem 3.1. The first step in the proof of the existence is to show the result when T is sufficiently small. According to

Theorem 3.1 and Proposition 3.5, let us define the following function Φ from $\mathcal{B}_{2p}$ into itself. For $(U,V) \in \mathcal{B}_{2p}$, $\Phi(U,V) = (Y,Z)$ where (Y,Z) is the unique solution in $\mathcal{B}_{2p}$ of the BSDE:

$$Y_t = \xi + \int_t^T f(s, Y_s, V_s)ds - \int_t^T Z_s dW_s, \qquad 0 \leq t \leq T.$$

Next we prove that Φ is a strict contraction provided that T is small enough. Indeed, if $\left(U^1, V^1\right)$ and $\left(U^2, V^2\right)$ are both elements of the space $\mathcal{B}_{2p}$, we have, applying Proposition 3.2 for $\left(Y^i, Z^i\right) = \Phi\left(U^i, V^i\right)$, $i = 1, 2$,

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |\delta Y_t|^{2p} + \left(\int_0^T \|\delta Z_t\|^2 dt\right)^p\right] \leq K_p \mathbb{E}\left[\left(\int_0^T |f(s, Y_s^2, V_s^1) - f(s, Y_s^2, V_s^2| ds\right)^{2p}\right],$$

where $\delta Y \equiv Y^1 - Y^2$, $\delta Z \equiv Z^1 - Z^2$ and K_p is a constant depending only on p. Using the Lipschitz assumption on f, (A 1).1, and Hölder's inequality we get the inequality

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |\delta Y_t|^{2p} + \left(\int_0^T \|\delta Z_t\|^2 dt\right)^p\right] \leq K_p \gamma^{2p} T^p \mathbb{E}\left[\left(\int_0^T \|V_s^1 - V_s^2\|^2 ds\right)^p\right].$$

Hence, if T is such that $K_p \gamma^{2p} T^p < 1$, Φ is a strict contraction and thus Φ has a unique fixed point in the space $\mathcal{B}_{2p}$ which is the unique solution of the BSDE (1). The general case is treated by subdividing the time interval $[0,T]$ into a finite number of intervals whose lengths are small enough and using the above existence and uniqueness result in each of the subintervals. $\square$

4 The Case of Random Terminal Times

In this section, we briefly explain how to extend the results of the previous section to the case of a random terminal time.

4.1 Notation and Assumptions

Let us recall that $(W_t)_{t\geq 0}$ is a d-dimensional Brownian motion, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and that $\left(\mathcal{F}_t\right)_{t\geq 0}$ is the complete σ-algebra generated by $(W_t)_{t\geq 0}$.

Let τ be a stopping time with respect to $\left(\mathcal{F}_t\right)_{t\geq 0}$ and let us assume that τ is finite $\mathbb{P}$–*a.s.* Let us consider also a random variable ξ $\mathcal{F}_\tau$–measurable and a function f defined on $\Omega \times \mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^{k\times d}$ with values in $\mathbb{R}^k$ and such that the process $\left(f(\cdot, y, z)\right)$ is progressively measurable for each (y,z).

We study the following BSDE with the random terminal time τ:

$$Y_t = \xi + \int_{t\wedge\tau}^\tau f(s, Y_s, Z_s)ds - \int_{t\wedge\tau}^\tau Z_s dW_s, \qquad t \geq 0. \tag{13}$$

By a solution of this equation, we always mean a progressively measurable process $\left((Y_t, Z_t)\right)_{t\geq 0}$ with values in $\mathbb{R}^k \times \mathbb{R}^{k\times d}$ such that $Z_t = 0$ if $t > \tau$. Moreover, since τ is finite $\mathbb{P}$–*a.s.*, (13) implies that $Y_t = \xi$ if $t \geq \tau$.

We need to introduce further notation. Let us consider $q > 1$ and $\alpha \in \mathbb{R}$. We say that a progressively measurable process ψ with values in $\mathbb{R}^n$ belongs to $\mathcal{H}_q^\alpha(\mathbb{R}^n)$ if

$$\mathbb{E}\left[\left(\int_0^\infty e^{\alpha t} \|\psi_t\|^2 dt\right)^{q/2}\right] < \infty.$$

Moreover, we say that ψ belongs to the space $\mathcal{S}_q^{\alpha,\tau}(\mathbb{R}^n)$ if

$$\mathbb{E}\Big[\sup_{t\geq 0} e^{(q/2)\alpha(t\wedge\tau)}|\psi_t|^q\Big] < \infty.$$

We are going to prove an existence and uniqueness result for the BSDE (13) under assumptions which are very similar to those made in section 2 for the study of the case of BSDEs with fixed terminal times. Precisely, we will suppose in the framework of random terminal times the following two assumptions:

(A 3). There exist constants $\gamma \geq 0$, $\mu \in \mathbb{R}$, $C \geq 0$, $p > 1$ and $\kappa \in \{0,1\}$ such that $\mathbb{P}-a.s.$, we have:

1. $\forall t,\ \forall y,\ \forall (z,z'),\quad \big|f(t,y,z)-f(t,y,z')\big| \leq \gamma\|z-z'\|;$
2. $\forall t,\ \forall z,\ \forall (y,y'),\quad (y-y')\cdot\big(f(t,y,z)-f(t,y',z)\big) \leq -\mu|y-y'|^2;$
3. $\forall t,\ \forall y,\ \forall z,\quad \big|f(t,y,z)\big| \leq \big|f(t,0,z)\big| + C(\kappa+|y|^p);$
4. $\forall t,\ \forall z,\quad y \longmapsto f(t,y,z)$ is continuous,

(A 4). ξ is $\mathcal{F}_\tau$-measurable and there exists a real number ρ such that $\rho > \gamma^2 - 2\mu$ and

$$\mathbb{E}\Big[\kappa e^{\rho\tau} + \{e^{\rho\tau}+e^{p\rho\tau}\}|\xi|^{2p} + \Big(\int_0^\tau e^{\rho s}\big|f(s,0,0)\big|^2 ds\Big)^p + \Big(\int_0^\tau e^{(\rho/2)s}\big|f(s,0,0)\big|ds\Big)^{2p}\Big] < \infty.$$

Remark. In the case $\rho < 0$, which may occur if τ is an unbounded stopping time, our integrability conditions are fulfilled if we assume that

$$\mathbb{E}\Big[e^{\rho\tau}|\xi|^{2p} + \Big(\int_0^\tau e^{(\rho/2)s}\big|f(s,0,0)\big|^2 ds\Big)^p\Big] < \infty.$$

For notational convenience, we will simply write, in the remaining of the paper, $\mathcal{S}_q^{\rho,\tau}$ and $\mathcal{H}_q^\rho$ instead of $\mathcal{S}_q^{\rho,\tau}(\mathbb{R}^k)$ and $\mathcal{H}_q^\rho(\mathbb{R}^{k\times d})$ respectively.

4.2 Existence and Uniqueness

In this section, we deal with the existence and uniqueness of the solutions of the BSDE (13). We claim the following proposition.

Proposition 4.1 *Under the assumptions (A 3) and (A 4), there exists at most a solution of the BSDE* (13) *in the space* $\mathcal{S}_2^{\rho,\tau}\times\mathcal{H}_2^\rho$.

Proof. Let (Y^1,Z^1) and (Y^2,Z^2) be two solutions of (13) in the space $\mathcal{S}_2^{\rho,\tau}\times\mathcal{H}_2^\rho$. Let us notice first that $Y_t^1 = Y_t^2 = \xi$ if $t\geq\tau$ and $Z_t^1 = Z_t^2 = 0$ on the set $\{t>\tau\}$. Applying Itô's formula, we get

$$\begin{aligned} e^{\rho(t\wedge\tau)}|\delta Y_{t\wedge\tau}|^2 + \int_{t\wedge\tau}^\tau e^{\rho s}\|\delta Z_s\|^2 ds &= 2\int_{t\wedge\tau}^\tau e^{\rho s}\delta Y_s\cdot\big(f(s,Y_s^1,Z_s^1)-f(s,Y_s^2,Z_s^2)\big)ds \\ &\quad -\int_{t\wedge\tau}^\tau \rho e^{\rho s}|\delta Y_s|^2 ds - 2\int_{t\wedge\tau}^\tau e^{\rho s}\delta Y_s\cdot\delta Z_s dW_s, \end{aligned}$$

where we have set $\delta Y \equiv Y^1 - Y^2$ and $\delta Z \equiv Z^1 - Z^2$. It is worth noting that, since f is Lipschitz in z and monotone in y, we have, for each $\varepsilon > 0$,

$$\forall (t,y,y',z,z'), \quad 2(y-y')\cdot\big(f(t,y,z)-f(t,y',z')\big) \le (-2\mu+\gamma^2/\varepsilon)|y-y'|^2+\varepsilon\|z-z'\|^2. \quad (14)$$

Moreover, by Burkholder's inequality the continuous local martingale

$$\Big\{\int_0^{t\wedge\tau} e^{\rho s}\delta Y_s\cdot\delta Z_s dW_s, \quad t\ge 0\Big\}$$

is a uniformly integrable martingale. Indeed,

$$\begin{aligned}\mathbb{E}\Big[\Big\langle\int_0^{\cdot\wedge t} e^{\rho s}\delta Y_s\cdot\delta Z_s dW_s\Big\rangle_\infty^{1/2}\Big] &= \mathbb{E}\Big[\Big(\int_0^\tau e^{2\rho s}|\delta Y_s|^2\|\delta Z_s\|^2 ds\Big)^{1/2}\Big]\\ &\le K\mathbb{E}\Big[\Big(\sup_{0\le t\le\tau} e^{\rho t}|\delta Y_t|^2\Big)^{1/2}\Big(\int_0^\tau e^{\rho s}\|\delta Z_s\|^2 ds\Big)^{1/2}\Big],\end{aligned}$$

and then,

$$\mathbb{E}\Big[\Big\langle\int_0^{\cdot\wedge t} e^{\rho s}\delta Y_s\cdot\delta Z_s dW_s\Big\rangle_\infty^{1/2}\Big] \le \frac{K}{2}\mathbb{E}\Big[\sup_{0\le t\le\tau} e^{\rho t}|\delta Y_t|^2+\int_0^\tau e^{\rho s}\|\delta Z_s\|^2 ds\Big],$$

which is finite since $(\delta Y,\delta Z)$ belongs to the space $\mathcal{S}_2^{\rho,\tau}\times\mathcal{H}_2^\rho$. Thanks to the inequality $\rho>\gamma^2-2\mu$, we can choose ε such that $0<\varepsilon<1$ and $\rho>\gamma^2/\varepsilon-2\mu$. Using the inequality (14), we deduce that, the expectation of the stochastic integral vanishing in view of the above computation, for each t,

$$\mathbb{E}\Big[e^{\rho(t\wedge\tau)}|\delta Y_{t\wedge\tau}|^2+(1-\varepsilon)\int_{t\wedge\tau}^\tau e^{\rho s}\|\delta Z_s\|^2 ds\Big]\le 0,$$

which gives the result. □

Before proving the existence part of the result, let us introduce a sequence of processes whose construction is due to R. W. R. Darling and E. Pardoux [5, pp. 1148–1149]. Let us set $\lambda=\gamma^2/2-\mu$ and let $(\widehat{Y}^n,\widehat{Z}^n)$ be the unique solution of the classical (the terminal time is deterministic) BSDE on $[0,n]$

$$\widehat{Y}_t^n=\mathbb{E}\{e^{\lambda\tau}\xi\mid\mathcal{F}_n\}+\int_{t\wedge\tau}^{n\wedge\tau}\{e^{\lambda s}f(s,e^{-\lambda s}\widehat{Y}_s^n,e^{-\lambda s}\widehat{Z}_s^n)-\lambda\widehat{Y}_s^n\}ds-\int_t^n\widehat{Z}_s^n dW_s.$$

Since $\mathbb{E}\Big[e^{2p\lambda\tau}|\xi|^{2p}\Big]\le\mathbb{E}\Big[e^{p\rho\tau}|\xi|^{2p}\Big]$ and since

$$\mathbb{E}\Big[\Big(\int_0^\tau e^{2\lambda s}\big|f(s,0,0)\big|^2 ds\Big)^p\Big]\le\mathbb{E}\Big[\Big(\int_0^\tau e^{\rho s}\big|f(s,0,0)\big|^2 ds\Big)^p\Big],$$

the assumption (A 4) and Theorem 3.6 ensure that $(\widehat{Y}^n,\widehat{Z}^n)$ belongs to the space $\mathcal{B}_{2p}$ (on the interval $[0,n]$). In view of [12, Proposition 3.1], we have

$$\widehat{Y}^n(t\wedge\tau)=\widehat{Y}_t^n, \quad \text{and,} \quad \widehat{Z}_t^n=0 \text{ on } \{t>\tau\}.$$

Since $e^{\lambda\tau}\xi$ belongs to $\mathrm{L}^{2p}(\mathcal{F}_\tau)$ there exists a process (η) in $\mathcal{H}_2^0$ such that $\eta_t=0$ if $t>\tau$ and

$$e^{\lambda\tau}\xi=\mathbb{E}\big[e^{\lambda\tau}\xi\big]+\int_0^\tau\eta_s dW_s.$$

We introduce still new notation. For each $t > n$ we set:

$$\widehat{Y}_t^n = \mathbb{E}\{e^{\lambda\tau}\xi \mid \mathcal{F}_t\} = \zeta_t, \quad \text{and,} \quad \widehat{Z}_t^n = \eta_t,$$

and for each nonnegative t:

$$Y_t^n = e^{-\lambda(t\wedge\tau)}\widehat{Y}_t^n, \quad \text{and,} \quad Z_t^n = e^{-\lambda(t\wedge\tau)}\widehat{Z}_t^n.$$

This process satisfies $Y_{t\wedge\tau}^n = Y_t^n$ and $Z_t^n = 0$ on $\{t > \tau\}$ and moreover (Y^n, Z^n) solves the BSDE

$$Y_t^n = \xi + \int_{t\wedge\tau}^{\tau} f_n(s, Y_s^n, Z_s^n)ds - \int_{t\wedge\tau}^{\tau} Z_s^n dW_s, \qquad t \geq 0, \tag{15}$$

where $f_n(t,y,z) = \mathbb{1}_{t\leq n} f(t,y,z) + \mathbb{1}_{t>n}\lambda y$ (cf [5]). We start with a technical lemma.

Lemma 4.2 *Let the assumptions (A 3) and (A 4) hold. Then, we have, with the notation*

$$K(\xi, f) = K\mathbb{E}\Big[e^{p\rho\tau}|\xi|^{2p} + \Big(\int_0^\tau e^{(\rho/2)s}\big|f(s,0,0)\big|ds\Big)^{2p}\Big],$$

$$\sup_{\mathbb{N}} \mathbb{E}\left[\sup_{t\geq 0} e^{p\rho(t\wedge\tau)}|Y_t^n|^{2p} + \Big(\int_0^\tau e^{\rho s}|Y_s^n|^2 ds\Big)^p + \Big(\int_0^\infty e^{\rho s}\|Z_s^n\|^2 ds\Big)^p\right] \leq K(\xi, f), \tag{16}$$

and, also, for $\sigma = \rho - 2\lambda$,

$$\mathbb{E}\left[\sup_{t\geq 0} e^{p\sigma(t\wedge\tau)}|\zeta_t|^{2p} + \Big(\int_0^\tau e^{\sigma s}|\zeta_s|^2 ds\Big)^p + \Big(\int_0^\infty e^{\sigma s}\|\eta_s\|^2 ds\Big)^p\right] \leq K\mathbb{E}\Big[e^{p\rho\tau}|\xi|^{2p}\Big]. \tag{17}$$

Proof. Firstly, let us remark that $Z_t^n = \eta_t = 0$ if $t > \tau$ and, since $Y_t^n = \xi$ if $t \geq \tau$, we have $\sup_{t\geq 0} e^{p\rho(t\wedge\tau)}|Y_t^n|^{2p} = \sup_{0\leq t\leq\tau} e^{p\rho t}|Y_t^n|^{2p}$. Moreover, since $\rho > 2\lambda$ we can find ε such that $0 < \varepsilon < 1$ and $\rho > \gamma^2/\varepsilon - 2\mu$. Applying Proposition 3.2 (actually a very mere extension to deal with bounded stopping times as terminal times), we get

$$\begin{aligned}\mathbb{E}&\left[\sup_{0\leq t\leq n\wedge\tau} e^{p\rho t}|Y_t^n|^{2p} + \Big(\int_0^{n\wedge\tau} e^{\rho s}|Y_s^n|^2 ds\Big)^p + \Big(\int_0^{n\wedge\tau} e^{\rho s}\|Z_s^n\|^2 ds\Big)^p\right] \\ &\leq KE\Big[e^{p\rho(n\wedge\tau)}|Y^n(n\wedge\tau)|^{2p} + \Big(\int_0^{n\wedge\tau} e^{(\rho/2)s}\big|f(s,0,0)\big|ds\Big)^{2p}\Big].\end{aligned}$$

We have $Y_{n\wedge\tau}^n = Y_n^n = e^{-\lambda(n\wedge\tau)}\mathbb{E}\{e^{\lambda\tau}\xi \mid \mathcal{F}_{n\wedge\tau}\}$ and then we deduce immediately that, since $\rho/2 - \lambda > 0$ and using Jensen's inequality,

$$\begin{aligned}\mathbb{E}\Big[e^{p\rho(n\wedge\tau)}|Y^n(n\wedge\tau)|^{2p}\Big] &= \mathbb{E}\Big[\big|\mathbb{E}\{e^{(\rho/2-\lambda)(n\wedge\tau)}e^{\lambda\tau}\xi \mid \mathcal{F}_{n\wedge\tau}\}\big|^{2p}\Big] \\ &\leq \mathbb{E}\Big[e^{p\rho\tau}|\xi|^{2p}\Big].\end{aligned} \tag{18}$$

Hence, for each integer n,

$$\mathbb{E}\left[\sup_{0\leq t\leq n\wedge\tau} e^{p\rho t}|Y_t^n|^{2p} + \Big(\int_0^{n\wedge\tau} e^{\rho s}|Y_s^n|^2 ds\Big)^p + \Big(\int_0^{n\wedge\tau} e^{\rho s}\|Z_s^n\|^2 ds\Big)^p\right] \leq K(\xi, f).$$

It remains to prove that we can find the same upper bound for

$$\mathbb{E}\left[\sup_{n\wedge\tau< t\leq\tau} e^{p\rho t}|Y_t^n|^{2p} + \Big(\int_{n\wedge\tau}^{\tau} e^{\rho s}|Y_s^n|^2 ds\Big)^p + \Big(\int_{n\wedge\tau}^{\tau} e^{\rho s}\|Z_s^n\|^2 ds\Big)^p\right].$$

But the expectation is over the set $\{n<\tau\}$ and coming back to the definition of $(\widehat{Y}_n,\widehat{Z}_n)$ for $t>n$, it is enough to check that

$$\mathbb{E}\Big[\sup_{t\geq 0} e^{p(\rho-2\lambda)(t\wedge\tau)}|\zeta_t|^{2p}+\Big(\int_0^\tau e^{(\rho-2\lambda)s}|\zeta_s|^2ds\Big)^p+\Big(\int_0^\tau e^{(\rho-2\lambda)s}\|\eta_s\|^2ds\Big)^p\Big]\leq K\mathbb{E}\Big[e^{p\rho\tau}|\xi|^{2p}\Big]$$

to get the inequality (16) of the lemma and thus to complete the proof since, in view of the definition of σ, the previous inequality is nothing but the inequality (17). But, for each n, (ζ,η) solves the the following BSDE:

$$\zeta_t=\mathbb{E}\{e^{\lambda\tau}\xi\,|\,\mathcal{F}_{n\wedge\tau}\}-\int_t^n\eta_s dW_s,\qquad 0\leq t\leq n,$$

and by Proposition 3.2, since $\sigma=\rho-2\lambda>0$,

$$\mathbb{E}\Big[\sup_{0\leq t\leq n\wedge\tau} e^{p\sigma t}|\zeta_t|^{2p}+\Big(\int_0^{n\wedge\tau} e^{\sigma s}|\zeta_s|^2ds\Big)^p+\Big(\int_0^{n\wedge\tau} e^{\sigma s}\|\eta_s\|^2ds\Big)^p\Big]\leq K\mathbb{E}\Big[e^{p\sigma(n\wedge\tau)}|\zeta_{n\wedge\tau}|^{2p}\Big].$$

We have already seen (cf (18)) that $\mathbb{E}\big[e^{p\sigma(n\wedge\tau)}|\zeta_{n\wedge\tau}|^{2p}\big]\leq\mathbb{E}\big[e^{p\rho\tau}|\xi|^{2p}\big]$ and thus the proof of this rather technical lemma is complete. □

With the help of this useful lemma we can construct a solution to the BSDE (13). This is the aim of the following theorem.

Theorem 4.3 *Under the assumptions (A 3) and (A 3), the BSDE* (13) *has a unique solution* (Y,Z) *in the space* $\mathcal{S}_2^{\rho,\tau}\times\mathcal{H}_2^\rho$ *which satisfies moreover*

$$\mathbb{E}\Big[\sup_{t\geq 0} e^{p\rho(t\wedge\tau)}|Y_t|^{2p}+\Big(\int_0^\tau e^{\rho s}|Y_s|^2ds\Big)^p+\Big(\int_0^\infty e^{\rho s}\|Z_s\|^2ds\Big)^p\Big]\leq K(\xi,f).$$

Proof. The uniqueness part of this claim is already proved in Proposition 4.1. We concentrate ourselves on the existence part. We split the proof into the two following steps: first we show that the sequence $\big((Y^n,Z^n)\big)_{\mathbb{N}}$ is a Cauchy sequence in the space $\mathcal{S}_2^{\rho,\tau}\times\mathcal{H}_2^\rho$ and then we shall prove that the limiting process is indeed a solution.

Let us first recall that for each integer n, the process (Y^n,Z^n) satisfies $Y^n_{t\wedge\tau}=Y^n_t$ and $Z^n_t=0$ on $\{t>\tau\}$ and moreover solves the BSDE (15) whose generator f_n is defined in the following way: $f_n(t,y,z)=\mathbb{1}_{t\leq n}f(t,y,z)+\mathbb{1}_{t>n}\lambda y$. If we fix $m\geq n$, Itô's formula gives, since we have also $Y^m_{m\wedge\tau}=Y^m_m=Y^n_{m\wedge\tau}=Y^n_m=e^{-\lambda(m\wedge\tau)}\zeta_m$, for $t\leq m$,

$$\begin{aligned}e^{\rho(t\wedge\tau)}|\delta Y_{t\wedge\tau}|^2+\int_{t\wedge\tau}^{m\wedge\tau}e^{\rho s}\|\delta Z_s\|^2ds &= 2\int_{t\wedge\tau}^{m\wedge\tau}e^{\rho s}\delta Y_s\cdot\big(f_m(s,Y^m_s,Z^m_s)-f_n(s,Y^n_s,Z^n_s)\big)ds\\ &\quad-\int_{t\wedge\tau}^{m\wedge\tau}\rho e^{\rho s}|\delta Y_s|^2ds-2\int_{t\wedge\tau}^{m\wedge\tau}e^{\rho s}\delta Y_s\cdot\delta Z_s dW_s,\end{aligned}$$

where we have set $\delta Y\equiv Y^m-Y^n$, $\delta Z\equiv Z^m-Z^n$. It follows from the definition of f_n,

$$\begin{aligned}e^{\rho(t\wedge\tau)}|\delta Y_{t\wedge\tau}|^2+\int_{t\wedge\tau}^{m\wedge\tau}e^{\rho s}\|\delta Z_s\|^2ds &= 2\int_{t\wedge\tau}^{m\wedge\tau}e^{\rho s}\delta Y_s\cdot\big(f(s,Y^m_s,Z^m_s)-f(s,Y^n_s,Z^n_s)\big)ds\\ &\quad-\int_{t\wedge\tau}^{m\wedge\tau}\rho e^{\rho s}|\delta Y_s|^2ds-2\int_{t\wedge\tau}^{m\wedge\tau}e^{\rho s}\delta Y_s\cdot\delta Z_s dW_s\\ &\quad+2\int_{t\wedge\tau}^{m\wedge\tau}\mathbb{1}_{s>n}e^{\rho s}\delta Y_s\cdot\big(f(s,Y^n_s,Z^n_s)-\lambda Y^n_s\big)ds.\end{aligned}$$

Since $\rho > \gamma^2 - 2\mu$, we can find ε such that $0 < \varepsilon < 1$ and $\nu = \rho - \gamma^2/\varepsilon + 2\mu > 0$. Using the inequality (14) with this ε, we deduce from the previous inequality,

$$\begin{aligned} e^{\rho(t\wedge\tau)}|\delta Y_{t\wedge\tau}|^2 + (1-\varepsilon)\int_{t\wedge\tau}^{m\wedge\tau} e^{\rho s}\|\delta Z_s\|^2\}ds \quad &\le \quad -\nu\int_{t\wedge\tau}^{m\wedge\tau} e^{\rho s}|\delta Y_s|^2 ds - 2\int_{t\wedge\tau}^{m\wedge\tau} e^{\rho s}\delta Y_s\cdot\delta Z_s dW_s \\ &\quad +2\int_{(t\vee n)\wedge\tau}^{m\wedge\tau} e^{\rho s}|\delta Y_s|.\big|f(s,Y_s^n,Z_s^n) - \lambda Y_s^n\big|ds. \end{aligned}$$

Now, using the inequality $2ab \le \varpi a^2 + b^2/\varpi$ for the second term of the right hand side of the previous inequality, with $\varpi < \nu$, we get, for each $t \le m$, noting $\beta = \min(1-\varepsilon, \nu - \varpi) > 0$,

$$\begin{aligned} e^{\rho(t\wedge\tau)}|\delta Y_{t\wedge\tau}|^2 + \beta\int_{t\wedge\tau}^{m\wedge\tau} e^{\rho s}\big\{|\delta Y_s|^2 + \|\delta Z_s\|^2\big\}ds \quad &\le \frac{1}{\varpi}\int_{n\wedge\tau}^{m\wedge\tau} e^{\rho s}\big|f(s,Y_s^n,Z_s^n) - \lambda Y_s^n\big|^2 ds \\ &\quad - 2\int_{t\wedge\tau}^{m\wedge\tau} e^{\rho s}\delta Y_s\cdot\delta Z_s dW_s. \end{aligned} \tag{19}$$

In particular, we have, the expectation of the stochastic integral vanishes (cf Lemma 4.2),

$$\mathbb{E}\Big[\int_0^{m\wedge\tau} e^{\rho s}\big\{|\delta Y_s|^2 + \|\delta Z_s\|^2\big\}ds\Big] \le K\mathbb{E}\Big[\int_{n\wedge\tau}^{m\wedge\tau} e^{\rho s}\big|f(s,Y_s^n,Z_s^n) - \lambda Y_s^n\big|^2 ds\Big].$$

Coming back to the inequality (19), Burkholder's inequality yields

$$\mathbb{E}\Big[\sup_{0\le t\le m\wedge\tau} e^{\rho t}|\delta Y_t|^2\Big] \le K\mathbb{E}\Big[\int_{n\wedge\tau}^{m\wedge\tau} e^{\rho s}\big|f(s,Y_s^n,Z_s^n) - \lambda Y_s^n\big|^2 ds + \Big(\int_0^{m\wedge\tau} e^{2\rho s}|\delta Y_s|^2\|\delta Z_s\|^2 ds\Big)^{1/2}\Big].$$

But, by an argument already used,

$$\begin{aligned} K\mathbb{E}\Big[\Big(\int_0^{m\wedge\tau} e^{2\rho s}|\delta Y_s|^2\|\delta Z_s\|^2 ds\Big)^{1/2}\Big] \quad &\le \quad K\mathbb{E}\Big[\Big(\sup_{0\le t\le m\wedge\tau} e^{\rho t}|\delta Y_t|^2\Big)^{1/2}\Big(\int_0^{m\wedge\tau} e^{\rho s}\|\delta Z_s\|^2 ds\Big)^{1/2}\Big] \\ &\le \quad \frac{1}{2}\mathbb{E}\Big[\sup_{0\le t\le m\wedge\tau} e^{\rho t}|\delta Y_t|^2\Big] + \frac{K^2}{2}\mathbb{E}\Big[\int_0^{m\wedge\tau} e^{\rho s}\|\delta Z_s\|^2 ds\Big]. \end{aligned}$$

As a consequence we obtain the inequality:

$$\mathbb{E}\Big[\sup_{0\le t\le m\wedge\tau} e^{\rho t}|\delta Y_t|^2 + \int_0^{m\wedge\tau} e^{\rho s}\big\{|\delta Y_s|^2 + \|\delta Z_s\|^2\big\}ds\Big] \le K\mathbb{E}\Big[\int_{n\wedge\tau}^{m\wedge\tau} e^{\rho s}\big|f(s,Y_s^n,Z_s^n) - \lambda Y_s^n\big|^2 ds\Big],$$

and since $Y_t^m = Y_t^n$ if $t \ge m$, $Y_t^i = \xi$ on $\{t \ge \tau\}$ for each i, $Z_t^m = Z_t^n = \eta_t$ as soon as $t \ge m$ and $\eta_t = 0$ on $\{t > \tau\}$ we deduce from the previous inequality

$$\mathbb{E}\Big[\sup_{t\ge 0} e^{\rho(t\wedge\tau)}|\delta Y_t|^2 + \int_0^\tau e^{\rho s}|\delta Y_s|^2 ds + \int_0^\infty e^{\rho s}\|\delta Z_s\|^2 ds\Big] \le \Gamma_n, \tag{20}$$

where we have set $\Gamma_n = \mathbb{E}\Big[\int_{n\wedge\tau}^{\tau} e^{\rho s}\big|f(s,Y_s^n,Z_s^n) - \lambda Y_s^n\big|^2 ds\Big]$. But the growth assumption on f (A 3).3 implies that, up to a constant, Γ_n is upper bounded by

$$\mathbb{E}\Big[\int_{n\wedge\tau}^{\tau} e^{\rho s}\big\{\big|f(s,0,0)\big|^2 + \kappa + |Y_s^n|^2 + \|Z_s^n\|^2 + |Y_s^n|^{2p}\big\}ds\Big].$$

Since, by assumption (A 4), $\mathbb{E}\Big[\int_0^\tau e^{\rho s}|f(s,0,0)|^2ds\Big]$ and $\mathbb{E}\big[\kappa e^{\rho\tau}\big]$ are finite, the first two terms of the previous upper bound tends to 0 as n goes to ∞. Moreover, coming back to the definition of $\big(\widehat{Y}^n,\widehat{Z}^n\big)$ for $t>n$, we have

$$\mathbb{E}\Big[\int_{n\wedge\tau}^\tau e^{\rho s}\big\{|Y_s^n|^2+\|Z_s^n\|^2\big\}ds\Big]=\mathbb{E}\Big[\int_{n\wedge\tau}^\tau e^{(\rho-2\lambda)s}\big\{|\zeta_s|^2+\|\eta_s\|^2\big\}ds\Big],$$

and by Lemma 4.2 (cf (17)) the quantity above tends also to 0 with n going to ∞. It remains to check that the same is true for

$$\mathbb{E}\Big[\int_{n\wedge\tau}^\tau e^{\rho s}|Y_s^n|^{2p}ds\Big]=\mathbb{E}\Big[\int_{n\wedge\tau}^\tau e^{(\rho-2p\lambda)s}|\zeta_s|^{2p}ds\Big],$$

where, let us recall it, ζ_s means $\mathbb{E}\big\{e^{\lambda\tau}\xi\;\big|\;\mathcal{F}_s\big\}$. By Jensen's inequality, it is enough to show the following:

$$\mathbb{E}\Big[\int_{n\wedge\tau}^\tau e^{(\rho-2\lambda p)s}\mathbb{E}\big\{e^{p\lambda\tau}|\xi|^p\;\big|\;\mathcal{F}_s\big\}^2ds\Big]\longrightarrow 0,\qquad\text{as}\quad n\to\infty.$$

If $\rho>2p\lambda$, since $\mathbb{E}\big[e^{2p\lambda\tau}|\xi|^{2p}\big]\le\mathbb{E}\big[e^{p\rho\tau}|\xi|^{2p}\big]<\infty$ and $\mathbb{E}\big[e^{\rho\tau}|\xi|^{2p}\big]<\infty$, Lemma 4.1 in [5] gives

$$\mathbb{E}\Big[\int_0^\tau e^{(\rho-2\lambda p)s}\mathbb{E}\big\{e^{p\lambda\tau}|\xi|^p\;\big|\;\mathcal{F}_s\big\}^2ds\Big]<\infty,$$

from which we get the result.

Now, we deal with the case $\rho\le 2p\lambda$ which implies $0<2\lambda<\rho\le 2p\lambda<p\rho$. Using once more time Jensen's inequality, we have

$$\begin{aligned}\mathbb{E}\Big[\int_{n\wedge\tau}^\tau e^{(\rho-2\lambda p)s}\mathbb{E}\big\{e^{p\lambda\tau}|\xi|^p\;\big|\;\mathcal{F}_s\big\}^2ds\Big] &\le \mathbb{E}\Big[\int_{n\wedge\tau}^\tau \mathbb{E}\big\{e^{2p\lambda\tau}|\xi|^{2p}\;\big|\;\mathcal{F}_s\big\}ds\Big]\\ &\le \mathbb{E}\Big[\int_{n\wedge\tau}^\tau \mathbb{E}\big\{e^{(2\lambda-\rho)p\tau}e^{p\rho\tau}|\xi|^{2p}\;\big|\;\mathcal{F}_s\big\}ds\Big],\end{aligned}$$

and since $\rho>2\lambda$ we have $\mathbb{E}\big\{e^{(2\lambda-\rho)p\tau}e^{p\rho\tau}|\xi|^{2p}\;\big|\;\mathcal{F}_s\big\}\le e^{(2\lambda-\rho)p(s\wedge\tau)}\mathbb{E}\big\{e^{p\rho\tau}|\xi|^{2p}\;\big|\;\mathcal{F}_s\big\}$. Hence, it follows,

$$\begin{aligned}\mathbb{E}\Big[\int_{n\wedge\tau}^\tau e^{(\rho-2\lambda p)s}\mathbb{E}\big\{e^{p\lambda\tau}|\xi|^p\;\big|\;\mathcal{F}_s\big\}^2ds\Big] &\le \mathbb{E}\Big[\int_{n\wedge\tau}^\tau e^{(2\lambda-\rho)ps}\mathbb{E}\big\{e^{p\rho\tau}|\xi|^{2p}\;\big|\;\mathcal{F}_s\big\}ds\Big]\\ &\le \mathbb{E}\Big[e^{p\rho\tau}|\xi|^{2p}\Big]\int_n^\infty e^{(2\lambda-\rho)ps}ds.\end{aligned}$$

Since $2\lambda-\rho<0$ and $\mathbb{E}\big[e^{p\rho\tau}|\xi|^{2p}\big]<\infty$, we complete the proof of the last case. Thus we have shown that Γ_n converges to 0 as n tends to ∞ and coming back to the inequality (20), we get

$$\mathbb{E}\Big[\sup_{t\ge0}e^{\rho(t\wedge\tau)}|\delta Y_t|^2+\int_0^\tau e^{\rho s}|\delta Y_s|^2ds+\int_0^\infty e^{\rho s}\|\delta Z_s\|^2ds\Big]\longrightarrow 0,$$

as n tends to ∞, uniformly in m. In particular the sequence $\big((Y^n,Z^n)\big)_{\mathbb{N}}$ is a Cauchy sequence in $\mathcal{S}_2^{\rho,\tau}\times\mathcal{H}_2^\rho$ and thus converges in this space to a process (Y,Z). Moreover, taking into account the inequality (16) of Lemma 4.2, Fatou's lemma implies

$$\mathbb{E}\Big[\sup_{t\ge0}e^{p\rho(t\wedge\tau)}|Y_t|^{2p}+\Big(\int_0^\tau e^{\rho s}|Y_s|^2ds\Big)^p+\Big(\int_0^\infty e^{\rho s}\|Z_s\|^2ds\Big)^p\Big]\le K(\xi,f).\tag{21}$$

It remains to check that the process (Y,Z) solves the BSDE (13). To do this, we follow the discussion of R. W. R. Darling, E. Pardoux [5, pp. 1150–1151]. Let us pick a real number α such that $\alpha < 0 \wedge \rho/2 \wedge p\rho$ (this implies that $\alpha < \rho$) and let us fix a nonnegative real number t. Since (Y_n, Z_n) solves the BSDE (15), we have, from Itô's formula, for $n \geq t$,

$$\begin{aligned} e^{\alpha(t\wedge\tau)}Y_t^n &= e^{\alpha\tau}\xi + \int_{t\wedge\tau}^{\tau} e^{\alpha s}\{f(s,Y_s^n,Z_s^n) - \alpha Y_s^n\}ds - \int_{t\wedge\tau}^{\tau} e^{\alpha s}Z_s^n dW_s \\ &\quad + \int_{n\wedge\tau}^{\tau} e^{\alpha s}\{\lambda Y_s^n - f(s,Y_s^n,Z_s^n)\}ds, \end{aligned}$$

and we want to pass to the limit in this equation knowing that

$$\mathbb{E}\Big[\sup_{t\geq 0} e^{\rho(t\wedge\tau)}|Y_t - Y_t^n|^2 + \int_0^{\tau} e^{\rho s}|Y_s - Y_s^n|^2 ds + \int_0^{\infty} e^{\rho s}\|Z_s - Z_s^n\|^2 ds\Big] \longrightarrow 0.$$

We have, $e^{\alpha(t\wedge\tau)}Y_t^n \longrightarrow e^{\alpha(t\wedge\tau)}Y_t$ in L^2. Moreover, Hölder's inequality gives

$$\mathbb{E}\Big[\int_0^{\tau} e^{\alpha s}|Y_s^n - Y_s|ds\Big] \leq \Big\{\mathbb{E}\Big[\int_0^{\tau} e^{\rho s}|Y_s^n - Y_s|^2 ds\Big]\Big\}^{1/2}\Big\{\mathbb{E}\Big[\int_0^{\tau} e^{(2\alpha-\rho)s}ds\Big]\Big\}^{1/2}$$

from which we deduce, since $2\alpha < \rho$, that $\int_{t\wedge\tau}^{\tau} e^{\alpha s}Y_s^n ds$ tends to $\int_{t\wedge\tau}^{\tau} e^{\alpha s}Y_s ds$ in L^1. We remark also that $\int_{t\wedge\tau}^{\tau} e^{\alpha s}Z_s^n dW_s$ converges to $\int_{t\wedge\tau}^{\tau} e^{\alpha s}Z_s dW_s$ in L^2 since, thanks to $2\alpha < \rho$,

$$\mathbb{E}\Big[\Big|\int_{t\wedge\tau}^{\tau} e^{\alpha s}\big(Z_s^n - Z_s\big)\cdot dW_s\Big|^2\Big] \leq \mathbb{E}\Big[\int_0^{\tau} e^{\rho s}\|Z_s^n - Z_s\|^2 ds\Big].$$

Using Hölder's inequality, we have

$$\mathbb{E}\Big[\int_{n\wedge\tau}^{\tau} e^{\alpha s}\big|\lambda Y_s^n - f(s,Y_s^n,Z_s^n)\big|ds\Big] \leq \frac{1}{\sqrt{\rho - 2\alpha}}\Big\{\mathbb{E}\Big[\int_{n\wedge\tau}^{\tau} e^{\rho s}\big|\lambda Y_s^n - f(s,Y_s^n,Z_s^n)\big|^2 ds\Big]\Big\}^{1/2},$$

and we have already proved that the right hand side tends to 0 (see the definition of Γ_n). It remains to study the term $\int_{t\wedge\tau}^{\tau} f(s,Y_s^n,Z_s^n)ds$. But, since f is Lipschitz in z, we have

$$\mathbb{E}\Big[\int_{t\wedge\tau}^{\tau} e^{\alpha s}\big|f(s,Y_s^n,Z_s^n) - f(s,Y_s^n,Z_s)\big|ds\Big] \leq \frac{\gamma}{\sqrt{\rho - 2\alpha}}\Big\{\mathbb{E}\Big[\int_{n\wedge\tau}^{\tau} e^{\rho s}\|Z_s^n - Z_s\|^2 ds\Big]\Big\}^{1/2},$$

and thus goes to 0 with n. So now, it suffices to show that

$$\mathbb{E}\Big[\int_0^{\tau} e^{\alpha s}\big|f(s,Y_s^n,Z_s) - f(s,Y_s,Z_s)\big|ds\Big] \longrightarrow 0,$$

to control the limit in the equation. We prove this by showing that each subsequence has a subsequence for which the above convergence hold. Indeed, if we pick a subsequence (still denoted by Y^n), since we have $\mathbb{E}\big[\sup_{t\geq 0} e^{\rho(t\wedge\tau)}|Y_t - Y_t^n|^2\big] \longrightarrow 0$ there exist a subsequence still denoted in the same way such that $\mathbb{P}$–a.s. $\big(\forall t, \quad Y_t^n \longrightarrow Y_t\big)$. By the continuity of the function f, $\mathbb{P}$–a.s. $\big(\forall t, \quad f(t,Y_t^n,Z_t) \longrightarrow f(t,Y_t,Z_t)\big)$. If we prove that

$$\sup_{\mathbb{N}} \mathbb{E}\Big[\int_0^{\tau} e^{\alpha s}\big|f(s,Y_s^n,Z_s) - f(s,Y_s,Z_s)\big|^2 ds\Big] < \infty,$$

then the sequence $\left|f(\cdot,Y_\cdot^n,Z_\cdot)-f(\cdot,Y_\cdot,Z_\cdot)\right|$ will be a uniformly integrable sequence for the finite measure $e^{\alpha s}\mathbf{1}_{s\le\tau}ds\otimes d\mathbb{P}$ (remember that $\alpha<0$)and thus converging in $\mathrm{L}^1(e^{\alpha s}\mathbf{1}_{s\le\tau}ds\otimes d\mathbb{P})$ which is the desired result. But from the growth assumption on f, we have

$$\begin{aligned}\mathbb{E}\Big[\int_0^\tau e^{\alpha s}\big|f(s,Y_s^n,Z_s)-f(s,Y_s,Z_s)\big|^2ds\Big] &\le K\mathbb{E}\Big[\int_0^\tau e^{\alpha s}\big\{|f(s,0,0)|^2+\|Z_s^n\|^2+\|Z_s\|^2\big\}ds\Big]\\ &\quad +K\mathbb{E}\Big[\int_0^\tau e^{\alpha s}\big\{\kappa+|Y_s^n|^{2p}+|Y_s|^{2p}\big\}ds\Big].\end{aligned}$$

Since $\rho>\alpha$, the inequalities (16)–(21), implies that

$$\sup_{\mathbb{N}}\mathbb{E}\Big[\int_0^\tau e^{\alpha s}\big\{|f(s,0,0)|^2+\kappa+\|Z_s^n\|^2+\|Z_s\|^2\big\}ds\Big]$$

is finite. Moreover,

$$\mathbb{E}\Big[\int_0^\tau e^{\alpha s}|Y_s^n|^{2p}ds\le\mathbb{E}\Big[\sup_{0\le t\le\tau}e^{p\rho t}|Y_t^n|^{2p}\Big]\int_0^\infty e^{(\alpha-p\rho)s}ds.$$

Since $p\rho>\alpha$, we conclude the proof of the convergence of the last term by using the first part of the inequalities (16)–(21). Passing to the limit when n goes to infinity, we get, for each t,

$$e^{\alpha(t\wedge\tau)}Y_t=e^{\alpha\tau}\xi+\int_{t\wedge\tau}^\tau e^{\alpha s}\big\{f(s,Y_s,Z_s)-\alpha Y_s\big\}ds-\int_{t\wedge\tau}^\tau e^{\alpha s}Z_sdW_s.$$

It then follows by Itô's formula that (Y,Z) solves the BSDE (13). $\square$

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