PRICING AND HEDGING SPREAD OPTIONS IN A LOG-NORMAL MODEL

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ABSTRACT. This paper deals with the pricing of spread options on the difference between correlated log-normal underlying assets. We introduce a new pricing paradigm based on a set of precise lower bounds. We also derive closed form formulae for the Greeks and other sensitivities of the prices. In doing so we prove that the price of a spread option is a decreasing function of the correlation parameter, and we analyze the notion of implied correlation. We use numerical experiments to provide an extensive analysis of the performance of these new pricing and hedging algorithms, and we compare the results with those of the existing methods.

1. INTRODUCTION

As Eric Reiner put it at a recent seminar, a multivariate version of the Black-Scholes' formula has yet to emerge. Nevertheless, there is a tremendous number of papers dealing with the issue. Whether they look at basket options or discrete-time average Asian options, all try to price and hedge multivariate contingent claims. In order to tackle this problem we start with the simplest case of bivariate contingent claims, and more precisely of spread options. Spread options already contain the essence of the difficulty in pricing multivariate contingent claims: the linear combination of log-normal underlying assets. To overcome this obstacle, we introduce a new pricing paradigm. Although it is original, our work was partly inspired by the article by Rogers and Shi [9] on Asian options. Our method has several advantages. It gives an extremely good approximation to the price, and computing the so-called Greeks is straightforward. More importantly it encapsulates the univariate case, that is to say, it is an extension of the classical results of Black-Scholes and Margrabe. This is certainly a desirable feature from a mathematical point of view, but also from a risk management point of view. Indeed, pricing and hedging a single very exotic product is often not the main issue for financial engineers, but aggregating different simple products as vanilla options together with basket options for example is a daily challenge. Pricing and hedging them in a consistent way is the main issue which our approach helps resolve.

In the next section, we introduce two examples of spread options: one in the equity markets and the other in the fixed income markets. These examples are intended to motivate the forthcoming computations. The main thrust of the paper is contained in Section 3 where we derive closed form formulae for our approximation and its error. Section 4 gives a short review of two existing methods that are closely related to ours. We derive pricing formulae in these models for further comparison with our methodology. Section 5 applies the results of Section 3 to the case of a spread between two correlated stocks whose price dynamics are given by geometric Brownian motions, and we derive closed form formulae for the so-called Greeks. Finally, to illustrate possible extensions of our method, we show how to include jumps in the dynamics of the underlying assets, and we discuss the notion of

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implied correlation. The accuracy of our approximate price formula, together with the performance of the subsequent hedging strategies are illustrated by the results of extensive numerical experiments based on Monte Carlo simulations from the geometric Brownian motion models.

2. TWO EXAMPLES OF SPREAD OPTIONS

In this section, we introduce the notation and the terminology which we use throughout the paper. In so doing we describe two examples which should be used as a motivation for the need for the kind of formulae we derive in the sequel.

2.1. **Spread Options in the Equity or Energy Markets.** We look at the classical setting where besides a riskless bank account with constant interest rate r, our arbitrage-free market model comprises two assets whose prices at time t are denoted by $S_1(t)$ and $S_2(t)$. We assume that their risk-neutral price dynamics are given by the following stochastic differential equations:

$$\begin{aligned} dS_1(t) &= S_1(t)[(r-q_1)dt + \sigma_1 dW_1(t)] \\ dS_2(t) &= S_2(t)[(r-q_2)dt + \sigma_2 dW_2(t)] \end{aligned}$$

where q_1 and q_2 are the instantaneous dividend yields, the volatilities σ_1 and σ_2 are positive constants and W_1 and W_2 are two Brownian motions with correlation ρ. The initial conditions will be denoted by $S_1(0) = x_1$ and $S_2(0) = x_2$. We mainly focus on the pricing of spread options on two stocks in which case the q_i's should be interpreted as dividend rates. In the case of a spread option on two commodity spot prices, the q_i's should be interpreted as instantaneous convenience yields, while we should set $q_1 = q_2 = r$ in the case of spread options on futures contracts.

In any case, the price p at time 0 of the spread option with date of maturity T and strike K is given by the risk-neutral expectation:

$$p = e^{-rT}\mathbb{E}\{(S_2(T) - S_1(T) - K)^+\}. \tag{1}$$

The special case $K = 0$ corresponds to an *exchange* since the pay-off $(S_2(T) - S_1(T))^+$ will provide the holder of the option with the difference $S_2(T) - S_1(T)$ when $S_2(T) > S_1(T)$. So if one would like to own either one of the indexes at time T, one could simply buy the second one and purchase the above call spread option with $K = 0$. This will guarantee that, short of the premium of the option, we do as well as if we had bought the one ending up being the cheaper of the two in all cases!

The price p of formula (1) can be rewritten in the form:

$$p = e^{-rT}\mathbb{E}\left\{\left(x_2 e^{(r-q_2-\sigma_2^2/2)T+\sigma_2 W_2(T)} - x_1 e^{(r-q_1-\sigma_1^2/2)T+\sigma_1 W_1(T)} - K\right)^+\right\} \tag{2}$$

which shows that the price p is given by the integral of a function of two variables with respect to a bivariate Gaussian distribution, namely the joint distribution of $W_1(T)$ and $W_2(T)$. Unfortunately, except in the case of an exchange option (*i.e.*, an option with strike $K = 0$), the price of the spread option cannot be given by a formula in closed form.

2.2. **Spread Options in the Fixed-Income Markets.** The results obtained in this paper can be applied in the context of fixed-income derivative pricing. Consider a call option on the difference between the 3-month and the 6-month Libor rates. More precisely if we let $\delta = 3$ months, we denote by $L(T,\delta)$ the 3-month Libor rate spanning the interval $[T, T+\delta]$, and similarly by $L(T,2\delta)$ the 6-month Libor rate spanning $[T, T+2\delta]$. We consider the option that pays the amount

$$\delta(L(T,\delta) - 2L(T,2\delta) - k)^+.$$

at time $T+\delta$. The price at time t is given by the conditional expectation with respect to the forward pricing measure $\mathbb{Q}^{T+\delta}$

$$p_t = B(t, T+\delta)\mathbb{E}^{\mathbb{Q}^{T+\delta}}_{\mathcal{F}_t}\left\{(\delta L(T,\delta) - 2\delta L(T,2\delta) - \delta k)^+\right\}$$

where $B(t,T)$ denotes the T-zero coupon bond price at time t, and where we use a superscript on the expectation to emphasize the probability measure with respect to which the expectation is computed, and a subscript to indicate that the expectation is in fact a conditional expectation with respect to the σ-field used as a subscript. The price p_t rewrites:

$$\begin{aligned}
p_t &= B(t,T+\delta)\mathbb{E}^{\mathbb{Q}^{T+\delta}}_{\mathcal{F}_t}\left\{(1+\delta L(T,\delta) - (1+2\delta L(T,2\delta)) - \delta k)^+\right\} \\
&= B(t,T+\delta)\mathbb{E}^{\mathbb{Q}^{T+\delta}}_{\mathcal{F}_t}\left\{\left(\frac{1}{B(T,T+\delta)} - \frac{1}{B(T,T+2\delta)} - \delta k\right)^+\right\} \\
&= B(t,T+\delta)\mathbb{E}^{\mathbb{Q}^{T+\delta}}_{\mathcal{F}_t}\left\{\frac{1}{B(T,T+\delta)}\left(1 - \frac{B(T,T+\delta)}{B(T,T+2\delta)} - \delta k B(T,T+\delta)\right)^+\right\} \\
&= B(t,T)\mathbb{E}^{\mathbb{Q}^{T}}_{\mathcal{F}_t}\left\{\left(1 - \frac{B(T,T+\delta)}{B(T,T+2\delta)} - \delta k B(T,T+\delta)\right)^+\right\} \\
&= B(t,T)\mathbb{E}^{\mathbb{Q}^{T}}_{\mathcal{F}_t}\left\{\left(1 - \frac{B_T(T,T+\delta)}{B_T(T,T+2\delta)} - \delta k B_T(T,T+\delta)\right)^+\right\}
\end{aligned}$$

where $B_t(T, T+\delta) = B(t, T+\delta)/B(t,T)$ is the price at time t of the forward zero coupon bond. Let us assume from now on that we are in a Gaussian Heath-Jarrow-Morton framework. This is merely saying that the volatility of the zero coupon $B(t,T)$, say $v(t,T)$, is deterministic i.e. non-random. It is well-known (see, *e.g.*, [6]) that $B_t(T,T+\delta)$ and $B_t(T,T+2\delta)$ are log-normal martingales under $\mathbb{Q}^T$. More precisely, we have:

$$\frac{dB_t(T,T+\delta)}{B_t(T,T+\delta)} = \sigma_t(T,T+\delta)dW^T_t$$

for some $\mathbb{Q}^T$-Brownian motion W^T provided we set $\sigma_t(T,T') = v(t,T) - v(t,T')$. Obviously, we have a similar formula for the dynamics of $B_t(T,T+2\delta)$. We can therefore infer the distributions of the forward bonds under $\mathbb{Q}^T$ since:

$$\begin{aligned} B_T(T,T+\delta) &= \frac{B(t,T+\delta)}{B(t,T)} \exp\left(-\frac{1}{2}\int_t^T \sigma_s^2(T,T+\delta)ds + \int_t^T \sigma_s(T,T+\delta)dW_s^T\right) \\ \frac{B_T(T,T+\delta)}{B_T(T,T+2\delta)} &= \frac{B(t,T+\delta)}{B(t,T+2\delta)} \exp\Bigg(-\frac{1}{2}\int_t^T \sigma_s^2(T,T+\delta) - \sigma_s^2(T,T+2\delta)ds \\ &\qquad + \int_t^T \sigma_s(T,T+\delta) - \sigma_s(T,T+2\delta)dW_s^T\Bigg). \end{aligned}$$

So if we restrict ourselves to $k<0$, the computation of the price reduces to the problem of the pricing of a spread option in the equity markets. This is indeed the case if we use the formulae

$$\begin{aligned} \sigma_1^2 &= \frac{1}{T-t}\int_t^T (\sigma_s(T,T+\delta) - \sigma_s(T,T+2\delta))^2\, ds \\ \sigma_2^2 &= \frac{1}{T-t}\int_t^T \sigma_s^2(T,T+\delta)ds \\ \rho &= \frac{1}{T-t}\int_t^T \sigma_s(T,T+\delta)\cdot(\sigma_s(T,T+\delta) - \sigma_s(T,T+2\delta))\, ds. \end{aligned}$$

for the volatilities and the correlation. The case $k>0$ can equivalently be treated by means of symmetry arguments as we derive below.

3. Main Result

The above pricing problems boil down to computing the following expectation:

$$\Pi = \Pi(\alpha,\beta,\gamma,\delta,\kappa,\rho) = \mathbb{E}\left\{\left(\alpha e^{\beta X_1 - \beta^2/2} - \gamma e^{\delta X_2 - \delta^2/2} - \kappa\right)^+\right\} \tag{3}$$

where α, β, γ, δ and κ are real constants and X_1 and X_2 are jointly Gaussian $N(0,1)$ random variables with correlation ρ. The purpose of this section is to compute this expectation. It can be expressed as a double integral, but in general, its value cannot be given in closed form. Our goal is to provide a very good approximation for its value, in such a way to retain all the advantages of closed form formulae such as the Black Scholes formula.

Without any loss of generality we assume that

$$\rho \neq 1 \qquad \text{or} \qquad \beta \neq \delta$$

which is equivalent to

$$\beta^2 - 2\rho\beta\delta + \delta^2 \neq 0.$$

Indeed, Π is explicitly computable with the Black-Scholes' formula whenever the above assumption does not hold.

3.1. **Symmetries of the Problem.** We first study the symmetries properties of (3). The simple formula:

$$\mathbb{E}\{X^+\} = \mathbb{E}\{(-X)^+\} + \mathbb{E}\{X\} \tag{4}$$

becomes very useful when $\mathbb{E}\{X\}$ can be computed explicitly. In the present situation, we have:

$$\mathbb{E}\left\{\left(\alpha e^{\beta X_1-\beta^2/2} - \gamma e^{\delta X_2-\delta^2/2} - \kappa\right)^+\right\} = \mathbb{E}\left\{\left(\gamma e^{\delta X_2-\delta^2/2} - \alpha e^{\beta X_1-\beta^2/2} + \kappa\right)^+\right\} + \alpha - \gamma - \kappa$$

or equivalently,

$$\Pi(\alpha,\beta,\gamma,\delta,\kappa,\rho) = \Pi(\gamma,\delta,\alpha,\beta,-\kappa,\rho) + \alpha - \gamma - \kappa$$

which can make some proofs simpler by reducing the parameter domain for which properties of the expectation need to be derived.

Formulae derived from (4) will be called parity formulae, and arguments based on parity formulae will be called parity arguments. This terminology is motivated by the standard call/put parity relating the price of a European call option to the price of the European put option with the same strike and expiry. Next, we introduce two independent $N(0,1)$ random variables Z_1 and Z_2 such that:

$$\begin{aligned} X_1 &= \sqrt{1-\rho^2}Z_1 + \rho Z_2 \\ X_2 &= Z_2 \end{aligned}$$

and we denote by ϕ the unique number $\phi \in [0,\pi]$ such that $\rho = \cos\phi$. With these notations, the expectation giving Π can be rewritten as:

$$\Pi = \mathbb{E}\left\{\left(\alpha e^{\beta[\sin\phi Z_1 + \cos\phi Z_2] - \beta^2/2} - \gamma e^{\delta Z_2 - \delta^2/2} - \kappa\right)^+\right\}. \tag{5}$$

Function (3) also enjoys symmetry properties as given by the following proposition.

Proposition 1. *The following symmetry relations hold*

$$\Pi(\alpha,\beta,\gamma,\delta,\kappa,\rho) = \Pi\left(\alpha,\sigma,\kappa,\delta,\gamma,\frac{\delta-\rho\beta}{\sigma}\right) = \Pi\left(-\kappa,\beta,\gamma,\sigma,-\alpha,\frac{\beta-\rho\delta}{\sigma}\right)$$

where $\sigma = \sqrt{\beta^2 - 2\rho\beta\delta + \delta^2}$.

Proof. The assumption $\beta \neq \delta$ or $\rho \neq 1$ allows us to divide by σ and therefore to define the new correlations. Let us let $\rho' = \sqrt{1-\rho^2}$ and let us write Π as follows.

$$\begin{aligned} \Pi(\alpha,\beta,\gamma,\delta,\kappa,\rho) &= \mathbb{E}\left\{\left(\alpha e^{\beta\rho' Z_1 + \beta\rho Z_2 - \beta^2/2} - \gamma e^{\delta Z_2 - \delta^2/2} - \kappa\right)^+\right\} \\ &= \mathbb{E}\left\{e^{\delta Z_2 - \delta^2/2}\left(\alpha e^{\beta\rho' Z_1 + (\beta\rho-\delta)Z_2 - \beta^2/2 + \delta^2/2} - \gamma - \kappa e^{-\delta Z_2 + \delta^2/2}\right)^+\right\} \\ &= \mathbb{E}\left\{\left(\alpha e^{\beta\rho' Z_1 + (\delta-\beta\rho)Z_2 - (\beta^2 - 2\rho\beta\delta + \delta^2)/2} - \kappa e^{\delta Z_2 - \delta^2/2} - \gamma\right)^+\right\} \\ &= \Pi\left(\alpha, \sqrt{\beta^2 - 2\rho\beta\delta + \delta^2}, \kappa, \delta, \gamma, \frac{\delta - \rho\beta}{\sqrt{\beta^2 - 2\rho\beta\delta + \delta^2}}\right) \end{aligned}$$

From the second to the third line, we used the Girsanov's transform. The second symmetry relation is obtained exactly in the same way. ∎

It is interesting to remark that the above symmetry relations give directly Margrabe's formula [4]. Indeed:

$$\begin{aligned}\Pi(\alpha,\beta,\gamma,\delta,0,\rho) &= \Pi\left(\alpha,\sigma,0,\delta,\gamma,\frac{\delta-\rho\beta}{\sigma}\right)\\ &= \mathbb{E}\left\{\left(\alpha e^{\sigma U-\sigma^2/2}-\gamma\right)^+\right\}\end{aligned}$$

where $U\sim N(0,1)$, and this last expectation is given by the Black-Scholes' formula.

Symmetry arguments also allow us to price options on the sum of two assets:

$$\mathbb{E}\left\{\left(\alpha e^{\beta X_1-\beta^2/2}+\gamma e^{\delta X_2-\delta^2/2}-\kappa\right)^+\right\}.$$

Indeed, if $\kappa\le 0$, the computation is straightforward. On the contrary if $\kappa<0$, a symmetry argument transforms the computation into that of a spread option.

We conclude this section by explaining how parity and symmetries interplay. Let us denote by n, s and t the following involution functions

$$\begin{aligned}n(\alpha,\beta,\gamma,\delta,\kappa,\rho) &= (\gamma,\delta,\alpha,\beta,-\kappa,\rho)\\ s(\alpha,\beta,\gamma,\delta,\kappa,\rho) &= \left(\alpha,\sqrt{\beta^2-2\rho\beta\delta+\delta^2},\kappa,\delta,\gamma,\frac{\delta-\rho\beta}{\sqrt{\beta^2-2\rho\beta\delta+\delta^2}}\right)\\ t(\alpha,\beta,\gamma,\delta,\kappa,\rho) &= \left(-\kappa,\beta,\gamma,\sqrt{\beta^2-2\rho\beta\delta+\delta^2},-\alpha,\frac{\beta-\rho\delta}{\sqrt{\beta^2-2\rho\beta\delta+\delta^2}}\right)\end{aligned}$$

Then one easily checks that:

$$n\circ s=t\circ n. \tag{6}$$

3.2. **The Approximation.** We now derive our approximate pricing formula from a set of lower bounds for the price.

A Preliminary Remark. Recall that the Black-Scholes' formula for valuing call options is given by:

$$C^{BS}=Se^{-qT}\Phi(d_0+\sigma\sqrt{T})-Ke^{-rT}\Phi(d_0)$$

where

$$d_0=\frac{\ln(Se^{-qT}/Ke^{-rT})}{\sigma\sqrt{T}}-\frac{\sigma\sqrt{T}}{2}.$$

Here and throughout the paper, we use the notation $\varphi(x)$ and $\Phi(x)$ for the density and the cumulative distribution function of the standard Gaussian $N(0,1)$ distribution, *i.e.*,

$$\varphi(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\qquad\text{and}\qquad\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-u^2/2}du.$$

C^{BS} can be seen as a function of d_0, the other parameters remaining fixed:

$$C^{BS} = C^{BS}(d_0)$$

One can then check (this will also be a simple consequence of Proposition 2 below) that the value d_0 is precisely that value of d at which the function $C^{BS}(d)$ is attains its maximum, so that:

$$C^{BS} = C^{BS}(d_0) = \sup_{d\in\mathbb{R}} C^{BS}(d).$$

With this remark in mind, the delta (*i.e.*, the derivative of C^{BS} with respect to S) is very easily computable, and:

$$\Delta = \frac{\partial C^{BS}}{\partial S} + \frac{\partial C^{BS}}{\partial d}\frac{\partial d_0}{\partial S} = e^{-qT}\Phi(d_0 + \sigma\sqrt{T})$$

As the reader can check, the brute force differentiation of the Black-Scholes' formula requires more than a single line of computation. Our approximation for the value (3) is based on this idea. We would like to have an approximation that looks like a Black-Scholes' formula. We introduce some degrees of freedom (like the parameter d in the one-dimensional case) and the values of the parameters we choose will be those that maximize the value of the approximation. Unlike the one-dimensional case, we cannot recover the exact value of (3) (except in some very special cases) but the approximation will turn out to be excellent. But like the one-dimensional case, we can easily compute the sensitivities of (3) with respect to its various parameters.

The next result is elementary. We state it as a proposition for future reference.

Proposition 2. *Let X be a real valued random variable on a probability space $(\Omega, \mathcal{H}, P)$, and let $\mathcal{A} \subset \mathcal{H}$ be any family of events such that $\{X \geq 0\} \in \mathcal{A}$. Then,*

$$\mathbb{E}\{X^+\} = \sup_{A\in\mathcal{A}} \mathbb{E}\{X\mathbf{1}_A\}$$

Proof. Let $A \in \mathcal{A}$,

$$\begin{aligned} \mathbb{E}\{X^+\} = \int_A X\, dP &= \int_{A\cap\{X\geq 0\}} X\, dP + \int_{A\cap\{X<0\}} X\, dP \\ &\leq \int_{A\cap\{X\geq 0\}} X\, dP \\ &\leq \int_{\{X\geq 0\}} X\, dP \\ &= \mathbb{E}\{X^+\} \end{aligned}$$

which proves that $\sup_{A\in\mathcal{A}} \mathbb{E}\{X\mathbf{1}_A\} \leq \mathbb{E}\{X^+\}$. To get the reverse inequality, note that $\{X \geq 0\} \in \mathcal{A}$ so that

$$\sup_{A\in\mathcal{A}} \mathbb{E}\{X\mathbf{1}_A\} \geq \mathbb{E}\{X\mathbf{1}_{\{X\geq 0\}}\} = \mathbb{E}\{X^+\}. \quad \blacksquare$$

Two Different Elementary Lower Bounds. Recall that we are interested in computing $\mathbb{E}\{X^+\}$ where X is the following random variable:

$$X = \alpha e^{\beta[\sin\phi Z_1 + \cos\phi Z_2] - \beta^2/2} - \gamma e^{\delta Z_2 - \delta^2/2} - \kappa.$$

The main obstruction to a closed formula for this expectation is that the exercise region (*i.e.* the set $\{X \geq 0\}$) does not have a simple shape. Proposition 2 tells us that any attempt to approximate the value of this integral by approximating the exercise region by simpler regions will lead to a lower bound. The two elementary lower bounds that we are about to introduce are based on simple approximations of the exercise region.

For each $\theta \in \mathbb{R}$ we introduce the random variable:

$$Y_\theta = \sin\theta Z_1 - \cos\theta Z_2.$$

By choosing for $\mathcal{A}$ the σ-field $\sigma\{Y_\theta\}$ generated by Y_θ, we get our first elementary lower bound:

$$\bar{\Pi}(\theta) = \sup_{A \in \sigma\{Y_\theta\}} \mathbb{E}\{X \mathbf{1}_A\}.$$

By choosing $\mathcal{A} = \big\{\{Y_\theta \leq d\}; d \in \mathbb{R}\big\}$, we get the second one:

$$\hat{\Pi}(\theta) = \sup_{d \in \mathbb{R}} \mathbb{E}\{X \mathbf{1}_{\{Y_\theta \leq d\}}\}.$$

The approximation $\hat{\Pi}$ which we propose is defined as the supremum of the lower bounds $\hat{\Pi}(\theta)$ when we vary the free parameter θ. We denote by θ^* a maximizer of $\hat{\Pi}(\theta)$.

$$\hat{\Pi} = \hat{\Pi}(\theta^*) = \sup_{\theta \in \mathbb{R}} \hat{\Pi}(\theta) = \sup_{\theta \in \mathbb{R}} \sup_{d \in \mathbb{R}} \mathbb{E}\{X \mathbf{1}_{\{Y_\theta \leq d\}}\}. \tag{7}$$

We will also need the approximation $\bar{\Pi}$ corresponding to $\bar{\Pi}(\theta)$. Although their values are presumably different, we will also denote by θ^* a maximizer for that lower bound. The context in which we use the notation θ^* will always make it clear which one we have in mind, and no confusion will arise.

Proposition 3. *We have:*

$$\bar{\Pi} := \sup_{\theta \in \mathbb{R}} \bar{\Pi}(\theta) = \sup_{A \in \bigcup_\theta \sigma\{Y_\theta\}} \int_A X\, dP = \sup_{\theta \in \mathbb{R}} \mathbb{E}\{\mathbb{E}\{X|Y_\theta\}^+\}$$

Proof. Since $\{\mathbb{E}\{X|Y_\theta\} \geq 0\} \in \sigma\{Y_\theta\}$, Proposition 2 yields

$$\sup_\theta \mathbb{E}\{\mathbb{E}\{X|Y_\theta\}^+\} = \sup_\theta \sup_{A \in \sigma\{Y_\theta\}} \int_A \mathbb{E}\{X|Y_\theta\}\, dP$$

By the definition of conditional expectations, we have

$$\begin{aligned} \sup_\theta \sup_{A \in \sigma\{Y_\theta\}} \int_A \mathbb{E}\{X|Y_\theta\}\, dP &= \sup_\theta \sup_{A \in \sigma\{Y_\theta\}} \int_A X\, dP \\ &= \sup_{A \in \bigcup_\theta \sigma\{Y_\theta\}} \int_A X\, dP \\ &= \bar{\Pi} \quad \blacksquare \end{aligned}$$

The above proposition shows that our approximations are very related to Jensen's inequalities. The function $x \mapsto x^+$ being convex, Jensen's inequality implies that:

$$\mathbb{E}\{X^+\} = \mathbb{E}\{\mathbb{E}\{X^+|Y_\theta\}\} \geq \mathbb{E}\{\mathbb{E}\{X|Y_\theta\}^+\} = \bar{\Pi}(\theta).$$

In fact, we even have:

$$\mathbb{E}\{X\}^+ \leq \hat{\Pi} \leq \bar{\Pi} \leq \mathbb{E}\{X^+\}$$

The first inequality is just a consequence of the fact that

$$\hat{\Pi} \geq \lim_{d\to-\infty} \mathbb{E}\{X\mathbf{1}_{\{Y_{\theta^*}\leq d\}}\} = 0$$

and

$$\hat{\Pi} \geq \lim_{d\to+\infty} \mathbb{E}\{X\mathbf{1}_{\{Y_{\theta^*}\leq d\}}\} = \mathbb{E}\{X\}.$$

The second one simply comes from the the trivial inclusion

$$\big\{\{Y_\theta \leq d\}; d \in \mathbb{R}\big\} \subset \sigma\{Y_\theta\}$$

and the last one has just been established.

Notice that the lower bounds are much simpler to handle than (3). After all, $\bar{\Pi}(\theta)$ is nothing but a one-dimensional Gaussian integral.

$$\bar{\Pi}(\theta) = \mathbb{E}\{\tilde{X}(\theta)^+\} \tag{8}$$

with:

$$\tilde{X}(\theta) = \mathbb{E}\{X|Y_\theta\} = \alpha e^{-\beta\cos(\theta+\phi)Y_\theta - \beta^2\cos^2(\theta+\phi)\ /2} - \gamma e^{-\delta\cos\theta Y_\theta - \delta^2\cos^2\theta\ /2} - \kappa. \tag{9}$$

The quantity $\hat{\Pi}(\theta)$ can be computed a bit more explicitly.

Proposition 4.

$$\hat{\Pi}(\theta) = \sup_{d\in\mathbb{R}} \left[\alpha\Phi\left(d + \beta\cos(\theta+\phi)\right) - \gamma\Phi\left(d + \delta\cos(\theta)\right) - \kappa\Phi(d)\right] \tag{10}$$

We will also denote by d^* a maximizer of the above supremum.

Proof. In view of (9), $\hat{\Pi}(\theta)$ can be rewritten as:

$$\begin{aligned}
\hat{\Pi}(\theta) &= \sup_{d\in\mathbb{R}} \int_{\{Y_\theta\leq d\}} X\ dP \\
&= \sup_{d\in\mathbb{R}} \int_{\{Y_\theta\leq d\}} \mathbb{E}\{X|Y_\theta\}\ dP \\
&= \sup_{d\in\mathbb{R}} \mathbb{E}\left\{\left(\alpha e^{-\beta\cos(\theta+\phi)Y_\theta - (\beta^2/2)\cos^2(\theta+\phi)} - \gamma e^{-\delta\cos\theta Y_\theta - (\delta^2/2)\cos^2\theta} - \kappa\right)\mathbf{1}_{\{Y_\theta\leq d\}}\right\}
\end{aligned}$$

and $\hat{\Pi}(\theta)$ consists in three terms which can be computed separately. The first one gives:

$$
\begin{aligned}
\mathbb{E}&\left\{\alpha e^{-\beta\cos(\theta+\phi)Y_\theta-(\beta^2/2)\cos^2(\theta+\phi)}\mathbf{1}_{\{Y_\theta\le d\}}\right\}\\
&=\alpha\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d}e^{-\beta\cos(\theta+\phi)x-(\beta^2/2)\cos^2(\theta+\phi)}e^{-x^2/2}dx\\
&=\alpha\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d+\beta\cos(\theta+\phi)}e^{-u^2/2}du\\
&=\alpha\Phi(d+\beta\cos(\theta+\phi))
\end{aligned}
$$

The other two terms can be computed similarly. This leads to the desired expression for $\hat{\Pi}(\theta)$ ∎

Explicit Computation of $\hat{\Pi}$. We are now in a position to explain how to compute $\hat{\Pi}$. In order to maximize the function $\hat{\Pi}(\theta)$, we need its derivative. A formula for the latter is given in the statement of the next result.

Proposition 5.

$$\hat{\Pi}'(\theta)=\alpha\beta\sin(\theta+\phi)\ \varphi(d^*+\beta\cos(\theta+\phi))-\gamma\delta\sin\theta\ \varphi(d^*+\delta\cos\theta)$$

Proof. Note that we have already established that

$$\hat{\Pi}(\theta)=\mathbb{E}\left\{\tilde{X}(\theta)\mathbf{1}_{\{Y_\theta\le d^*\}}\right\}.$$

We compute the derivative of $\hat{\Pi}(\theta)$ by differentiating both sides of this equation. The derivative of the function $x\mapsto\mathbf{1}_{\{x\le 0\}}$ is a Dirac function which is equal to $-\infty$ when $x=0$ and 0 otherwise. Consequently, taking derivative under the integral/expectation sign gives:

(11)

$$\frac{d}{d\theta}\mathbb{E}\left\{\tilde{X}(\theta)\mathbf{1}_{\{Y_\theta\le d^*\}}\right\}=\mathbb{E}\left\{\frac{d}{d\theta}\tilde{X}(\theta)\mathbf{1}_{\{Y_\theta\le d^*\}}\right\}=\mathbb{E}\left\{\frac{d\tilde{X}(\theta)}{d\theta}\mathbf{1}_{\{Y_\theta\le d^*\}}+\tilde{X}(\theta)\frac{d\mathbf{1}_{\{Y_\theta\le d^*\}}}{d\theta}\right\}.$$

The second term in the most right expression is zero since the Lebesgue measure has no atom. Formula (11) can easily be justified by convolving the function $x\mapsto\mathbf{1}_{\{x\le 0\}}$ with an approximate identity, taking the derivative under the expectation sign, and removing the effect of the regularization by taking a limit controlled by the Lebesgue's dominated convergence theorem. This gives:

$$
\begin{aligned}
\hat{\Pi}'(\theta)\;&=\;\mathbb{E}\Big\{\Big(-\alpha\beta\sin(\theta+\phi)[\beta\cos(\theta+\phi)+Y_\theta]e^{-\beta\cos(\theta+\phi)Y_\theta-(\beta^2/2)\cos^2(\theta+\phi)}\\
&\qquad+\ \gamma\delta\sin\theta[\delta\cos\theta+Y_\theta]e^{-\delta\cos\theta\,Y_\theta-(\delta^2/2)\cos^2\theta}\Big)\mathbf{1}_{\{Y_\theta\le d^*\}}\Big\}\\
&=\;-\alpha\beta\sin(\theta+\phi)\int_{-\infty}^{d^*+\beta\cos(\theta+\phi)}u\frac{e^{-u^2/2}}{\sqrt{2\pi}}\\
&\qquad+\gamma\delta\sin\theta\int_{-\infty}^{d^*+\delta\cos\theta}u\frac{e^{-u^2/2}}{\sqrt{2\pi}}\\
&=\;\alpha\beta\sin(\theta+\phi)\ \varphi(d^*+\beta\cos(\theta+\phi))-\gamma\delta\sin\theta\ \varphi(d^*+\delta\cos\theta)
\end{aligned}
$$

which is the desired result. ∎

So computing explicitly $\hat{\Pi}$ reduces to computing d^* and θ^*. To that end we need a system of two equations. The equation $\hat{\Pi}'(\theta^*) = 0$ gives a first one. In view of Proposition 2, it is clear that the d^* that achieves the maximum in (10) has to be a zero of the function:

$$\alpha e^{-\beta\cos(\theta+\phi)\,x-\beta^2\cos^2(\theta+\phi)\,/2} - \gamma e^{-\delta\cos\theta\,x-\delta^2\cos^2\theta\,/2} - \kappa.$$

This leads to a second equation. Therefore d^* and θ^* are necessarily solutions to the following system:

(12)

$$\left\{\begin{array}{rcl} 0 &=& \alpha e^{-\beta\cos(\theta^*+\phi)\,d^*-\beta^2\cos^2(\theta^*+\phi)\,/2} - \gamma e^{-\delta\cos\theta^*\,d^*-\delta^2\cos^2\theta^*\,/2} - \kappa \\ 0 &=& \alpha\beta\sin(\theta^*+\phi)\,e^{-\beta\cos(\theta^*+\phi)\,d^*-\beta^2\cos^2(\theta^*+\phi)\,/2} - \gamma\delta\sin\theta^*\,e^{-\delta\cos\theta^*\,d^*-\delta^2\cos^2\theta^*\,/2}. \end{array}\right.$$

From this system we get

$$\begin{array}{rcl} e^{-\beta\cos(\theta^*+\phi)\,d^*-\beta^2\cos^2(\theta^*+\phi)\,/2} &=& -\dfrac{\delta\kappa\sin\theta^*}{\alpha[\beta\sin(\theta^*+\phi)-\delta\sin\theta^*]} \\ e^{-\delta\cos\theta^*\,d^*-\delta^2\cos^2\theta^*\,/2} &=& -\dfrac{\beta\kappa\sin(\theta^*+\phi)}{\gamma[\beta\sin(\theta^*+\phi)-\delta\sin\theta^*]}. \end{array}$$

In solving for d^* in each of the above equations, we see that θ^* is necessarily a solution to:

$$\begin{array}{rl} & \dfrac{1}{\delta\cos\theta}\ln\left(-\dfrac{\beta\kappa\sin(\theta+\phi)}{\gamma[\beta\sin(\theta+\phi)-\delta\sin\theta]}\right) - \dfrac{\delta\cos\theta}{2} \\ = & \dfrac{1}{\beta\cos(\theta+\phi)}\ln\left(-\dfrac{\delta\kappa\sin\theta}{\alpha[\beta\sin(\theta+\phi)-\delta\sin\theta]}\right) - \dfrac{\beta\cos(\theta+\phi)}{2} \end{array} \tag{13}$$

and d^* is equal to either the left or the right hand side of equation (13).

We could be more precise and bracket the solution θ^* that corresponds to the maximum. Unfortunately, we would need to distinguish several cases. Since we have not found a nice way of putting the result, we shall refrain from doing so.

Proposition 6. *Let θ^* be the solution of (13) corresponding to the maximum. Let*

$$d^* = \frac{1}{\sigma\cos(\theta^*-\psi)}\ln\left(\frac{\alpha\beta\sin(\theta^*+\phi)}{\gamma\delta\sin\theta^*}\right) - \frac{1}{2}(\beta\cos(\theta^*+\phi)+\delta\cos\theta^*)$$

where

$$\psi = \arccos\left(\frac{\delta-\rho\beta}{\sigma}\right),$$

then

$$\hat{\Pi} = \alpha\Phi(d^*+\beta\cos(\theta^*+\phi)) - \gamma\Phi(d^*+\delta\cos\theta^*) - \kappa\Phi(d^*) \tag{14}$$

Special Cases. We now look at which cases our approximations are in fact the true value. We also explain why in the other cases, our approximation is strictly less than the true value. In view of Proposition 2, we only need look for the cases where $\{X \geq 0\} \in \bigcup_\theta \sigma Y_\theta$ or $\{X \geq 0\} \in \big\{\{Y_\theta \leq d\}; d \in \mathbb{R}\big\}$.

Proposition 7. $\bar{\Pi} = \Pi$ *whenever* $\kappa = 0$, *or* $\alpha = 0$, *or* $\gamma = 0$, *or* $\rho = +1$ *or* $\rho = -1$.

Proof. When $\alpha = 0$,

$$\{X \geq 0\} = \left\{-\gamma e^{\delta Z_2 - \delta^2/2} - \kappa \geq 0\right\} \in \sigma Z_2 = \sigma\{Y_\pi\}$$

when $\gamma = 0$,

$$\{X \geq 0\} = \left\{\alpha e^{\beta[\sin\phi Z_1 + \cos\phi Z_2] - \beta^2/2} - \kappa \geq 0\right\} \in \sigma\{Y_{\pi-\phi}\}$$

when $\kappa = 0$,

$$\begin{aligned}\{X \geq 0\} &= \left\{\alpha e^{\beta[\sin\phi Z_1 + \cos\phi Z_2] - \beta^2/2} - \gamma e^{\delta Z_2 - \delta^2/2} \geq 0\right\} \\ &= \left\{\beta \sin\phi Z_1 + (\beta\cos\phi - \delta) Z_2 \geq \ln(\gamma/\alpha) + \beta^2/2 - \delta^2/2\right\} \in \sigma\{Y_\psi\}\end{aligned}$$

for $\psi = \arccos\left(\frac{\delta - \rho\beta}{\sigma}\right)$, when $\rho = \pm 1$,

$$\{X \geq 0\} = \left\{\alpha e^{\pm\beta Z_2 - \beta^2/2} - \gamma e^{\delta Z_2 - \delta^2/2} - \kappa \geq 0\right\} \in \sigma\{Y_\pi\}. \blacksquare$$

In any other case, the exercise region $\{X \geq 0\}$ will exhibit some *curvature* and the lower bounds will therefore be strict lower bounds.

Proposition 8. $\hat{\Pi} = \Pi$ *whenever* $\kappa = 0$, *or* $\alpha = 0$, *or* $\gamma = 0$, *or* $\rho = -1$, *or*

$$\rho = +1 \qquad \textit{and} \qquad \kappa(\beta - \delta) > 0$$

or

$$\rho = +1 \qquad \textit{and} \qquad \kappa(\beta - \delta) < 0 \qquad \textit{and} \qquad \left(\frac{\gamma\delta}{\alpha\beta}\right)^{\frac{\delta}{\beta-\delta}} \leq \frac{\kappa\beta}{\gamma(\delta - \beta)} e^{-\delta(\beta+\delta)/2}$$

Proof. By simple inspection of the proof of the last proposition, we see that except in the case $\rho = 1$, $\{X \geq 0\}$ will be either of the form $\{Y_\theta \leq x\}$ or of the form $\{Y_\theta \geq x\}$. The later can be rewritten as $\{Y_{\theta+\pi} \leq -x\}$ which is of the desired form. The only problem can arise when $\rho = 1$ and when the function

$$\alpha e^{\beta x - \beta^2/2} - \gamma e^{\delta x - \delta^2/2} - \kappa$$

has two zeroes. This gives the conditions of the proposition. $\blacksquare$

Numerical evidence suggests that the lower bound $\hat{\Pi}$ is extremely accurate. We shall illustrate this claim later in the paper. They are several cases where this approximation is indeed the true price as shown in the Proposition 8.

3.3. **Upper Bound and Control of the Error.**

In order to control the error of the approximation provided by $\hat{\Pi}$, we derive upper bounds for Π. In view of Proposition 2, it is easy to get lower bounds for the expectation, but much more difficult to get good upper bounds. Indeed, trying to simplify the exercise region can only lead to lower bounds.

Elementary Upper Bound. Since the standard Gaussian distribution is rotation-invariant, for every $\theta \in \mathbb{R}$ we can replace the couple (Z_1, Z_2) by the couple $(-\cos\theta Z_1 + \sin\theta Z_2, -\sin\theta Z_1 - \cos\theta Z_2)$ without changing the distribution of X.

$$\Pi = \mathbb{E}\left\{\left(\alpha e^{-\beta[\sin(\phi+\theta)Z_1+\cos(\phi+\theta)Z_2]-\beta^2/2} - \gamma e^{-\delta[\sin\theta Z_1+\cos\theta Z_2]-\delta^2/2} - \kappa\right)^+\right\} \tag{15}$$

Recall that we have a nice expression for $\bar\Pi(\theta)$ given in Proposition 3. We see that for each $\theta \in \mathbb{R}$, we have:

$$\begin{aligned}\Pi - \bar\Pi(\theta) &= \mathbb{E}\{X^+\} - \mathbb{E}\{\mathbb{E}\{X|Y_\theta\}^+\}\\ &= \mathbb{E}\{X^+ - \mathbb{E}\{X|Y_\theta\}^+\}\\ &\leq \mathbb{E}\{(X - \mathbb{E}\{X|Y_\theta\})^+\}.\end{aligned}$$

We used the elementary inequality $(a+b)^+ \leq a^+ + b^+$ in the last step. This last expression rewrites

$$\mathbb{E}\left\{\left[\alpha e^{-\beta\cos(\theta+\phi)Z_2-(\beta^2/2)\cos^2(\theta+\phi)}\left(e^{-\beta\sin(\theta+\phi)Z_1-(\beta^2/2)\sin^2(\theta+\phi)} - 1\right)\right.\right.$$
$$\left.\left.-\gamma e^{-\delta\cos\theta\, Z_2-(\delta^2/2)\cos^2\theta}\left(e^{-\delta\sin\theta\, Z_1-(\delta^2/2)\sin^2\theta} - 1\right)\right]^+\right\}.$$

Using again the elementary inequality $(a+b)^+ \leq a^+ + b^+$ and the independence of Z_1 and Z_2, we get:

$$\alpha\mathbb{E}\left\{\left(e^{-\beta\sin(\theta+\phi)Z_1-(\beta^2/2)\sin^2(\theta+\phi)} - 1\right)^+\right\} + \gamma\mathbb{E}\left\{\left(e^{-\delta\sin\theta\, Z_1-(\delta^2/2)\sin^2\theta} - 1\right)^+\right\}$$
$$= \alpha\left[\Phi\left(\frac{|\beta\sin(\theta+\phi)|}{2}\right) - \Phi\left(-\frac{|\beta\sin(\theta+\phi)|}{2}\right)\right] + \gamma\left[\Phi\left(\frac{|\delta\sin\theta|}{2}\right) - \Phi\left(-\frac{|\delta\sin\theta|}{2}\right)\right].$$

The one-dimensional Gaussian integral computation is similar to the derivation of the Black-Scholes' formula. This upper bound can also be optimized over θ. Let Ψ be the following *normalized Black-Scholes'* function

$$\Psi(x) = \Phi(|x/2|) - \Phi(-|x/2|) = |\Phi(x/2) - \Phi(-x/2)|, \quad x \in \mathbb{R}.$$

We have the following upper bound.

Proposition 9.

$$\Pi \leq \min\left\{\bar\Pi(0) + \alpha\Psi(\beta\sin\phi); \quad \bar\Pi(-\phi) + \gamma\Psi(\delta\sin\phi)\right\}$$

It remains to explain how we compute $\bar\Pi(\theta)$. This is done in the following proposition.

Proposition 10. *For each $\theta \in \mathbb{R}$, the lower bound $\hat\Pi(\theta)$ is given by:*

$$\begin{aligned}\bar\Pi(\theta) &= \frac{1-\epsilon}{2}(\alpha - \gamma - \kappa) + \epsilon\Big(\alpha[\Phi(d_2 + \beta\cos(\theta+\phi)) - \Phi(d_1 + \beta\cos(\theta+\phi))]\\ &\qquad - \gamma[\Phi(d_2 + \delta\cos\theta) - \Phi(d_1 + \delta\cos\theta)] - \kappa[\Phi(d_2) - \Phi(d_1)]\Big)\end{aligned}$$

where, d_1 and d_2 are the zeroes of the function $x \mapsto D(x)$ defined by:

$$D(x) = \alpha e^{-\beta\cos(\theta+\phi)x-(\beta^2/2)\cos^2(\theta+\phi)} - \gamma e^{-\delta\cos\theta x-(\delta^2/2)\cos^2\theta} - \kappa. \tag{16}$$

when the latter has two zeroes, with the convention that $d_1 = d_2 = \infty$ *when the equation* $D(x) = 0$ *has no solution, and* $d_2 = \infty$ *when the equation* $D(x) = 0$ *has exactly one solution* d_1*.* $\epsilon = \pm 1$ *is related to the sign of* D *between its zeroes.*

Proof. Since:

$$\bar{\Pi}(\theta) = \mathbb{E}\{D(Y_\theta)^+\} = \mathbb{E}\{D(Y_\theta)\mathbf{1}_{\{D(Y_\theta)>0\}}\}$$

the first task is to properly describe the set $\mathcal{E} = \mathcal{E}(\theta)$ defined by:

$$\mathcal{E} = \{x \in \mathbb{R};\ \ D(x) > 0\}.$$

The function $x \mapsto D(x)$ is continuously differentiable on $\mathbb{R}$ and may have 0, 1 or 2 zeroes. Let us denote by d_1 the unique zero when the function D has exactly one zero, and by $d_1 < d_2$ the values of the two zeroes when they exist. Therefore, $\mathcal{E}$ can be any of the following six sets.

$$\emptyset, \quad (d_1, d_2), \quad (d_1, +\infty), \quad (-\infty, d_1), \quad (-\infty, d_1) \cup (d_2, +\infty), \quad \mathbb{R}.$$

A third constant ϵ is defined by setting $\epsilon = +1$ in the first three cases, and $\epsilon = -1$ in the remaining cases. Without any loss of generality we can restrict ourselves to the case $\epsilon = +1$, because the case $\epsilon = -1$ can be derived from the case $\epsilon = 1$ and the parity formula (4).

With our convention on the values of d_1 and d_2, we always have $\mathcal{E} = (d_1, d_2)$ in the first three cases we are concentrating on, *i.e.*, when $\epsilon = 1$. Consequently the definition (10), (9) of $\bar{\Pi}(\theta)$ can be rewritten as:

$$\bar{\Pi}(\theta) = \mathbb{E}\left\{\left(\alpha e^{-\beta\cos(\theta+\phi)Y_\theta - (\beta^2/2)\cos^2(\theta+\phi)} - \gamma e^{-\delta\cos\theta Y_\theta - (\delta^2/2)\cos^2\theta} - \kappa\right)\mathbf{1}_{\{d_1<Y_\theta<d_2\}}\right\}.$$

As $\hat{\Pi}(\theta)$, $\bar{\Pi}(\theta)$ consists in three terms which can be computed separately. The first one gives:

$$\begin{aligned}
&\mathbb{E}\left\{\alpha e^{-\beta\cos(\theta+\phi)Z_2 - (\beta^2/2)\cos^2(\theta+\phi)}\mathbf{1}_{\{d_1<Y_\theta<d_2\}}\right\} \\
&\qquad = \alpha\frac{1}{\sqrt{2\pi}}\int_{d_1}^{d_2} e^{-\beta\cos(\theta+\phi)x - (\beta^2/2)\cos^2(\theta+\phi)} e^{-x^2/2}dx \\
&\qquad = \alpha\frac{1}{\sqrt{2\pi}}\int_{d_1+\beta\cos(\theta+\phi)}^{d_2+\beta\cos(\theta+\phi)} e^{-u^2/2}du \\
&\qquad = \alpha[\Phi(d_2 + \beta\cos(\theta+\phi)) - \Phi(d_1 + \beta\cos(\theta+\phi))]
\end{aligned}$$

The other two terms can be computed similarly. This leads to the following expression for $\bar{\Pi}(\theta)$:

$$\begin{aligned}
\bar{\Pi}(\theta) \quad = \quad & \alpha[\Phi(d_2 + \beta\cos(\theta+\phi)) - \Phi(d_1 + \beta\cos(\theta+\phi))] \\
& - \gamma[\Phi(d_2 + \delta\cos\theta) - \Phi(d_1 + \delta\cos\theta)] \\
& - \kappa[\Phi(d_2) - \Phi(d_1)]
\end{aligned}$$

which is the desired result when $\epsilon = +1$. The case $\epsilon = -1$ is obtained similarly, the fact that the set $\mathcal{E}$ is now the complementary set of what it was before accounts for the changes in the formula. The proof of the proposition is now complete. ∎

Improvement of the Upper Bound via Symmetry Arguments. The upper bound of Proposition 9 can be improved. As we can see in this proposition, when $\alpha = 0$, we have

$$\bar{\Pi}(0) \leq \bar{\Pi} \leq \min\left(\bar{\Pi}(0); \quad \bar{\Pi}(-\phi) + \gamma\Psi(\delta \sin\phi)\right) \leq \bar{\Pi}(0)$$

The upper bound is equal to the lower bound and hence is the true value. The same is true when $\gamma = 0$ or $\rho = \pm 1$ (*i.e.*, $\sin\phi = 0$). However, in the case $\kappa = 0$, it does not give us that the lower bound is in fact exact. The previous symmetry relations will allow us to get a better upper bound that will equal the lower bound also when $\kappa = 0$.

This leads to the following improvement of our upper bound.

Proposition 11.

$$\Pi \quad \leq \quad \min\left\{ \min\left(\bar{\Pi}(0); \overline{\Pi \circ s}(0)\right) + \alpha\Psi(\beta\rho'); \quad \min\left(\bar{\Pi}(-\phi); \overline{\Pi \circ t}(0)\right) + \gamma\Psi(\delta\rho'); \right.$$
$$\left. \min\left(\overline{\Pi \circ s}(-\psi); \overline{\Pi \circ t}(\psi - \pi)\right) + |\kappa|\,\Psi\left(\frac{\beta\delta\rho'}{\sigma}\right) \right\}$$

Improvement of the Upper Bound via Monotonicity Arguments. It turns out that the previous upper bound is not always decreasing in ρ whereas the true price is. Obviously, the largest decreasing function below our upper bound will be a better upper bound. The fact that (3) is decreasing with respect to ρ seems to be known but we did not find a proof of it in the financial literature. This is the topic of the next proposition.

Proposition 12. $\rho \mapsto \Pi(\rho)$ *is decreasing on* $[-1, 1]$.

Proof. Let us denote by f the bivariate standard Gaussian density with correlation ρ:

$$f(\rho, u, v) = \frac{\exp\left(-\frac{1}{2}\frac{u^2 - 2\rho uv + v^2}{1-\rho^2}\right)}{2\pi\sqrt{1-\rho^2}}$$

We will use the following identity:

$$\frac{\partial f}{\partial \rho} = \frac{\partial^2 f}{\partial u \partial v}.$$

Let us rewrite Π using f.

$$\begin{aligned} \Pi &= \int_{\mathbb{R}^2} \left(\alpha e^{\beta u - \beta^2/2} - \gamma e^{\delta v - \delta^2/2} - \kappa\right)^+ f(\rho, u, v)\, du dv \\ &= \int_{\mathbb{R}^2} (e^x - e^y - \kappa)^+ f\left(\rho, \frac{x - \ln\alpha + \beta^2/2}{\beta}, \frac{y - \ln\gamma + \delta^2/2}{\delta}\right) \frac{dx dy}{\beta\delta} \end{aligned}$$

Since f is $C^1((-1,1)\times\mathbb{R}\times\mathbb{R})$, we can use the Lebesgue's differentiation theorem to compute the derivative of Π with respect to ρ.

$$\begin{aligned}
\frac{\partial \Pi}{\partial \rho} &= \int_{\mathbb{R}^2} (e^x - e^y - \kappa)^+ \frac{\partial f}{\partial \rho}\left(\rho, \frac{x-\ln\alpha+\beta^2/2}{\beta}, \frac{y-\ln\gamma+\delta^2/2}{\delta}\right)\frac{dxdy}{\beta\delta} \\
&= \int_{\mathbb{R}^2} (e^x - e^y - \kappa)^+ \frac{\partial^2 f}{\partial u \partial v}\left(\rho, \frac{x-\ln\alpha+\beta^2/2}{\beta}, \frac{y-\ln\gamma+\delta^2/2}{\delta}\right)\frac{dxdy}{\beta\delta} \\
&= \alpha\beta\gamma\delta \frac{\partial^2}{\partial\alpha\partial\gamma}\int_{\mathbb{R}^2} (e^x - e^y - \kappa)^+ f\left(\rho, \frac{x-\ln\alpha+\beta^2/2}{\beta}, \frac{y-\ln\gamma+\delta^2/2}{\delta}\right)\frac{dxdy}{\beta\delta} \\
&= \alpha\beta\gamma\delta \frac{\partial^2}{\partial\alpha\partial\gamma}\int_{\mathbb{R}^2} \left(\alpha e^{\beta u-\beta^2/2} - \gamma e^{\delta v - \delta^2/2} - \kappa\right)^+ f(\rho,u,v)\, dudv
\end{aligned}$$

To show that this last expression is always negative, notice that it is the limit as h and k go to zero of

$$\frac{1}{hk}\int_{\mathbb{R}^2} \left(g(\alpha+h,\gamma+k) - g(\alpha,\gamma+k) - g(\alpha+h,\gamma) + g(\alpha,\gamma)\right) f(\rho,u,v)\, dudv$$

where $g(\alpha,\gamma) = \left(\alpha e^{\beta u-\beta^2/2} - \gamma e^{\delta v-\delta^2/2} - \kappa\right)^+$. Note that for any reals x and $h > k > 0$, we have

$$\begin{aligned}
&\frac{1}{hk}\left((x+h-k)^+ - (x+h)^+ - (x-k)^+ + x^+\right) \\
&= \frac{1}{hk}\left(-k\mathbf{1}_{\{x+h\geq k\}} - (x+h)\mathbf{1}_{\{0\leq x+h\leq k\}} + k\mathbf{1}_{\{x\geq k\}} + x\mathbf{1}_{\{0\leq x\leq k\}}\right) \\
&= \frac{1}{hk}\left(k\left(\mathbf{1}_{\{x\geq k\}} - \mathbf{1}_{\{x+h\geq k\}}\right) + x\mathbf{1}_{\{0\leq x\leq k\}} - (x+h)\mathbf{1}_{\{0\leq x+h\leq k\}}\right) \\
&\leq \frac{1}{hk}\left(k\left(\mathbf{1}_{\{x\geq k\}} - \mathbf{1}_{\{x+h\geq k\}}\right) + x\mathbf{1}_{\{0\leq x\leq k\}}\right) \\
&\leq \frac{1}{h}\left(\mathbf{1}_{\{x\geq k\}} - \mathbf{1}_{\{x+h\geq k\}} + \mathbf{1}_{\{0\leq x\leq k\}}\right) \\
&= \frac{1}{h}\left(\mathbf{1}_{\{x\geq 0\}} - \mathbf{1}_{\{x\geq k-h\}}\right) \\
&\leq 0.
\end{aligned}$$

The fact that $h > k$ was only used at the last line. This implies that the quantity we are studying as $k > h > 0$ go to zero is always negative and since the limit exists by the previous argument, we have $\frac{\partial\Pi}{\partial\rho} \leq 0$. ∎

Numerical examples show that this improved upper bound is quite accurate (about 10%). However this upper bound does not catch the extreme accuracy of the lower bound as the numerical experiments that we ran (see below) suggest.

4. OTHER ANALYTICAL APPROXIMATIONS

There have already existed attempts to give an efficient *analytical* approximation of Π. We present two of them: the Bachelier's model and the Kirk's model, both of them have been considered by market's practitioners.

4.1. **The Bachelier's Model.** In most typical applications, all the underlying indexes are modelled by means of log-normal distributions or at least, exponential transformations of standard distributions. This is motivated in part by the inherent positivity of asset prices. But the positivity restriction does not apply to the spreads themselves, since the latter can be negative as differences of positive quantities. Indeed, computing histograms of historical spread values shows that the marginal distribution of a spread at a given time extends on both tails, and surprisingly enough, that the Gaussian distribution can give a reasonable fit. This simple remark is the starting point of a series of papers proposing to use arithmetic Brownian motion (as opposed to the geometric Brownian motion leading to the log-normal distribution of the indexes) for the dynamics of spreads. In so doing, prices of options can be derived by computing simple Gaussian integrals leading to simple closed form formulae. It was originally advocated by Shimko in the early nineties. See [7] for a detailed exposé of this method.

If we approximate the distribution of the spread

$$S = \alpha e^{\beta X_1 - \beta^2/2} - \gamma e^{\delta X_2 - \delta^2/2}$$

by the Gaussian distribution, the least we can ask is that it matches the first two moments. Therefore:

$$S \sim N(\mathbb{E}\{S\}, \text{var}\{S\})$$

and classical computations give

$$\begin{aligned} \mathbb{E}\{S\} &= \alpha - \gamma \\ \text{var}\{S\} &= \alpha^2 \left(e^{\beta^2} - 1\right) - 2\alpha\gamma \left(e^{\rho\beta\delta} - 1\right) + \gamma^2 \left(e^{\delta^2} - 1\right). \end{aligned}$$

Proposition 13. *If the value of the spread at maturity is assumed to have the Gaussian distribution, the Bachelier's approximation* Π^B *is given by:*

$$\Pi^B = (\alpha - \gamma - \kappa)\,\Phi\left(\frac{\alpha - \gamma - \kappa}{\sigma^B}\right) + \sigma^B \varphi\left(\frac{\alpha - \gamma - \kappa}{\sigma^B}\right) \tag{17}$$

where we used the notation:

$$\sigma^B = \sqrt{\alpha^2 \left(e^{\beta^2} - 1\right) - 2\alpha\gamma \left(e^{\rho\beta\delta} - 1\right) + \gamma^2 \left(e^{\delta^2} - 1\right)}$$

Proof. Plainly,

$$\begin{aligned} \Pi^B &= \mathbb{E}\{(S - \kappa)^+\} \\ &= \mathbb{E}\{(\alpha - \gamma - \kappa + \sigma^B \xi)^+\} \end{aligned}$$

for some Gsn$(0,1)$ random variable ξ. Consequently:

$$\Pi^B = \frac{1}{\sqrt{2\pi}} \int_{-\frac{\alpha-\gamma-\kappa}{\sigma^B}}^{\infty} (\alpha - \gamma - \kappa + \sigma^B u) e^{-u^2/2} du$$

from which we easily get the desired result. ∎

Based on Edgeworth expansions, Jarrow and Rudd [2] improved the Bachelier's model in taking into account higher order moments (skew and kurtosis). We refer to Mbafeno [5] for the formula giving an approximate price for a spread option.

4.2. **The Kirk's Model.** More recently, Kirk in [3] proposed the formula

$$\Pi^K = \alpha\Phi\left(\frac{\ln\left(\frac{\alpha}{\gamma+\kappa}\right)}{\sigma^K} + \frac{\sigma^K}{2}\right) - (\gamma+\kappa)\Phi\left(\frac{\ln\left(\frac{\alpha}{\gamma+\kappa}\right)}{\sigma^K} - \frac{\sigma^K}{2}\right)$$

where

$$\sigma^K = \sqrt{\beta^2 - 2\rho\beta\delta\frac{\gamma}{\gamma+\kappa} + \delta^2\left(\frac{\gamma}{\gamma+\kappa}\right)^2}$$

As the reader can immediately see, the approximation is exact when $\kappa = 0$ and when $\gamma = 0$. However unlike our lower bound, it cannot be exact when neither $\rho = \pm 1$ nor when $\alpha = 0$. The numerical approximation of the price is very good but, as we will see later, the resulting hedging strategy does not always perform very well.

5. Pricing Options on the Spread of Geometric Brownian Motions

5.1. **Pricing Formula.** In this section we apply the results obtained in section 3 to the case of a option on the difference of two assets. Recall formula (2). This expectation is a particular case of the expectations computed before provided we set

$$\alpha = x_2 e^{-q_2 T} \quad \beta = \sigma_2\sqrt{T} \quad \gamma = x_1 e^{-q_1 T} \quad \delta = \sigma_1\sqrt{T} \quad \text{and} \quad \kappa = Ke^{-rT}. \tag{18}$$

So for each $\theta \in \mathbb{R}$, the number $\hat{p}(\theta) = \hat{\Pi}(\theta)$ is a lower bound for the price p. Following the discussion of Section 3 we introduce the approximation $\hat{p}$ given by the supremum of the lower bounds $\hat{p}(\theta)$ when we vary the free parameter θ.

$$\hat{p} = \sup_{\theta\in\mathbb{R}} \hat{p}(\theta) \tag{19}$$

According to Proposition 6 we have:

Proposition 14. *Let θ^* be the solution of (13) corresponding to the maximum. Let*

$$d^* = \frac{1}{\sigma\cos(\theta^* - \psi)\sqrt{T}} \ln\left(\frac{x_2 e^{-q_2 T}\sigma_2 \sin(\theta^* + \phi)}{x_1 e^{-q_1 T}\sigma_1 \sin\theta^*}\right) - \frac{1}{2}(\sigma_2\cos(\theta^* + \phi) + \sigma_1\cos\theta^*)\sqrt{T}$$

then

$$\hat{p} = x_2 e^{-q_2 T}\Phi\left(d^* + \sigma_2\cos(\theta^* + \phi)\sqrt{T}\right) - x_1 e^{-q_1 T}\Phi\left(d^* + \sigma_1 \sin\theta^* \sqrt{T}\right) - Ke^{-rT}\Phi(d^*) \tag{20}$$

Note that this formula is as close to the Black-Scholes' formula as we could hope. We will see later that they have many features in common.

Proposition 8 takes the form:

Proposition 15. *The approximation $\hat{p}$ is equal to the true price p when $K = 0$, or $x_1 = 0$, or $x_2 = 0$, or $\rho = -1$. In particular, $\hat{p}$ is given by Margrabe's formula when $K = 0$, and by the classical Black-Scholes' formula when $x_1 x_2 = 0$.*

For the cases of equality when $\rho = +1$, we refer to Proposition 8.

Proof. There is nothing to be proven but we show how we recover Margrabe's formula in the case $K = 0$. Since it encompasses the Black-Scholes' formula, this will also show how we deal with case $x_1 x_2 = 0$. In order to do that, we take the θ^* that we computed in the proof of proposition 7, that is

$$\theta^* = \pi + \psi = \pi + \arccos\left(\frac{\sigma_1 - \rho\sigma_2}{\sigma}\right).$$

where σ is now

$$\sigma = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$$

We easily compute that

$$\sigma_2 \sin(\theta^* + \phi) = \sigma_1 \sin\theta^*$$

so that

$$d^* = \frac{1}{\sigma\sqrt{T}} \ln\left(\frac{x_2 e^{-q_2 T}}{x_1 e^{-q_1 T}}\right) - \frac{1}{2}(\sigma_2 \cos(\theta^* + \phi) + \sigma_1 \cos\theta^*)\sqrt{T}.$$

Furthermore

$$\sigma_2 \cos(\theta^* + \phi) - \sigma_1 \cos\theta^* = \sigma$$

hence,

$$\begin{aligned} d^* + \sigma_2 \cos(\theta^* + \phi)\ \sqrt{T} &= \frac{1}{\sigma\sqrt{T}} \ln\left(\frac{x_2 e^{-q_2 T}}{x_1 e^{-q_1 T}}\right) + \frac{\sigma\sqrt{T}}{2} \\ d^* + \sigma_1 \cos\theta^*\ \sqrt{T} &= \frac{1}{\sigma\sqrt{T}} \ln\left(\frac{x_2 e^{-q_2 T}}{x_1 e^{-q_1 T}}\right) - \frac{\sigma\sqrt{T}}{2}. \end{aligned}$$

Finally, we get

$$p = x_2 e^{-q_2 T}\Phi\left(\frac{1}{\sigma\sqrt{T}} \ln\left(\frac{x_2 e^{-q_2 T}}{x_1 e^{-q_1 T}}\right) + \frac{\sigma\sqrt{T}}{2}\right) - x_1 e^{-q_1 T}\Phi\left(\frac{1}{\sigma\sqrt{T}} \ln\left(\frac{x_2 e^{-q_2 T}}{x_1 e^{-q_1 T}}\right) - \frac{\sigma\sqrt{T}}{2}\right)$$

which is Margrabe's formula, see [4]. ∎

5.2. **Hedging and the Computation of the Greeks.**

First Order Derivatives. Exactly as in the case with a single asset, our derivation gives as a side effect a sub-hedge for the option.

Proposition 16. *Holding at each time $t \le T$*

$$\Delta_1 = -e^{-q_1 T}\Phi\left(d^* + \sigma_1 \cos\theta^*\ \sqrt{T}\right)$$

and

$$\Delta_2 = e^{-q_2 T}\Phi\left(d^* + \sigma_2 \cos(\theta^* + \phi)\sqrt{T}\right)$$

*of the underlying assets will provide a sub-hedge for the option (*i.e.*, such a portfolio gives a sub-replication of the pay-off of the option.)*

Proof. We need to compute $d\hat{p}/dx_1$ and $d\hat{p}/dx_2$. Note that the price functional depends on x_2 through both x_2 and θ^*. Therefore

$$\frac{d\hat{p}}{dx_2} = \frac{\partial\hat{p}}{\partial x_2} + \frac{\partial\hat{p}}{\partial\theta}\frac{\partial\theta^*}{\partial x_2}$$

but since we take as the price the optimal lower bound $\partial\hat{p}/\partial\theta = 0$ at θ^*. The remaining term is computed easily:

$$\begin{aligned}\frac{\partial\hat{p}}{\partial x_2} &= \frac{\partial}{\partial x_2}\mathbb{E}\bigg\{\bigg(x_2 e^{-q_2T-\sigma_2\cos(\theta^*+\phi)W(T)-\sigma_2^2\cos^2(\theta^*+\phi)T/2} \\ &\qquad - x_1 e^{-q_1T-\sigma_1\cos\theta^*\,W(T)-\sigma_1^2\cos^2\theta^*\,T/2} - Ke^{-rT}\bigg)\mathbf{1}_{\{W(T)\le d^*\}}\bigg\} \\ &= e^{-q_2T}\mathbb{E}\left\{e^{-\sigma_2\cos(\theta^*+\phi)W(T)-\sigma_2^2\cos^2(\theta^*+\phi)T/2}\mathbf{1}_{\{W(T)\le d^*\}}\right\} \\ &= e^{-q_2T}\Phi\left(d^*+\sigma_2\cos(\theta^*+\phi)\sqrt{T}\right)\end{aligned}$$

The computation is similar for $\partial\hat{p}/\partial x_1$. ∎

In the same manner we can get other partial derivatives of the price with respect to various interesting parameters. These are the so-called Greeks of the financial literature. We give some of them in the following proposition.

Proposition 17. *Let ϑ_1 and ϑ_2 denote the sensitivities of the price functional (20) with respect to the volatilities of each asset, χ be the sensitivity with respect to their correlation parameter ρ, κ be the sensitivity with respect to the strike price K and Θ be that with respect to the maturity time T.*

$$\begin{aligned}\vartheta_1 &= x_1e^{-q_1T}\varphi\left(d^*+\sigma_1\cos\theta^*\,\sqrt{T}\right)\cos\theta^*\,\sqrt{T} \\ \vartheta_2 &= -x_2e^{-q_2T}\varphi\left(d^*+\sigma_2\cos(\theta^*+\phi)\,\sqrt{T}\right)\cos(\theta^*+\phi)\,\sqrt{T} \\ \chi &= -x_1e^{-q_1T}\varphi\left(d^*+\sigma_1\cos\theta^*\,\sqrt{T}\right)\sigma_1\frac{\sin\theta^*}{\sin\phi}\sqrt{T} \\ \kappa &= -\Phi\left(d^*\right)e^{-rT} \\ \Theta &= \frac{\sigma_1\vartheta_1+\sigma_2\vartheta_2}{2T} - q_1x_1\Delta_1 - q_2x_2\Delta_2 - rK\kappa\end{aligned}$$

Proof. These formulae are derived exactly in the same way as we did for the deltas. Again, the key observation is that

$$\frac{\partial\hat{p}}{\partial\theta}(\theta^*) = 0$$

so that we only need differentiate as if θ^* were constant. The formula for Θ comes from the following *homogeneity* property:

$$\hat{p}(x_1,x_2,\sigma_1,\sigma_2,q_1,q_2,K,r,T) = \hat{p}(x_1e^{-q_1T},x_2e^{-q_2T},\sigma_1\sqrt{T},\sigma_2\sqrt{T},0,0,Ke^{-rT},0,1).$$

We can express it using the Greeks we have just computed. ∎

Second Order Derivatives. The second order derivatives are also of fundamental importance. Let Γ_{11}, Γ_{12} and Γ_{22} be the second derivatives with respect to the current values of the assets. The simplification that arose for the first order derivatives does not hold anymore. Indeed,

$$\begin{aligned}
\frac{d^2\hat{p}}{dx_1^2} &= \frac{\partial^2\hat{p}}{\partial x_1^2} + 2\frac{\partial^2\hat{p}}{\partial x_1\partial\theta}\frac{\partial\theta^*}{\partial x_1} + \frac{\partial^2\hat{p}}{\partial\theta^2}\left(\frac{\partial\theta^*}{\partial x_1}\right)^2 \\
&= \frac{\partial^2\hat{p}}{\partial x_1^2} - \frac{\partial^2\hat{p}}{\partial\theta^2}\left(\frac{\partial\theta^*}{\partial x_1}\right)^2 + 2\left(\frac{\partial^2\hat{p}}{\partial x_1\partial\theta} + \frac{\partial^2\hat{p}}{\partial\theta^2}\frac{\partial\theta^*}{\partial x_1}\right)\frac{\partial\theta^*}{\partial x_1} \\
&= \frac{\partial^2\hat{p}}{\partial x_1^2} - \frac{\partial^2\hat{p}}{\partial\theta^2}\left(\frac{\partial\theta^*}{\partial x_1}\right)^2 + 2\frac{d}{dx_1}\left(\frac{\partial\hat{p}}{\partial\theta}\right)\frac{\partial\theta^*}{\partial x_1} \\
&= \frac{\partial^2\hat{p}}{\partial x_1^2} - \frac{\partial^2\hat{p}}{\partial\theta^2}\left(\frac{\partial\theta^*}{\partial x_1}\right)^2.
\end{aligned}$$

However, we propose to approximate the second order derivative by the corresponding partial derivative. This is supported by different observations. The first one is that in case the approximation is the true price, the corresponding Γ will also be exact since in those case, θ^* does not depend neither on x_1 nor on x_2. The second observation is that θ^* does not vary very much with respect to those parameters. Finally, we have the surprising fact that this approximation on the Γ's exactly balances the approximation on $\hat{p}$ so that the Greeks satisfy the Black-Scholes' equation!

Instead of explaining how we computed the Γ's, we give them and show that they satisfy the Black-Scholes' equation.

Proposition 18. *Suppose*

$$\begin{aligned}
\Gamma_{11} &= \frac{e^{-2q_1T}}{\sigma_2\vartheta_2 + \sigma_1\vartheta_1}\varphi\left(d^* + \sigma_1\cos\theta^*\sqrt{T}\right)^2 \\
\Gamma_{12} &= -\frac{e^{-(q_1+q_2)T}}{\sigma_2\vartheta_2 + \sigma_1\vartheta_1}\varphi\left(d^* + \sigma_1\cos\theta^*\sqrt{T}\right)\varphi\left(d^* + \sigma_2\cos(\theta^*+\phi)\sqrt{T}\right) \\
\Gamma_{22} &= \frac{e^{-2q_2T}}{\sigma_2\vartheta_2 + \sigma_1\vartheta_1}\varphi\left(d^* + \sigma_2\cos(\theta^*+\phi)\sqrt{T}\right)^2,
\end{aligned}$$

then

$$-\Theta + \frac{1}{2}\sigma_1^2x_1^2\Gamma_{11} + \rho\sigma_1\sigma_2x_1x_2\Gamma_{12} + \frac{1}{2}\sigma_2^2x_2^2\Gamma_{22} + (r-q_1)x_1\Delta_1 + (r-q_2)x_2\Delta_2 - r\hat{p} = 0$$

Proof. Let us denote

$$\begin{aligned}
a_1 &= \sigma_1\cos\theta^*\sqrt{T} \\
a_2 &= \sigma_2\cos(\theta^*+\phi)\sqrt{T},
\end{aligned}$$

we also let $\tilde{x}_1 = x_1 e^{-q_1 T}$ and $\tilde{x}_2 = x_2 e^{-q_2 T}$. According to Proposition 17 and equation (20),

$$\begin{aligned}
&-\Theta + \frac{1}{2}\sigma_1^2 x_1^2 \Gamma_{11} + \rho\sigma_1\sigma_2 x_1 x_2 \Gamma_{12} + \frac{1}{2}\sigma_2^2 x_2^2 \Gamma_{22} + (r-q_1)x_1\Delta_1 + (r-q_2)x_2\Delta_2 - r\hat{p} \\
&= \frac{1}{2}\sigma_1^2 x_1^2 \Gamma_{11} + \rho\sigma_1\sigma_2 x_1 x_2 \Gamma_{12} + \frac{1}{2}\sigma_2^2 x_2^2 \Gamma_{22} - \frac{\sigma_1\vartheta_1 + \sigma_2\vartheta_2}{2T} \\
&= \frac{1}{2T(\sigma_1\vartheta_1 + \sigma_2\vartheta_2)} \Big[\sigma_1^2 \tilde{x}_1^2 T \varphi(d^* + a_1)^2 - 2\rho\sigma_1\sigma_2 \tilde{x}_1 \tilde{x}_2 T \varphi(d^* + a_1)\varphi(d^* + a_2) \\
&\quad + \sigma_2^2 \tilde{x}_2^2 T \varphi(d^* + a_2)^2 - a_1^2 \tilde{x}_1^2 \varphi(d^* + a_1)^2 + 2a_1 a_2 \tilde{x}_1 \tilde{x}_2 \varphi(d^* + a_2)\varphi(d^* + a_1) \\
&\quad - a_2^2 \tilde{x}_2^2 \varphi(d^* + a_2)^2 \Big] \\
&= \frac{1}{2(\sigma_1\vartheta_1 + \sigma_2\vartheta_2)} \Big[\sigma_1^2 \sin^2\theta^* \; \tilde{x}_1^2 \varphi(d^* + a_1)^2 + \sigma_2^2 \sin^2(\theta^* + \phi) \tilde{x}_2^2 \varphi(d^* + a_2)^2 \\
&\quad - 2\sigma_1\sigma_2 (\cos\phi - \cos(\theta^* + \phi)\cos\theta^*) \tilde{x}_1 \tilde{x}_2 \varphi(d^* + a_1)\varphi(d^* + a_2) \Big] \\
&= \frac{[\sigma_1 \sin\theta^* \; \tilde{x}_1 \varphi(d^* + a_1) - \sigma_2 \sin(\theta^* + \phi)\tilde{x}_2 \varphi(d^* + a_2)]^2}{2(\sigma_1\vartheta_1 + \sigma_2\vartheta_2)} \\
&= 0
\end{aligned}$$

The last equality comes from equation (12). ∎

Since these formulae are the true achievement of our method, we plotted two of them: ϑ_1 and χ in Figure 1. As Garman pointed out in [1], it is worth noticing that spread options exhibit the unusual feature of negative vegas. Indeed it is not very common that the price of a derivative product decreases when the volatility of the underlying increases. Here is an heuristic argument for that: imagine that the two assets are highly correlated so that the value of the spread is very likely to stay (for instance) out of the money. The corresponding price is expected to be low. Suppose σ_1 decreases, the spread increases its variance, the spread increases its probability of being in the money and therefore the price of the option increases. The reader can also check that $\chi \leq 0$. This phenomenon has the same rationale, as the correlation increases, the variance of the spread decreases.

6. Numerical Performance

6.1. **Approximation Error.** In order to illustrate the accuracy of our approximation, we present the result of a Monte Carlo analysis in the case of the geometric Brownian motion. We fix the values of the parameters of the marginal dynamical equations according to Table 1, and we vary the values of the correlation coefficient ρ and of the strike K. Results are reported in Table 2. As we can see the agreement is excellent. This strongly supports our formula, the price to pay is extremely cheap since we only need to compute numerically a zero of a given function. This is done very efficiently by a Newton-Raphson method. The computation time is immediate.

We plot the relative error between our approximation and a Monte Carlo simulation against the volatility parameters and against strike/correlation in Figure 2. On the right panel, we clearly see the bias of our lower bound when both volatilities are very high. On the contrary, on the left panel, we observe an increase of the relative error as K becomes large but the peaks are either negative or

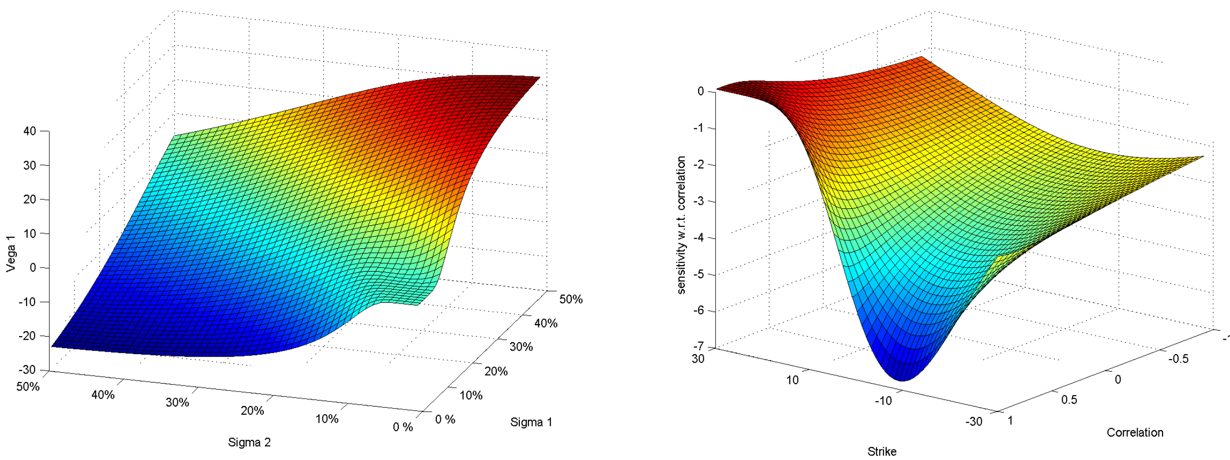

FIGURE 1. Left: Price sensitivity with respect to σ_1 for different volatilities σ_1 and σ_2; here $\rho = 70\%$. Right: Price sensitivity with respect to the correlation for different strikes and correlations; here $\sigma_1 = 20\%$ and $\sigma_2 = 10\%$.

positive, which means that we see the error due to Monte-Carlo and not the bias of our approximation.

	Asset 1	Asset 2
x	100	110
q	2%	3%
σ	15%	10%

TABLE 1. Model data together with $r = 5\%$ and $T = 1$.

6.2. **Comparisons with Bachelier's and Kirk's Approximations.** Let us now compare our formula with these two other models. As we have already pointed out, the main interesting feature of our model comes from the easy computation of the Greeks which in turn gives sensible hedging portfolios. The Kirk's formula (and also the Bachelier's formula) also gives rise to two deltas and we can therefore see how the two hedging strategies perform along a given path (or scenario.)

To this end, we estimate the standard deviation of the tracking error for both models. What we call tracking error is the difference between the payoff of the option at maturity and the value of the discretely re-balanced replicating portfolio. In Figure 3, we plot the standard deviation of the tracking error against the number of re-hedging times in log-scales. Those are typical examples: in most cases the agreement between Kirk's model and the lower bound is very good but in some cases our lower bound performs much better. In any cases, these two models clearly outperform the Bachelier's model. The reason is that the Bachelier's model is a one-factor model that cannot capture the whole structure of the true two-factor model we are dealing with.

K \ ρ	-1	-0.5	0	0.3	0.8	1
	29.656	28.994	28.381	28.070	27.770	**27.754**
-20	*29.653*	*28.995*	*28.381*	*28.069*	*27.770*	*27.754*
	29.656	29.442	28.773	28.311	27.790	**27.754**
	21.868	20.904	19.888	19.270	18.381	**18.244**
-10	*21.867*	*20.907*	*19.892*	*19.271*	*18.382*	*18.244*
	21.868	21.129	20.200	19.516	18.431	**18.244**
	15.133	**13.917**	**12.523**	**11.561**	**9.632**	**8.821**
0	*15.133*	*13.914*	*12.525*	*11.559*	*9.632*	*8.821*
	15.133	**13.917**	**12.523**	**11.561**	**9.632**	**8.821**
	12.244	10.956	9.445	8.367	5.967	**4.454**
5	*12.242*	*10.953*	*9.442*	*8.366*	*5.968*	*4.454*
	12.244	11.068	9.601	8.542	6.148	**4.454**
	7.521	6.242	4.744	3.679	1.342	**0.049**
15	*7.521*	*6.243*	*4.743*	*3.678*	*1.344*	*0.049*
	7.521	6.574	5.202	4.188	1.858	**0.049**
	4.201	3.129	1.961	1.219	0.103	**0.000**
25	*4.203*	*3.129*	*1.962*	*1.220*	*0.105*	*0.000*
	4.201	3.680	2.718	2.062	0.926	**0.000**

TABLE 2. The number appearing in italic in the center of each box is the result of a stratified Monte-Carlo computation with 100,000 trials, the number on top is our lower bound approximation (20), the improved upper bound appearing at the bottom. We used bold faces when the lower bound and the upper bound are equal to the true value. The values of the parameters used for these runs are given in Table 1.

Note also that from these plots it is *impossible* to see that the hedging strategy induced by our formula is merely sub-replicating!

7. IMPLIED CORRELATION

7.1. **A Tractable Jump-Diffusion Model.** We are now going to show that our previous result can fit in the more general case of a jump-diffusion process. The process we are looking at is often referred as the Merton's jump model in the financial literature. In the same spirit, we allow for jumps in the risk-neutral dynamics of S_1 and S_2.

$$\frac{dS_i(t)}{S_i(t-)} = (r - q_i - \lambda_i\mu_i - \lambda_0\mu_0)dt + \sigma_i dW_i(t) + (e^{J_i(t)} - 1)dN_i(t) + (e^{J_0(t)} - 1)dN_0(t) \tag{21}$$

where N_0, N_1 and N_2 are three independent Poisson processes with intensity λ_0, λ_1 and λ_2. They are also independent of W_2 and W_1. $(J_i(t))_{t\geq 0, i=0,1,2}$ is a sequence of independent Gaussian random variables (m_i, s_i^2). We will often need the following quantities

$$\mu_i = e^{m_i + s_i^2/2} - 1.$$

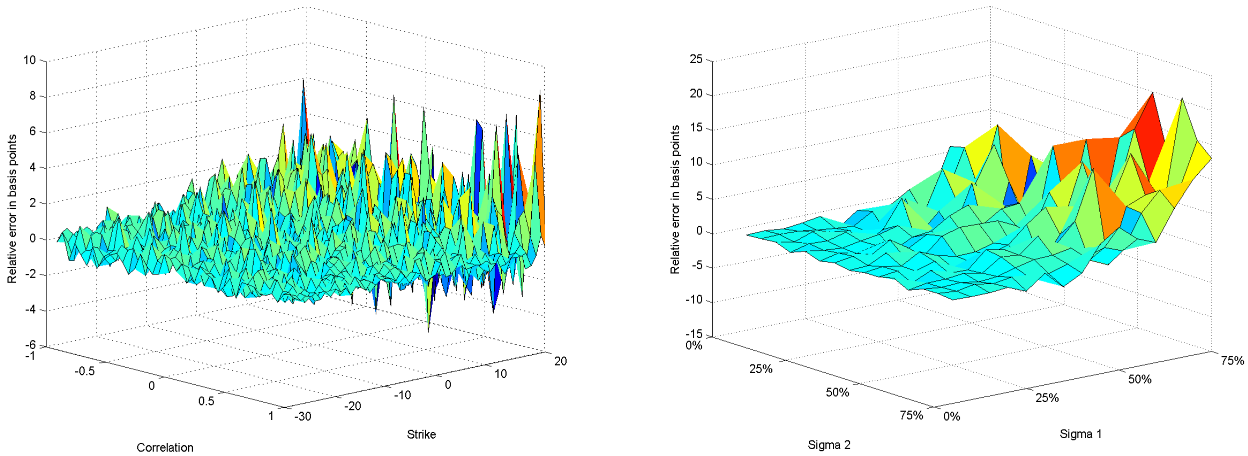

FIGURE 2. Relative errors between our approximation (20) and Monte-Carlo. Data are in given in Table 1.

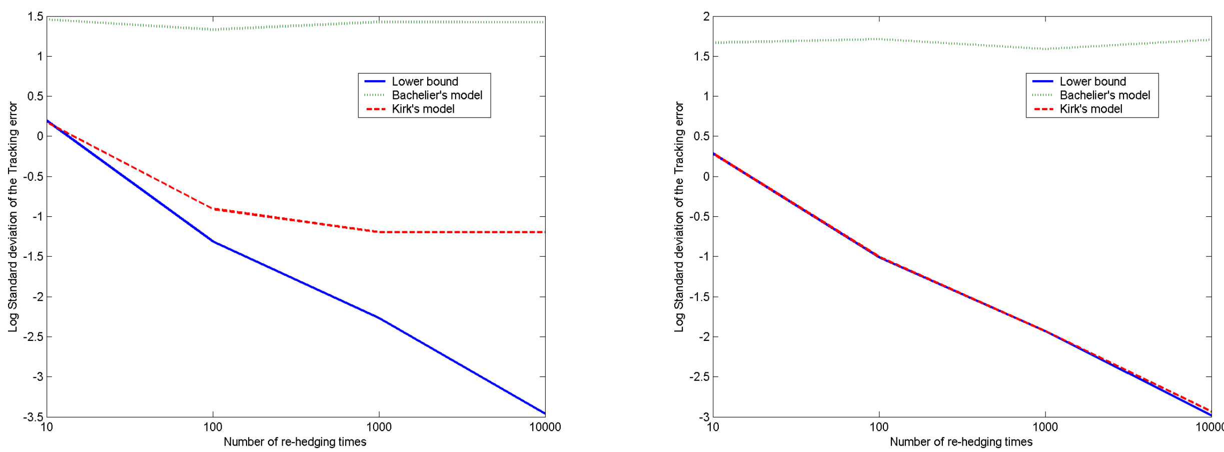

FIGURE 3. Behavior of the tracking error as the number of re-hedging times increases. The model data are $x_1 = 100$, $x_2 = 110$, $\sigma_1 = 10\%$, $\sigma_2 = 15\%$ and $T = 1$. $\rho = 0.9$, $K = 30$ (left) and $\rho = 0.6$, $K = 20$ (right).

We introduced three Poisson processes for the sake of generality but we shall consider two special cases. The first case is $\lambda_1 = \lambda_2 = 0$. This case would be more appropriate in the context of equity markets where the jumps (generally downwards) account for the sudden moves of the whole market. In such a case the jump component can be taken as being the same for each stock in a first approximation. A second case of interest is $\lambda_0 = 0$. This is more realistic in the case of spark-spread options in the energy markets where the two components of the spread are gas and electricity future contracts. They come from very different markets and the sudden jumps in the price dynamics have no reason for being correlated and we can take them independent.

By a simple use of Itô's formula, we get the integrated price dynamics

$$S_i(T) = x_i \exp\left((r - q_i - \sigma_i^2/2 - \lambda_0\mu_0 - \lambda_i\mu_i)T + \sigma_i W_i(T) + \sum_{k=1}^{N_0(T)} J_0(k) + \sum_{k=1}^{N_i(T)} J_i(k) \right)$$

Given $N_0(T)$, $N_1(T)$ and $N_2(T)$, $S_1(T)$ and $S_2(T)$ will still have the log-normal distribution so that our lower bound can be computed.

Proposition 19. *If we denote by $\hat{p}(x_1, x_2, \sigma_1, \sigma_2, \rho)$ our lower bound computed with log-normal distributions, the price of the spread option in Merton's model is*

$$\hat{p}^{jumps} = e^{-(\lambda_1+\lambda_2+\lambda_3)T} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(\lambda_1 T)^i(\lambda_2 T)^j(\lambda_3 T)^k}{i!j!k!} \hat{p}\left(\tilde{x}_1, \tilde{x}_2, \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\rho}\right) \tag{22}$$

with

$$\begin{aligned}
\tilde{x}_1 &= x_1 e^{-(\lambda_0\mu_0+\lambda_1\mu_1)T + i\left(m_1+s_1^2/2\right) + k\left(m_0+s_0^2/2\right)} \\
\tilde{x}_2 &= x_2 e^{-(\lambda_0\mu_0+\lambda_2\mu_2)T + j\left(m_2+s_2^2/2\right) + k\left(m_0+s_0^2/2\right)} \\
\tilde{\sigma}_1 &= \sqrt{\sigma_1^2 + (is_1^2 + ks_0^2)/T} \\
\tilde{\sigma}_2 &= \sqrt{\sigma_2^2 + (js_2^2 + ks_0^2)/T} \\
\tilde{\rho} &= \frac{\rho\sigma_1\sigma_2 + ks_0^2/T}{\sqrt{\sigma_1^2 + (is_1^2 + ks_0^2)/T}\sqrt{\sigma_2^2 + (js_2^2 + ks_0^2)/T}}
\end{aligned}$$

Proof. Formula (22) is obtained by conditioning on the numbers of jumps $N_0(T)$, $N_1(T)$ and $N_2(T)$ prior to T and by computing simple Gaussian integrals to derive the values of the parameters $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{\sigma}_1$, $\tilde{\sigma}_2$, and $\tilde{\rho}$ to use in (22). ∎

7.2. **Implied correlation.** As a simple illustration the tractability of our method, we compute the implied correlation in this model. It is well-known in the one-dimensional case that jumps in the stock price dynamics give rise to the so-called volatility smile. In other words the volatilities that need to be plugged in in the Black-Scholes' formula to give the observed prices vary when strike or time-to-maturity change. In our case, we compute the price of spread options for different strikes and maturities when the underlying model is a jump-diffusion one. Then, we look for the correlation parameter that would give the same price. Note that we have here to decide with which volatility parameters our trader is inverting our formula. Since we have assumed that he is pricing spread options without knowing that the underlying process has jumps, it is not fair to assume that he has access to the parameters σ_1 and σ_2. However, we can assume that he has observed the stock prices and that he can estimate the variance of their log-return based on these data. In the case when there were no jumps, this would lead to an unbiased estimator of the volatility. He will therefore take

$$\hat{\sigma_i} = \sqrt{\sigma_i^2 + \lambda_0(s_0^2 + m_0^2) + \lambda_i(s_i^2 + m_i^2)}$$

as inputs in his pricing formula.

We plot the implied volatility in Figure 4 and the implied correlation 5 for a given set of data (Table 3). It is interesting to note that the two cases we are considering lead to two radically different correlation smiles.

Independent jumps have the tendency to decrease the implied correlation for out-of-the-money options. At-the-money options have a relatively constant implied correlation. Note that since the two jump processes we are adding are independent and independent of the Brownian motions, they tend to decrease the correlation between the two assets. Let us now look at the term structure of the implied correlation. It is increasing in maturity for both negative strikes and positive strikes whereas it is relatively constant for small strikes.

On the other hand, simultaneous jumps tend to increase the implied correlation for out-of-the-money options. The implied correlation for at-the-money options is extremely flat is it should. The implied correlation also exhibit a very different term structure behavior as it is decreasing for both negative strikes and positive strikes. Like in the previous case it flattens for long maturities.

The volatility and correlation structure is consistent with that found by Mbafeno in [5] in the case of independent jumps, which is reasonable to assume here as we have already argued. The implied correlation is computed there, following Shimko, via

$$\hat{\rho} = \frac{\sigma_{spread}^2 - \sigma_{1,imp}^2 - \sigma_{2,imp}^2}{2\sigma_{1,imp}\sigma_{2,imp}}$$

where σ_{spread} is the volatility of the spread (computed through the historical variance of the spread) and $\sigma_{1,imp}$, $\sigma_{2,imp}$ are the implied volatilities of each of the assets. With the data given in Table 3, such an implied correlation for a spread option with maturity T and strike K would be given by

$$\hat{\rho}(K,T) = \frac{\sigma_{spread}^2 - \sigma_{1,imp}^2(x_2 - K, T) - \sigma_{2,imp}^2(x_1 + K, T)}{2\sigma_{1,imp}(x_2 - K, T)\sigma_{2,imp}(x_1 + K, T)}.$$

As we can see in Figure 6, this clearly underestimates the implied correlation (in a jump-diffusion world) but has a similar time-to-maturity structure for deep in- or out-of-the-money spread options.

	Asset 1	Asset 2
x	100	100
σ	15%	10%
λ	0.2	0.01
m	-0.04	-0.03
s^2	0.01	0.01
$\hat{\sigma}$	15.75%	10.05%

TABLE 3. Model data together with $\rho = 70\%$, $r = 5\%$ and $\lambda_0 = 0$.

8. CONCLUSION

This paper discussed a new pricing paradigm for spread options. The originality of this article relies on the introduction of this new algorithm to compute accurate prices for options written on the difference of two underlying indexes. We compared the performance of our algorithm to some of the

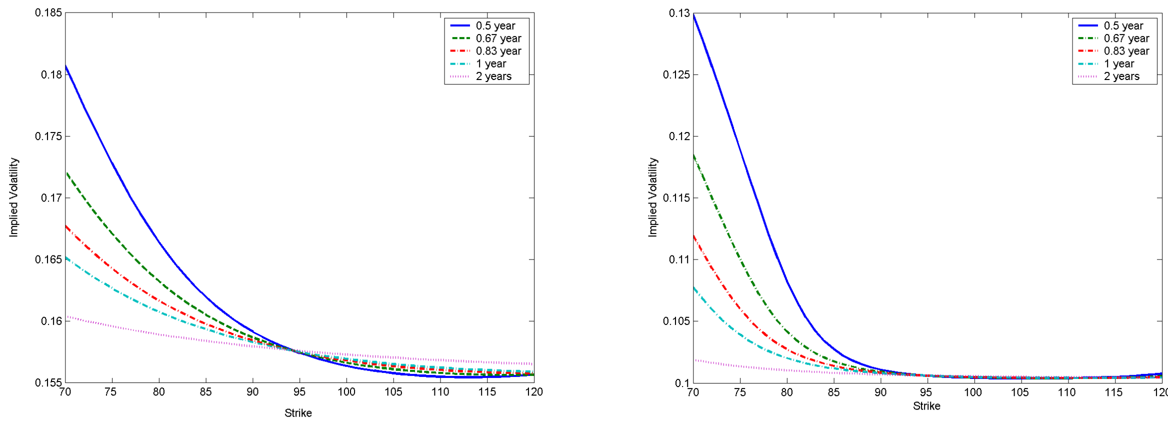

FIGURE 4. Implied volatility smile for a call option on the first asset (left) and on the second one (right.) The data are given in Table 3.

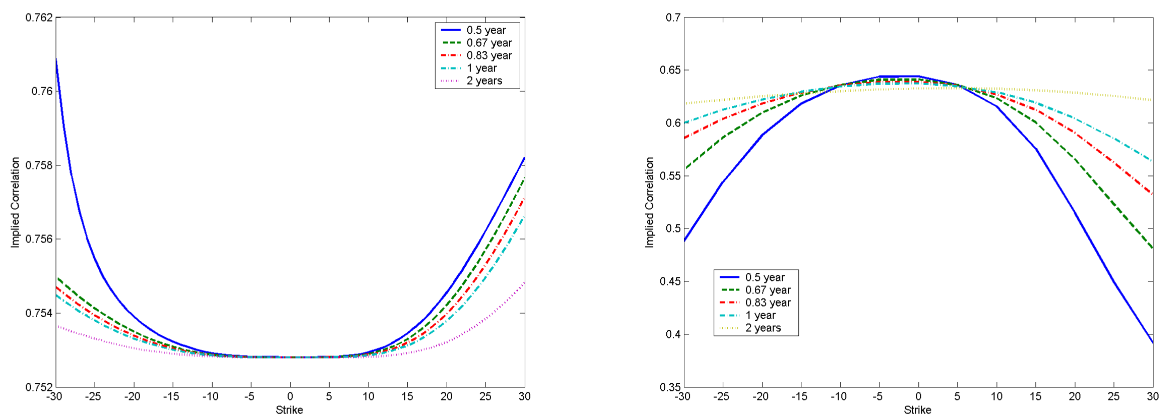

FIGURE 5. On the left panel: implied correlation smile for a spread option in the case of perfectly correlated jumps. The data are given in Table 3. On the right panel is the case where the jumps are independent. In that case $\lambda_0 = 0.2$, $m_0 = -0.1$ and $s_0 = 0.01$.

popular existing methods, both from a numerical and an analytical points of view. The advantages of our method are clear: its implementation is easy, run time is short, and its results are very accurate. Moreover, along with the price we can easily compute all the sensitivities, so that hedges and exposures are precisely known. In fact, all the hedging techniques derived from the Black-Scholes' formula can be implemented in our framework. Finally, unlike with the other approximations (whether they are numerical or analytical) we always know on which side of the true price we are.

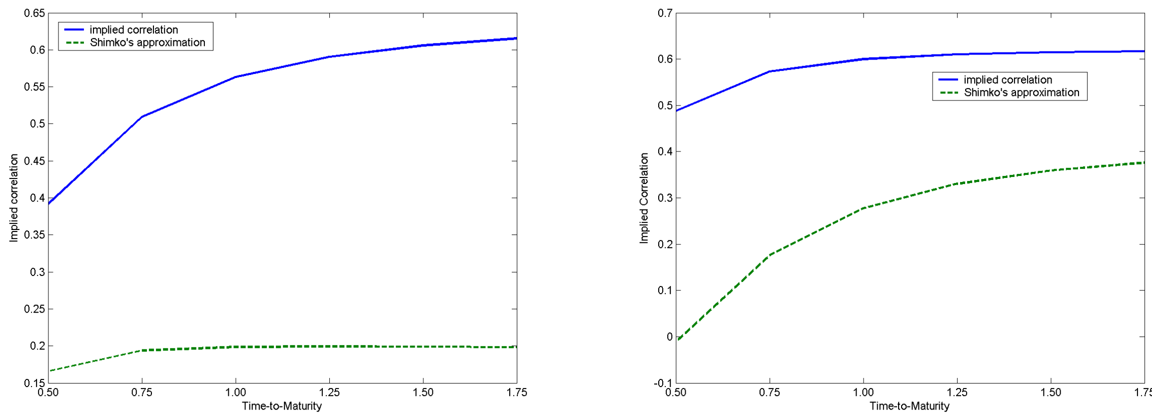

FIGURE 6. Comparison of Shimko's implied correlation with our jump-diffusion implied correlation in case of a positive strike (left) and a negative one (right).

As an added bonus, our formulae lead to efficient computions of implied quantities such as volatilities and correlation so that one is now able to build more complex models (stochastic volatility and/or correlation, jumps). These improved models will hopefully better fit the market reality. We believe that a numerical study of the implied correlation based on our results can lead to a better understanding of this quantity.

Finally, we would like to mention some possible extensions of our results. Our method can clearly be generalized to the case of an option on a linear combination of assets (basket, rainbow options . . .) or any linear combination of prices of a single asset at different times (discrete-time average Asian option for example.) Such extensions would provide efficient algorithms to compute prices and hedges for these options.

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