PRICING AND HEDGING SPREAD OPTIONS

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ABSTRACT. We survey the theoretical and the computational problems associated with the pricing of spread options. These options are ubiquitous in the financial markets, whether they be equity, fixed income, foreign exchange, commodities, or energy markets. As a matter of introduction, we present a general overview of the common features of all the spread options by discussing in detail their roles as speculation devices and risk management tools. We describe the mathematical framework used to model them and we review the numerical algorithms used to actually price and hedge them. There is already an extensive literature on the pricing of spread options in the equity and fixed income markets, and our contribution there is mostly to put together material scattered across a wide spectrum of recent text books and journal articles. On the other hand, information about the various numerical procedures which can be used to price and hedge spread options on physical commodities is more difficult to find. For this reason, we make a systematic effort to choose examples from the energy markets in order to illustrate the numeric challenges associated with these instruments. This gives us a chance to venture in the poorly understood world of asset valuation and real options which are the object of a frenzy of active mathematical research. In this spirit, we review the two major avenues to modeling energy prices dynamics, and we explain how the pricing and hedging algorithms can be implemented both in the framework of models for the spot prices dynamics as well as for the forward curves dynamics.

1. INTRODUCTION

Whether the motivation comes from speculation, basis risk mitigation, or even asset valuation, the use of spread options [1] is widespread despite the fact that the development of pricing and hedging techniques has not followed at the same pace. These options can be traded on an exchange, but the bulk of the volume comes from over the counter trades. They are designed to mitigate adverse movements of several indexes, hence their popularity. Because of their generic nature, spread options are used in markets as different as the fixed income markets, the currency and foreign exchange markets, the commodity futures markets and the energy markets.

One of our goals is to review the literature existing on the subject, including a self-contained discussion of all the pricing and hedging methodologies known to us. We implemented all the pricing algorithms whose existence we are aware of, and for the purpose of comparison, we report on their numerical performances and we give evidence of their relative accuracy and computing times.

As evidenced by the title of the paper, we intend to concentrate on the energy markets. Standard stock market theory relies on probabilistic models for the dynamics of stock prices, and uses arbitrage arguments to price derivatives. In most models, futures and forward contract prices are simply the current (spot) price of the stock corrected for growth at the current interest rate. This simple

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[1]The spread option is a set play in American football, and a lot of write ups have been devoted to its analysis and to its merits. Despite its importance in the life of football fans, we shall ignore this popular type of spread option and concentrate instead on the analysis of the spread options traded in the financial markets.

relationship between spot and forward prices does not hold in the commodity markets, and we will repeatedly mention seasonality and mean reversion as main culprits. In order to reconcile the physical commodity market models with its equity relatives, researchers have used several tricks to resolve this apparent anomaly, and consistency with the no-arbitrage theory is restored most often by adding cost of storage and convenience yields to the stochastic factors driving the models. See, *e.g.*, [20], [8], [41], [24] and [36]. But the main limitation of these methods is the inherent difficulties in modeling these unobserved factors (storage costs and especially convenience yield for example) and the proposal to use stochastic filtering techniques to estimates them, even though very attractive, did not fully succeed in resolving these problems.

Beyond the synthesis of results from a scattered technical literature, our contribution to the subject matter is the introduction of a new pricing algorithm based on closed form formulae providing lower bounds to the exact values of the spread options when the distributions of the underlying indexes are log-normal. We construct our approximate prices rigorously, we derive all the formulae necessary to the numerical implementation of our algorithm, and we demonstrate its efficiency on simulations and practical examples. All of the practical applications considered in this paper for the sake of illustration are from the energy markets.

The energy markets have seen rapid changes in the last decade, mostly because of the introduction of electricity trading and the restructuring of the power markets. The diversity in the statistical characteristics of the underlying indexes on which the financial instruments are defined, together with the extreme complexity of the derivatives traded, make the analysis of these markets an exciting challenge to the mathematically inclined observer. This paper came out of our curiosity in these markets, and our desire to better understand their idiosyncrasies. The reader is referred to [15] for a clear initiation to the intricacies of the energy markets, and to the recent texts [26], [43] and [3] for the economic and public policy issues specific to the electricity markets. But our emphasis will be different since we are only concerned with the technical aspects of energy trading and risk management. Several textbooks are devoted to the mathematical models of and risk management issues in the energy markets. The most frequently quoted are [37] and [7], but this may change with the publication of the forthcoming book [16]. Even though this paper concentrates on the spread options and [7] does not deal with cross-commodity instruments, we shall use many of the models and procedures presented in [7]. We close this introduction with a summary of the contents of the paper.

The paper starts with a review of the various forms of spread options in Section 2. We give examples of instruments traded in the equity, fixed income and commodity markets. Understanding this diversity is paramount to understanding the great variety of mathematical models and of pricing recipes which have appeared in the technical literature. In preparation for our discussion of the practical examples discussed in the last sections of the paper, we devote Section 3 to a detailed discussion of the data available to the energy market participants. The special characteristics of these data should not only justify the kind of notation and assumptions we use, but it should also serve as a yardstick to quantify how well the pricing algorithms do.

The mathematical framework for risk-neutral pricing of spread option is introduced in Section 4. Even though most of the spread options require only the statistics of the underlying indexes at one single time, namely the time of maturity of the option, these statistics are usually derived from a model of the time evolution of the values of the indexes. See nevertheless our short discussion of the calendar spreads where the joint distribution of the same underlying index at different times is needed. We introduce the stochastic differential equations used to model the dynamics of the underlying indexes. The price of a spread option is given by an expectation over the sample paths of the solution of this system of stochastic differential equations. One usually assumes that the coefficients of the stochastic differential equations are Markovian. In this case, the price is easily seen to be the solution of a parabolic partial differential equation. This connection between solutions of stochastic differential equations and solutions of partial differential equations is a cornerstone of Ito's stochastic calculus, and it has been exploited in many financial applications. Only in very exceptional situations do these equations have solutions given by closed form formulae. PDE solvers, tree methods and Monte Carlo methods are most commonly used to produce numerical values approximating the price of a spread. Because the applied mathematics community is more familiar with PDE solvers than with the other two, we spend more time reviewing the tree and Monte Carlo methods and the specifics of their implementations in the pricing of spread options. Also, we address the issue of the quantification of the dependencies of the price with respect to the various parameters of the model. We emphasize the crucial role of these sensitivities in a risk management context by explaining their roles as hedging tools.

As seen from the discussion of that section, the model of utmost importance is the Samuelson's model in which the distributions at the time of maturity of the indexes underlying the spread option are log-normal. We shall concentrate most of our efforts in understanding the underpinnings of this assumption on the statistics of the indexes.

The following Section 5 presents the first approximation procedure leading to a full battery of closed form expressions for the price and the hedging portfolios of a spread option with general strike price K. It is based on a simple minded remark: as evidenced by a quick look at empirical samples, the distribution of the difference between two random variables with log-normal distributions looks pretty much like a normal distribution. This is the rationale for the first of the three approximation methods which we review. In this approach, one refrains from modeling the distributions of the indexes separately and instead, one models the distribution of their difference. As we just argued, it is then reasonable to assume that the latter is normally distributed. This model is called the Bachelier model because it is consistent with a model of the spread dynamics based on a single Brownian motion, in the same way Bachelier originally proposed to model the dynamics of the value of a stock by a continuous time process generalizing the notion of random walk. Little did he knew he was ahead of Einstein introducing the process of Brownian motion. We give a complete analysis of this model. We derive explicit formulae for the option prices in the original form of the model, and when the model is adjusted for consistency with observed forward curves. In this section, we also examine in detail the numerical performance of the pricing formula, by comparing its results to the exact values when the driving dynamics are actually given by geometric Brownian motions as in the Samuelson's model which we study next.

In Section 6 we turn our attention to the particular case of a spread option with log-normal indexes and strike $K = 0$. Like in the case of the Bachelier's model, it is possible to give a Black-Scholes type formula for the price of the option. This formula was first derived by Margrabe in [33]. It cannot be extended to the general case $K > 0$, and this is the main reason for the investigations which we review in this paper. Besides the fact that the case $K = 0$ leads to a solution in closed form, it has also a practical appeal to the market participants. Indeed, it can be viewed as an option to exchange a product for another. Let us imagine for the sake of illustration, that we are interested in owning at a given time T in the future, either one of two instruments whose prices at time t we denote by $S_1(t)$ and $S_2(t)$, and that the choice of which one to buy has to be done now at time $t = 0$. If we fear that the difference in price may be significant at time T, choosing the second instrument and buying the spread option with strike $K = 0$ is the best way to guarantee that we will end up financially in the same situation as we had chosen in hindsight the instrument which will end up been the cheaper at time T. The only cost to us will be the purchase of the spread option. Indeed, if the second instrument ends up being the most expensive, i.e. if $S_2(T) > S_1(T)$, then the pay-off $S_2(T) - S_1(T)$ of the option will compensate us for our wrong choice.

The main mathematical thrust of the paper is contained in Section 7 where we review the recent results of Carmona and Durrleman [5], and where we compare the numerical performance of their method to the approximations based on the Bachelier's approach and the Kirk's approximate pricing formula. The basic problem is the pricing and hedging of the simplest spread option (*i.e.*, an European call option on the difference of two underlying indexes) when the risk-neutral dynamics of the values of the underlying indexes are given by correlated geometric Brownian motions. The results of Carmona and Durrleman are based on a systematic analysis of expectations of functions of linear combinations of log-normal random variables. The motivation for this analysis comes from the growing interest in basket options, whose pricing involves the computation of these expectations when the number of log-normal random variables is large. These products are extremely popular, as they are perceived as a safe diversification tool. But a rigorous pricing methodology is still missing. The authors of [5] derive lower bounds in closed form, and they propose an approximation to the exact value of these expectations by optimizing over these lower bounds. The performance of their numerical scheme is always as good as the results of Kirk's formula. But the main advantage of their approach is the fact that it also provides a set of approximations for all the sensitivities of the spread option price, an added bonus making possible risk management at the same time. We review the properties of these approximations, both from a theoretical and a numerical point of view by quantifying the accuracy on numerical simulations. The reader interested in detailed proofs and extensive numerical tests is referred to [5].

The geometric Brownian motion assumption of the Samuelson's theory is not realistic for most of the spread options traded in the energy markets. Indeed, most energy commodity indexes have a strong seasonal component, and they tend to revert to a long term mean level, this mean level having the interpretation of *cost of production.* These features are not accounted for by the plain geometric Brownian motion model of Samuelson. Section 8 deals with the extension of the results of Section 7 to the case of spread options on the difference of indexes whose risk-neutral dynamics include these features. We also show how to include jumps in the dynamics of these indexes. This is motivated by the pricing of spark spread options which involve electricity as one of the two underlying indexes, or the pricing of calendar spread options on electric power.

Up until Section 9, we only work with stochastic differential equation models for the indexes underlying the spread. In the case of the energy markets, the natural candidates for these underlying indexes are the commodity spot prices, and these models are usually called spot price models. See *e.g.*, Chapters 6 and 7 of [7]. According to the prevailing terminology, they are one-factor models for the term structure of forward prices. But it should be emphasized that our analysis extends easily, and without major changes, to the multi-factor models, at least as long as the distributions of the underlying indexes can be constructed from log-normal building blocks. This is the case for most of the models used in the literature on commodity markets. See, *e.g.*, [20], [8], [41], [24], [36] or [37] and [7].

Most of the energy commodities do not behave much differently than the other physical commodities. They share the mean reversion feature which we will mention quite often in this paper, but surprisingly enough, some do not exhibit much seasonality. This is the case for crude oil for example. But beyond natural gas whose historical data are readily available and which exhibits strong seasonality and mean reversion, one of these commodities does stand out because of its very special features: electric power. Indeed, its price is function of factors as diverse as 1) instant perishableness, 2) strong demand variations due to seasonality and geographic location, 3) extreme volatility and sudden fluctuations caused by weather changes in temperature, precipitation, . . . 4) physical constraints in production (start-ups, ramp-ups) and transmission (capacity constraints). It is by far the most difficult commodity index to model and predict. Derivative pricing and risk management present challenges of a new dimension: but what appears to be a nightmare for policy makers and business executives, is in fact a tremendous opportunity for the academic community, and the need for realistic mathematical models and rigorous analytics is a very attractive proposition for the scientific community at large.

The last section of the paper is concerned with forward curve models. Using ideas from the HJM theory developed for the fixed income markets, the starting point of Section 9 is a set of equations for the stochastic dynamics of the entire forward curve. This is a departure from the approach used in the previous sections, where the dynamics of the spot prices were modeled, and where the consistency with the existing forward curves was only an after thought. We give a detailed account of the fitting procedure based on Principal Components Analysis (PCA for short) and we illustrate the numerical performance of this calibration method using real data. Restricting the coefficients of the stochastic differential equations to be deterministic leads again to log-normal distributions and the results reviewed in this paper can be applied. We show how to price calendar spreads and spark spreads in this framework.

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2. Zoology of the Spread Options

Even though it is sometime understood as the difference between the bid and ask prices (for example one often says that liquid markets are characterized by narrow bid/ask spreads), the term spread is most frequently used for the difference between two indexes: the spread between the yield of a corporate bond and the yield of a Treasury bond, the spread between two rates of returns, . . . are typical examples. Naturally, a spread option is an option written on the difference between the values of two indexes. But as we are about to see, its definition has been loosened to include all the forms of options written on a linear combination of a finite set of indexes. In the currency and fixed income markets, spread options are based on the difference between two interest or swap rates, two yields, ... In the commodity markets, spread options are based on the differences between the prices of the same commodity at two different locations (location spreads), or between the prices of the same commodity at two different points in time (calendar spreads), or between the prices of inputs to, and outputs from, a production process (processing spreads) as well as between the prices of different grades of the same commodity (quality spreads). The New York Mercantile Exchange (NYMEX for short) offers the only exchange-traded options on energy spreads: the heating oil/crude oil and gasoline/crude oil crack spread options.

The following review is far from exhaustive. It is merely intended to give a flavor of the diversity of spread instruments in order to justify the variety of mathematical models and pricing algorithms found in the technical literature on spreads. In this paper, most of the emphasis is placed on cross-commodity spreads because of the tougher mathematical challenges they present. As we shall see, single commodity spreads (typically calendar spreads) are usually easier to price.

2.1. **Spread Options in the Currency and Fixed Income Markets.** Spread options are quite common in the foreign exchange markets where spreads involve interest rates in different countries. The French-German and the Dutch-German bond spreads are used because the economies of these countries are intimately related. A typical example is the standard cross-currency spread option which pays at maturity T the amount $(\alpha Y_1(T) - \beta Y_2(T) - K)^+$ in currency 1. Here α, β and K are positive constants, and we use the notation x^+ for the positive part of x, *i.e.*, $x^+ = \max\{x, 0\}$. The underlying indexes Y_1 and Y_2 are swap rates in possibly different currencies, say 2 and 3. The pricing of these spread options is usually done under some form of log-normality assumption via numerical integration of Margrabe formulae derived in Section 6 below. The more elaborate forms of this approach are used to price quanto-swaptions as described for example in [4].

In the US fixed income market, the most liquid spread instruments are spreads between maturities, such as the NOB spread (Notes - Bonds) and spreads between quality levels, such as the TED spread (Treasury Bills - EuroDollars). The MOB spread measures the difference between Municipal Bonds and Treasury Bonds. See [1] for an econometric analysis of the market efficiency of these instruments. Spreads between Treasury Bills and Treasury Bonds have been studied in [29] and [14]. A detailed analysis of a spread option between the three months and the six months LIBOR's (London Inter Bank Overnight Rate) is given in [5] where some of the mathematical tools reviewed of this paper were introduced.

2.2. **Spread Options in the Agricultural Futures Markets.** There are several spread options traded in the agricultural futures markets. For the sake of definiteness, we decided to concentrate on the so-called *crush spread* traded on the Chicago Board of Trade (CBOT). It is also known as the *soybean complex spread.* The underlying indexes comprise futures contracts of soybean, soybean oil and soybean meal. The unrefined product is the soybean, and the derivative products are the meal and the oil. This spread is known as the *crush spread* because soybeans are processed by crushing. The soybean crush spread is defined as the value of meal and oil extracted from a bushel of soybeans, minus the price of a bushel of soybeans. Notice that the computation of the spread requires three prices as well as the yield of oil and meal per bushel. The crush spread gives market participants an indication of the average gross processing margin. It is used by processors to hedge cash positions or for pure speculation.

The crush spread relates the cash market price of the soybean products (meal and oil) to the cash market price of soybeans. Since soybeans, soybean meal, and soybean oil are priced differently, conversion factors are needed to equate them when calculating the spread. On the average, crushing one bushel (*i.e.*, 60 pounds) of soybeans produces 48 pounds of meal and 11 pounds of oil. Consequently, the value $[CS]_t$ at time t of the crush spread in dollars per bushel can be defined as:

$$[CS]_t = 48[SM]_t/2000 + 11[SO]_t/100 - [S]_t \tag{1}$$

where $[S]_t$ is the futures price at time t of a soybean contract in dollars per bushel, $[SO]_t$ is the futures price at time t of a contract of soy oil in dollars per 100 pounds, and $[SM]_t$ is the price at time t of a soy meal contract in dollars per ton. In the terminology and the notation introduced below for the crack spreads, the spread described above is a 10:11:9 spread, *i.e.*, 10 Soybean futures, 11 Soybean Meal futures, and 9 Soybean Oil futures. So if we think of the crushing cost as a constant K, then crushing soybeans is profitable when the spread $[CS]_t$ is greater than K. The crush spread was analyzed from the point of view of market efficiency in [28].

2.3. **Spread Options in the Energy Markets.** In the energy markets, beside the *temporal spread* traders who try to take advantage of the differences in the prices of the same commodity at two different dates in the future, and the *locational spread* traders who try to hedge transportation/transmission risk exposure from futures contracts on the same commodity with physical deliveries at two different locations, most of the spread traders deal with at least two different physical commodities. In the energy markets spreads are typically used as a way to quantify the cost of production of refined products from the complex of raw material used to produce them. The most frequently quoted spread options are the crack spread options and the spark options which we review in detail in this section. Crack spreads are often called paper refineries while spark spreads are sometimes called paper plants.

Crack Spreads. A *crack spread* is the simultaneous purchase or sale of crude against the sale or purchase of refined petroleum products. These spread differentials which represent refining margins are normally quoted in dollars per barrel by converting the product prices into dollars per barrel and subtracting the crude price. They were introduced in October 1994 by the NYMEX with the intend to offer a new risk management tool to oil refiners.

For the sake of illustration, we describe the detailed structure of the most popular crack spread contracts. These spreads are computed on the daily futures prices of crude oil, heating oil and unleaded gasoline.

- The $3:2:1$ **crack spread** involves three contracts of crude oil, two contracts of unleaded gasoline, and one contract of heating oil. Using self-explanatory notation, the defining formula for such a spread can be written as:

$$[CS]_t = \frac{2}{3}[UG]_t + \frac{1}{3}[HO]_t - [CO]_t \tag{2}$$

  which means that at any given time t, the value (in US \$) $[CS]_t$ of the $3:2:1$ crack spread underlying index is given by the right hand side of formula (2) where $[UG]_t$, $[HO]_t$ and $[CO]_t$ denote the prices at time t of a futures contract of unleaded gasoline, heating oil and crude oil respectively. A modicum of care should be taken in the numerical implementation of formula (2) with real data. Indeed, crude oil prices are usually quoted in "dollars per barrel" while unleaded gasoline and heating oil prices are quoted in "dollars per gallon". A simple conversion needs to be applied to the data using the fact that there are 42 gallons per barrel, but it should not be overlooked.

- The $1:1:0$ **gasoline crack spread** involves one contract of crude oil and one contract of unleaded gasoline. Its value is given by the formula:

$$[GCS]_t = [UG]_t - [CO]_t \tag{3}$$

- The $1:0:1$ **heating oil crack spread** involves one contract of crude oil and one contract of heating oil. It is defined as:

$$[HOCS]_t = [HO]_t - [CO]_t \tag{4}$$

Notice that the first example is computed from three underlying indexes while the remaining two examples involve only two underlying indexes. Most of our analysis will concentrate on spread options written on indexes computed from two underlying indexes.

Crack spread options are the subject of a large number of papers attempting to demonstrate the stationarity of the spread time series by means of a statistical quantification of the co-integration properties of underlying index time series comprising the spread. Most of these papers are also concerned with the profitability of spread based trading strategies, subject which we would not dare to consider here. The interested reader is referred for example to [21], and [22] and the references therein for further information on these topics.

Spark Spreads. A *spark spread* is a proxy for the cost of converting a specific fuel (most of the time, natural gas) into electricity at a specific facility. It is the primary cross-commodity transaction in the electricity markets. Mathematically, it can be defined as the difference between the price of electricity sold by a generator and the price of the fuel used to generate it, provided these prices are expressed in appropriate units. The most commonly traded standardized contracts include:

- The $4:3$ **spark spread** involves four electric contracts and three contracts of natural gas. Its value is given by:

$$[SS]^{4,3}_t = 4[E]_t - 3[NG]_t \tag{5}$$

- The $5:3$ **spark spread** which involves five electric contracts and three contracts of natural gas. Its value is given by:

$$[SS]^{5,3}_t = 5[E]_t - 3[NG]_t. \tag{6}$$

But whether or not they are traded in this form, the most interesting spread options are European calls on an underlying index of the form:

$$S_t = F_E(t) - H_{eff} F_G(t)$$

where $F_E(t)$ and $F_G(t)$ denote the prices of futures contracts on electric power and natural gas respectively, and where H_{eff} is the heat rate, or the efficiency factor of a power plant. One of the most intriguing use of spark spread options is in real asset valuation or capacity valuation. This encapsulates the economic value of the generation asset used to produce the electricity. The spark spread can be expressed in \$/MWh (US dollar per Mega Watt hour) or any other applicable unit. It is calculated by multiplying the price of gas, (for example in \$/MMBtu), by the *heat rate* (in Btu/KWh), dividing by $1,000$, and then subtracting the electricity price (in \$/MWh). The heat rate is often called the *efficiency*. Indeed, a natural gas fired unit can be viewed as a series of spark spread options:

- when the heat rate implied by the spot prices of power and gas is above the operating heat rate of the plant, then the plant owner should buy gas, produce power, and sell it for profit.

- the plant owner should shut down its operation otherwise, *i.e.*, when the heat rate implied by the spot prices of power and gas is below the operating heat rate of its plant.

If an investor/producer wonders how much to bid for a power plant, he can easily estimate and predict the real estate and the hardware values of the plant with standard methods. But the operational value of the plant is better captured by the sum of the prices of spark spread options, than with the present value method based on the computation of discounted future cash flows (the so-called DCF method in the jargon of the business.) This *real option* approach to the plant valuation is one of the strongest incentive to develop a better understanding of the basis risk of spark spread options.

3. Market Data

The purpose of this section is to demonstrate why some of the financial derivatives used in the energy markets need to be treated with a modicum of care, by which we mean that applying blindly the tools developed for the equity or fixed income markets may not be appropriate. Most of the mathematical models used in the equity markets are based on generalizations of the geometric Brownian motion model first proposed by Samuelson. We shall use these models in several instances, mostly for the sake of completeness since they are not of great use in the applications we consider. Energy market models bare more resemblance to the models for the fixed income markets where there is a division between the models for the dynamics of the short interest rate, and the models for the dynamics of the entire yield curve. This dichotomy will appear below where we divide the energy markets models into two classes, the first one based on the dynamics the spot market prices, and the second class based on models for the dynamics of the entire forward curve. But in order to justify the specific assumptions we use, it is important to get a good understanding of the kind of data analysts, risk managers, traders, . . . are dealing with.

For most physical commodities, price discovery takes two different forms. The first one is backward looking. It is based on the analysis of a time series of historical prices giving the values observed in the past of the so-called *spot price* of the commodity. The spot market is a market where goods are traded for immediate delivery. Figure 1 shows a couple of examples of energy spot prices. The left panel of the figure displays the daily values of the propane gas spot price while the right panel contains a plot of the daily values of the Palo Verde firm on peak spot price. Obviously, these series do not look anything like stock prices or equity index values. The sudden increases in value and the high levels of volatility set them apart. But except for that, they have more in common with plots of instantaneous interest rates. Indeed, these series look more stationary than equity price series. This is usually explained by appealing to the *mean reversion* property of the energy prices which tend, despite the randomness of their evolution, to return to a local or asymptotic mean level. This mean reversion property is shared with interest rates. But the latter do not have the seasonality structure which appears in Figure 1. Gas prices are higher during the winters because of heating needs in the northern hemisphere, and slightly higher in the summer as well. Moreover, energy prices are much more volatile than equity prices. But as we already noticed, the important singularity which sets these data apart is the extreme nature of the fluctuations. This is obvious on the plot of the electricity spot price given in Figure 1.

For the sake of simplicity, we shall only consider *daily* time series in this paper, but these high levels of volatility are also found in hourly data. Except for the special case of the electric power markets, working with daily data is not a restrictive assumption. Indeed, most energy price quotes are

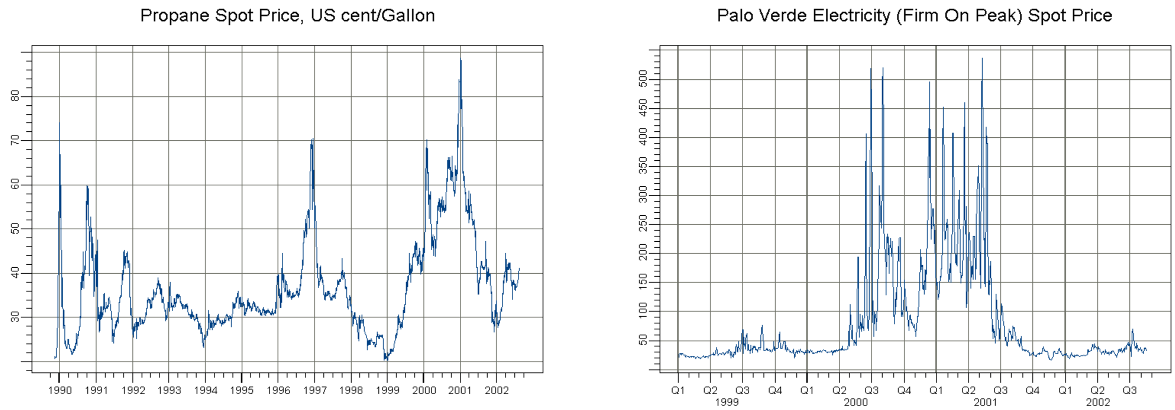

FIGURE 1. Time series plots of two energy commodity daily spot prices: propane (left) and Palo Verde firm on peak electric power (right).

recorded as "daily close", and using daily time series is quite appropriate. But electricity prices are very different. They have significant variations at scales much smaller than one day, and as a result prices are quoted more frequently (hourly, or even every half hour), and distinctions are made between *on-peak* and *off-peak* periods, weekdays and weekends, ... But one of the main distinctive feature of the power markets remains its inelasticity: the fact that for all practical purposes electricity cannot be stored in a flexible manner, hinders rapid responses to sudden changes in demand, and wild price fluctuations can follow. The weather is one of the culprits. Indeed, changes in the temperature affect the demand for power (*load*), creating sudden bursts in price volatility. The analysis of electricity prices at higher frequencies is a challenging problem which we will not consider in this paper. We are interested in multi-commodity instruments, and for this reason, we will restrict ourselves to daily sampling of the prices.

But like in the fixed income markets, another type of data is to be reckoned with. These data encapsulate the current market expectations for the future evolution of the prices. On any given day t, we have at our disposal the prices of a wide range of forward and/or future energy contracts. These contracts guarantee the delivery of the commodity at a given location and at a given date, or over a given period in the future. For the sake of simplicity, we shall assume that the delivery takes place at a given date which we denote by T and which we call the date of maturity of the contract. As in the case of the yield curve, or the discount rate curve, or the instantaneous forward rate curve used in the fixed income markets, the natural way to model the data is to assume the existence for each day t, of a function $T \hookrightarrow F(t,T)$ giving the price at time t of a forward/futures contract with maturity date T. Unfortunately, the domain of definition of the mathematical function $T \hookrightarrow F(t,T)$ changes with t. This is very inconvenient when it comes to statistical analysis of the characteristics of the forward curves. Even the more mundane issue of plotting becomes an issue because of that fact. A natural fix to that annoying problem is to parameterize the forward curves by the "time-to-maturity" $\tau = T - t$ instead of the "time-of-maturity" T. This simple suggestion showed far reaching consequences in the case of the fixed income models. We discuss below the advantages and the shortcoming of this new parametrization in the case of the energy markets.

The existence of a continuum of maturity dates T is a convenient mathematical idealization. In practice, on any given day t, the maturity dates of the outstanding contracts form a finite set $\{T_1, T_2, \cdots, T_n\}$, typically the first days of the n months following t for which forward/futures contracts are traded. The T_j's are often regularly spaced, one contract per month, and n is in the range 12 to 18 for most commodities, even though it can be as large as $7 \times 12 = 84$ as in the case of natural gas (even though n was not as large in the past.) Unfortunately, available data varies dramatically from one commodity to the other, from one location to another, even from one source to the other. And as one can easily imagine, historical data is often sparse, and sprinkled with erroneous entries and missing values.

Despite the data integrity problems specific to the energy markets, the main challenge remains the fact that the dates at which the forward curves are sampled vary from one day to the next. Let us illustrate this simple statement with an illustration. On day $t = 11/10/1989$ the $T_1 = Dec.89$, $T_2 = Jan.90$, $T_3 = Feb.90$, $\ldots$ contracts are open for trading, and quotes for their prices are available. For the sake of simplicity we shall not worry about the *bid-ask* spread, and we assume that a sharp price is quoted at which we can sell and/or buy the contract. In other words, on day $t = 11/10/1989$, we have observations of the values of the forward curve for the times-to-maturity $\tau_1 = 21$ days, $\tau_2 = 52$ days, $\tau_3 = 83$ days, $\cdots$. The following trading day is $t = 11/13/1989$, and on that day, we have observations of the prices of the same contracts with dates-of-maturity $T_1 = Dec.89$, $T_2 = Jan.90$, $T_3 = Feb.90$, $\ldots$, and we have now sample values of the forward curve for the times-to-maturity $\tau_1 = 18$ days, $\tau_2 = 49$ days, $\tau_3 = 80$ days, $\cdots$. Still the following trading day is $t = 11/14/1989$, and on that day we have observations of the prices of the same contracts but the values of the corresponding times-to-maturity are now $\tau_1 = 17$ days, $\tau_2 = 48$ days, $\tau_3 = 79$ days, $\cdots$. So the values τ_1, τ_2, $\cdots$, τ_n at which the forward function $\tau \hookrightarrow F(t, \tau)$ is sampled change from day to day. Even though the times of maturity T_1, T_2, T_3, $\cdots$ do not seem to vary with t in the above discussion, this is not so in general. Indeed when the date t approaches the end of the month of November, the December contract suddenly stops being traded and the nearest traded contract becomes January, and an extra month is added to the list. This switch typically takes place three to four days before the end of each month.

As we already mentioned, this state of affairs is especially inconvenient for plotting purposes and for the statistical analysis of the forward curves. So whenever we manipulate forward curve data, it should be understood that we pre-processed the data to get samples of these forward curves computed on a fixed set $\{\tau_1, \tau_2, \cdots, \tau_n\}$ which does not change with t. We do that by first switching to the time-to-maturity parametrization, then by *smoothing* the original data provided by the financial services, and finally by re-sampling the smoothed curve at the chosen sampling points. We sometimes fear that these manipulations are not always innocent, but we cannot quantify their influence, so we shall take their results for granted.

Figure 2 gives plots of the Henry Hub natural gas forward contract prices before and after such a processing. The left panel gives the raw data. Despite the rather poor quality of the plot, one sees clearly the structure of the data. Indeed, the domain of definition of the forward function $T \hookrightarrow F(t, T)$ is an interval of the form $[T_b(t), T_e(t)]$ where $T_b(t)$ is the date of maturity of the contract nearest to t and $T_e(t)$ is the maturity of the last contract quoted on day t. Hence, this domain of definition changes from day to day. In principle, the left hand point of the curve should give the spot price of the commodity. On any given day, the length of the forward curve depends upon the number of contracts traded on that day. Notice that in the case of natural gas displayed in Figure 2, the length recently went up to seven years. Also, the seasonality of the forward prices appears clearly on this plot. High ridges parallel to the time t-axis correspond to the contracts maturing in winter months when the price of gas is expected to be higher. The right panel of the figure also displays the natural gas forward surface, but the parametrization changed to the time-to-maturity $\tau = T - t$. There are nevertheless several obvious points to make. First, the forward curves are defined on the same time interval, and in particular, they have the same lengths which we chose to be three years in this particular instance. But the most noticeable change comes from the different pattern of the ridges corresponding to the periods with higher prices. Because of the parametrization by the time-to-maturity τ, the parallel ridges of high prices move toward the t axis when t increases, instead of remaining parallel to this time axis.

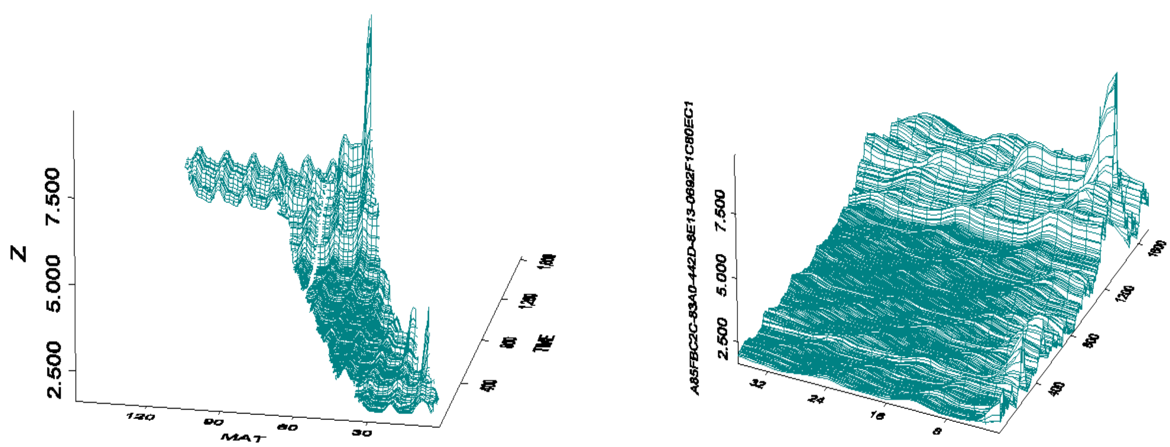

FIGURE 2. Surface plots of the historical time evolution of the forward curves of the Henry Hub natural gas contracts, in the time-of-maturity T parametrization (left) and in the time-to-maturity x parametrization (right).

Figure 3 shows the results of a similar processing in the case of the Palo Verde forward electric contracts. The simple linear interpolation procedure which we chose does not smooth much of the erratic behavior of the data, hence the rough look of these surfaces.

We would not want the reader to believe that we are proponents of a blind implementation of the parametrization of the forward curves by the "time-to-maturity" $\tau = T - t$ instead of the "time-of-maturity" T. Indeed, because of their physical nature, most energy commodities exhibit strong seasonality features, and the latter are more obvious in the time-of-maturity parametrization. This temporal nature of the physical commodities makes the time-to-maturity parametrization less helpful than in the fixed income markets.

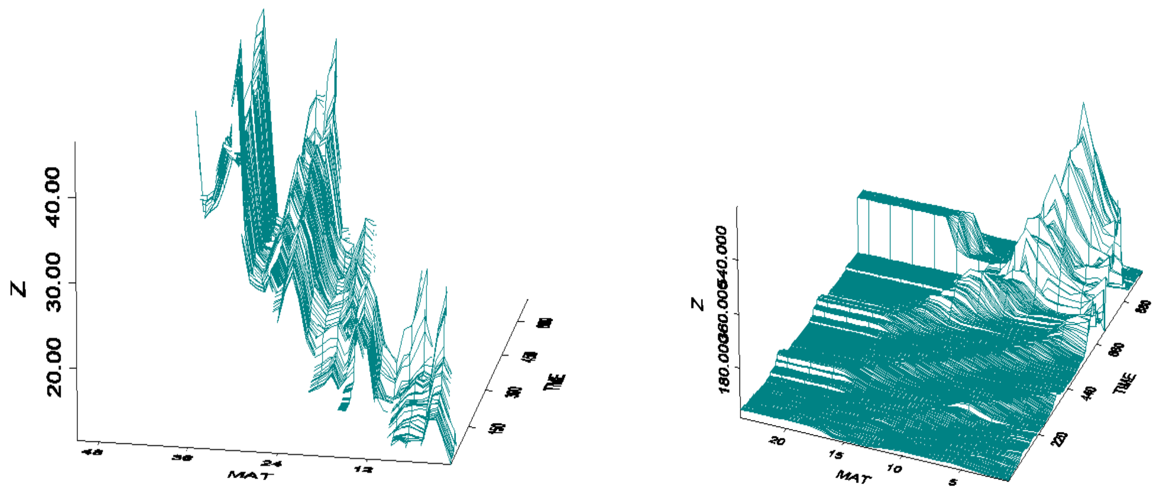

FIGURE 3. Surface plots of the historical time evolution of the forward price curves of the Palo Verde forward electricity contracts when plotted as functions of the time-of-maturity (left), and when plotted as functions of the time-to-maturity variable. We had to take a subset of the original period because of holes in the data due to missing values. In particular, the forward ridge for the long maturities in the recent days is an artifact of our re-sampling method given these missing values.

One can for instance be electricity delivery for the month of April 2006, another one for the month of November 2004. This fact although extremely natural is rather annoying for statistical analysis purposes. Indeed, at different times the market does not look the same. (For example, on April 1st and April 25th the future contract for the month of May will probably exhibit very different feature because the time-to-maturity is very different.)

To perform a sensible statistical analysis, we need some stationarity, that is we need to think of each day being identical to each other. We thus have at a given day t to interpolate between the different contract prices to get a time-to-maturity term structure. From now on, we will assume that we are given n such prices $F(t, \tau)$ where τ is the time-to-maturity (1 month, 2 months, 3 months, ...).

The following illustration is intended to show that despite our plea for considering seriously the effects of seasonality in the energy forward prices, it is important to keep in mind that not all the energy commodity prices have a strong seasonal component. The left panel of Figure 4 gives a surface plot of the crude oil forward prices from 11/10/1989 to 8/16/2002, as parameterized by the time-to-maturity of the contracts. Clearly, the bumps and the ridges indicative of seasonal effects are not present in this plot. The right panel of Figure 4 gives line plots of four crude oil forward curves. They have been chosen at random, and they are typical of what we should expect for crude oil forward curves: they are all monotone functions of the time-to-maturity. When a forward curve is monotone decreasing, the future prices of the commodity are expected to be lower than the current (spot) price:

we say the forward curve is in *backwardation.* When a forward curve is monotone increasing, the prices to come are expected to be higher than the spot price: we say the forward curve is in *contango.*

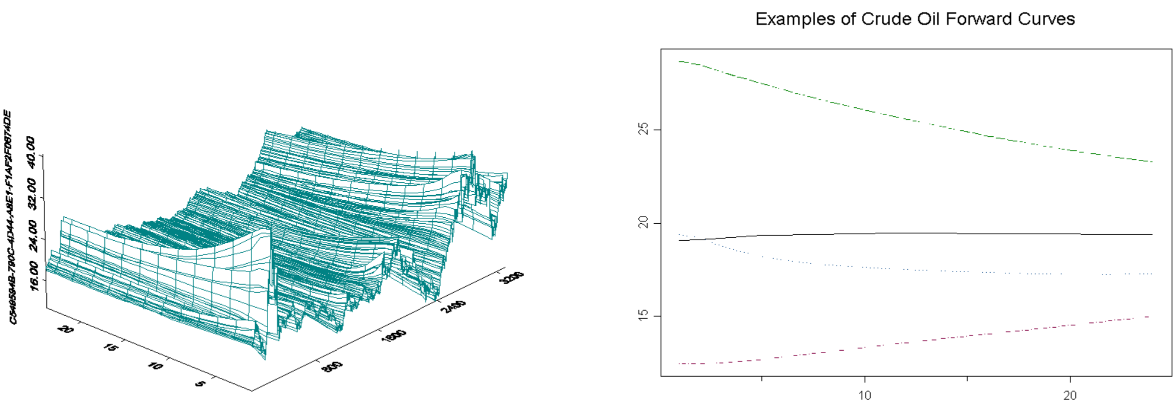

FIGURE 4. Surface plot of the crude oil forward prices from 11/10/1989 to 8/16/2002 (left), and four typical individual forward curves giving examples of forward curves in contango and in backwardation (right).

In the rest of the paper, we shall often discuss the consistency of a spot price model with the observed forward curves. This is done by computing the theoretical values of the forward curve from the model: indeed since we assume deterministic interest rates, and since we shall not model the convenience yield as a stochastic factor, the values of the forward contracts on any given day should be given by the conditional expectation of the future values of the spot prices. See for example [10] and [36] for details. At an intuitive level, this means that, at least in a least square sense, the values of the forward curve are nothing but the best predictors/guesses for the future values of the spot. The energy markets are so volatile that this fact has to be taken with a grain of salt. We chose the example of crude oil in order to illustrate this fact. We picked (essentially at random), 5 regularly spaced trading days separated by 200 trading days, and we superimposed the forward curves observed these days on the plot of the spot curve. The result is given in Figure 5. This graph demonstrates in a dramatic fashion how poor a predictor of the spot price the forward curve can be. The situation is not always as bad as our next illustration shows. Indeed, in stable periods of (relatively) low volatility, the forward curves can be a reasonable predictor of the future values of the spot prices. We illustrate this fact by plotting the forward curves of the Henry Hub natural gas contracts on the same five days we picked for the crude oil forward curves. As we can see in Figure 6, despite their greater lengths, there is a certain consistency between the forward curves in the tranquil periods. But still, they missed completely the sharp price increase of the 2000 crisis.

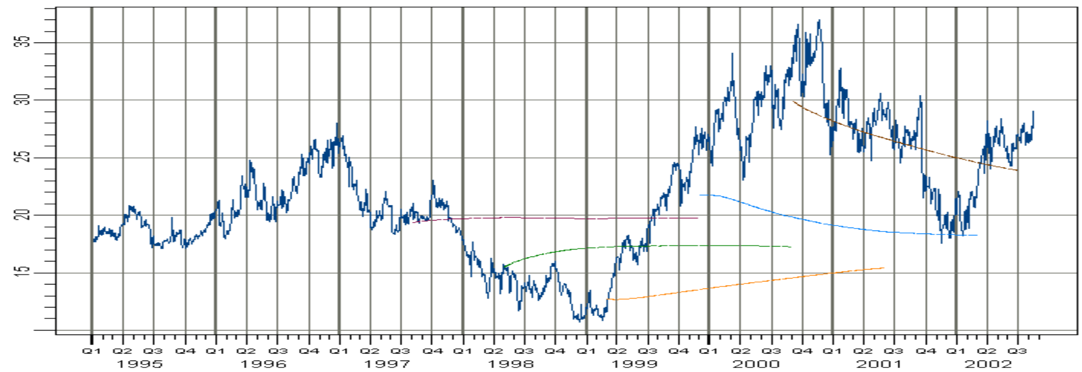

FIGURE 5. Crude Oil spot with a small set of forward curves superimposed to illustrate how poor a predictor of the spot prices can the forward curves be.

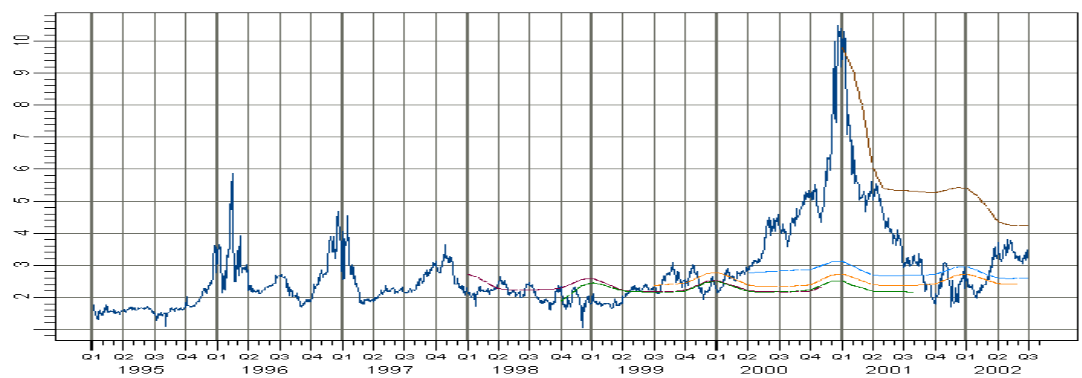

FIGURE 6. Henry Hub natural gas spot price with the forward curves computed on the same days as in the above crude oil example.

4. SPREAD OPTIONS PRICING: MATHEMATICAL SET-UP

In this section we introduce the standard definitions and the technical notation which we will use throughout the paper. For the sake of simplicity we restrict ourselves to the case of spread between two underlying asset prices, leaving the discussion of the more general case of the linear combinations used for the so-called basket options to side remarks. So we consider two indexes $S_1 = \{S_1(t)\}_{t\geq 0}$ and $S_2 = \{S_2(t)\}_{t\geq 0}$ evolving in time. We call them indexes instead of prices because, even though $S_1(t)$ and $S_2(t)$ will usually be the prices of stocks or commodities at time t, they could as well be interest rates, exchange rates, or compound indexes computed from the aggregation of other financial instruments. The spread is naturally defined as the instrument $S = \{S(t)\}_{t\geq 0}$ whose value at time t

is given by the difference:

$$S(t) = S_2(t) - S_1(t), \qquad t \geq 0.$$

4.1. **The Spread Option.** Our goal is to price European options on this spread. A European call option is defined by a date T called to date of maturity, a positive number K called the strike, and it gives the right to its owner to acquire at time T one unit of the underlying instrument at the unit price K. Assuming that this underlying instrument can be re-sold immediately on the market for its price $S(T)$ at that time, this means that the owner of the option will secure the amount $S(T) - K$ when the value of the underlying instrument at time T is greater than K, i.e. when $S(T) > K$, and nothing otherwise since in that case, she will act rationally and she will not exercise the option. So the owner of the option is guaranteed to receive the pay-out:

$$(S(T) - K)^+ = (S(T) - K)\mathbf{1}_{S(T)>K} \tag{7}$$

at maturity T. We denote by p the price at time 0 of this European call option with date of maturity T and strike K. More generally, we shall denote by p_t its price at time $t < T$. The Black-Scholes formula gives a value for p when $S(T)$ is a log-normal random variable for a probability structure called risk-neutral. The Black-Scholes pricing paradigm was extended using no-arbitrage arguments to more general classes of random variables, and even to situations where the dynamics of the values of the two underlying indexes S_1 and S_2 are given by stochastic processes, possibly with jumps. In any case, the price of the option is given by the risk-neutral expectation of the discounted pay-out of the option at maturity. So according to this pricing paradigm, the price p is given by a risk-neutral expectation:

$$p = \mathbb{E}\{e^{-rT}\,(S_2(T) - S_1(T) - K)^+\} \tag{8}$$

where the exponential factor e^{-rT} takes care of the discounting. In general, the discounting rate $r \geq 0$ is nothing but the short interest rate. But in some cases, it can contain corrections taking into account the rate of dividend payments, or the convenience yield in the case of physical commodities. For the sake of the present discussion, we shall assume that the discounting rate is the short interest rate which is assumed to be deterministic and constant throughout the life of the option (i.e. before the maturity date T.)

Pricing by Computing a Double Integral. It is important to remark that, even though we shall most of the time define the mathematical models by the prescriptions they give for the dynamics of the indexes S_1 and S_2, the fact that we are considering options with European exercises implies that we do not really need the full dynamics to price a spread option. Indeed, the pay-out at maturity depends only upon the values of the indexes at time T, i.e. of the values $S_1(T)$ and $S_2(T)$ (never mind how they got there.) So in order to compute the expectation giving the price p, the only thing we need is the joint density of the couple $S_1(T), S_2(T))$ of random variables. So ignoring momentarily the dynamics of the underlying indexes, we write the price of a call spread option as a double integral. More precisely:

$$e^{-rT}\mathbb{E}\{(S_2(T) - S_1(T) - K)^+\} = e^{-rT}\int\int (s_2 - s_1 - K)^+ f_T(s_1, s_2)\; ds_1 ds_2$$

if we denote by $f_T(s_1, s_2)$ the joint density of the random variables $S_1(T)$ and $S_2(T)$. Computing the expectation by conditioning first by the knowledge of $S_1(T)$ we get:

$$\begin{aligned} &e^{-rT}\mathbb{E}\{(S_2(T) - S_1(T) - K)^+\} \\ &\quad = e^{-rT}\mathbb{E}\{\mathbb{E}\{(S_2(T) - S_1(T) - K)^+ | S_1(T)\}\} \\ &\quad = \int \mathbb{E}\{(S_2(T) - (s_1 + K))^+ | S_1(T) = s_1\} f_{1,T}(s_1) ds_1 \\ &\quad = \int \left(\int (s_2 - s_1 - K)^+ f_{2,T|S_1(T)=s_1}(s_2) ds_2 \right) f_{1,T}(s_1) ds_1 \end{aligned}$$

where we used the notation $f_{1,T}(s_1)$ for the density of the first index $S_1(T)$ at the time T of maturity, and the notation $f_{2,T|S_1(T)=s_1}(s_2)$ for the conditional density of the second index $S_2(T)$ at maturity, given that the first index is equal to s_1 at that time. The intermediate result shows that the price of the call spread is the integral over s_1 of the prices of European calls on the second index with strikes $s_1 + K$.

In the log-normal models, the conditional density $f_{2,T|S_1(T)=s_1}(s_2)$ is still lognormal, so the value of the inner-most integral is given by the classical Black-Scholes formula for an appropriate choice of the strike. This shows that the price of the call spread is an integral of Black-Scholes formulae with respect to the (log-normal) density of the first index. Pricing the option on the spread by computing these integrals numerically can *always* be done. But even a good approximation of the price p is not sufficient in practice. Indeed, and this fact is too often ignored by the newcomers to financial mathematics, a pricing algorithm has to produce much more than a price if it is to be of any practical use, and this the main reason why the search for closed form formulae is still such an active research area, even in these days of fast and inexpensive computers. It is difficult to explain why without getting into details which would sidetrack our presentation, but we nevertheless justify our claim by a few remarks, leaving the details to asides which we will sprinkle throughout the rest of the paper as appropriate.

A Couple of Important Remarks. The above discussion may lead the reader to believe that having a pricing formula in closed form may not be of such a crucial importance. The following bullet points should diffuse this misconception.

1. Let us for a moment put ourselves in the shoes of the seller of the option. From the moment of the sale, she is exposed to the risk of having to pay (**??**) at the date of maturity T. This payout is random and cannot be predicted with certainty. The whole basis of the Black-Scholes pricing paradigm is to set up a portfolio and to devise a trading strategy which, whatever the final outcome at maturity T, will have the same exact value as the payout at that time. The thrust of the discovery of Black and Scholes lies in proving that such a replication of the payoff was possible, and once this stunning statement was proved, the price of the option had to be the cost of initially setting up such a replicating portfolio: that's definitely worth a Nobel prize! Replication of the payout of an option is obviously the best way to get a perfect hedge for the risk associated with the sale of this option. But what is even more remarkable is the fact that the components of the replicating portfolio are explicitly given by the derivatives of the price with respect to the initial value of the underlying index. Obviously, the partial derivatives of the price of the option with respect to the parameters of the model (initial value of the underlying instrument, interest rate, volatility or instantaneous standard deviation, ...) give the sensitivities of the price with respect to these parameters, and as such, they quantify the sizes of the price fluctuations produced by small changes in these economic parameters. These partial derivatives are of great importance to the trader and the risk manager who both rely on their values. For this reason they are given special names *delta, gamma, rho, vega,* ... and they are generically called the *Greeks.* Having a closed formula for the price of the option usually yields closed formulae for the Greeks which can then be evaluated rapidly and accurately. This is of great value to the practitioners, and this is one of the reasons alluded to earlier why people are searching so frantically for pricing formulae in closed forms.

2. Hedging is not the only reason why a pricing formula in closed form is far superior to a numerical algorithm. When a pricing formula can be inverted, one can infer values of the parameters (volatility, correlations, ...) of the pricing model from the quotes of the prices of the options with different maturities and different strikes already traded on the market. The values inferred in this way are called *implied.* They are of great significance and they are used by the market makers to price new instruments. This is the reason for the fame of the so-called *implied volatility* which we will encounter later in the paper.

A Parity Formula. The classical parity argument gives:

$$e^{-rT}\mathbb{E}\{(S_2(T)-S_1(T)-K)^+\} = e^{-rT}\mathbb{E}\{(S_1(T)-S_2(T)+K)^+\} + x_2 - x_1 - Ke^{-rT} \tag{9}$$

if we use the notation x_1 and x_2 for the initial values $S_1(0)$ and $S_2(0)$. Recall that $e^{-rT}\mathbb{E}\{S_1(T)\} = x_1$ and $e^{-rT}\mathbb{E}\{S_2(T)\} = x_2$ since we are using risk-neutral expectations. This call-put parity formula (valid under no other assumption than the absence of arbitrage) allows us to restrict ourselves to the case of European call options, ignoring altogether the pricing of put options.

4.2. **Markovian Models and Partial Differential Equations.** In the previous section we saw that the price p of the spread option is given by the risk-neutral expectation given in formula (8). In order to compute this expectation, we need to specify the risk-neutral dynamics of the underlying indexes. Let us assume that they satisfy a two-dimensional system of Itô stochastic differential equations of the type:

$$\begin{cases} \frac{dS_1(t)}{S_1(t)} = \mu_1(t,\mathbf{S}(t))dt + \sigma_1(t,\mathbf{S}(t))\rho(t,\mathbf{S}(t))d\tilde{W}_1(t) + \sigma_1(t,\mathbf{S}(t))\sqrt{1-\rho(t,\mathbf{S}(t))}d\tilde{W}_2(t) \\ \frac{dS_2(t)}{S_2(t)} = \mu_2(t,\mathbf{S}(t))dt + \sigma_2(t,\mathbf{S}(t))d\tilde{W}_2(t) \end{cases} \tag{10}$$

where we use the notation $\mathbf{S}$ for the couple (S_1, S_2), and where $\{\tilde{W}_1(t)\}_t$ and $\{\tilde{W}_2(t)\}_t$ are independent standard real valued Brownian motions. We also assume that the coefficients μ_i's, σ_i's and ρ are smooth enough for existence and uniqueness of a strong solution of this stochastic differential system. It is well known that a Lipschitz assumption with linear growth will do, but rather than giving technical conditions under which these assumptions are satisfied, we go on to explain how one can compute the expectation giving the price. This can be done by solving a partial differential equation. This link is known under the name of Feynman-Kac representation. Even though we shall not need this level of generality in the sequel, we state it in the general case of a time dependent stochastic short interest rate $r = r(t, \mathbf{S}(t))$ given by a deterministic function of $(t, \mathbf{S}(t))$.

Proposition 1. *Let u be a $C^{1,2,2}$-function in (t, x_1, x_2) with bounded partial derivatives in t, x_1 and in x_2 satisfying the terminal condition:*

$$\forall x_1, x_2 \in \mathbb{R} \quad u(T, x_1, x_2) = f(x_1, x_2)$$

for some nonnegative function f and the partial differential equation:

(11)
$$\left(\frac{\partial}{\partial t} + \frac{1}{2}\sigma_1^2 x_1^2 \frac{\partial^2}{\partial x_1^2} + \rho\sigma_1\sigma_2 x_1 x_2 \frac{\partial^2}{\partial x_1 \partial x_2} + \frac{1}{2}\sigma_2^2 x_2^2 \frac{\partial^2}{\partial x_2^2} + \mu_1 x_1 \frac{\partial}{\partial x_1} + \mu_2 x_2 \frac{\partial}{\partial x_2} - r\right) u = 0$$

on $[0, T] \times \mathbb{R} \times \mathbb{R}$. Then for all $(t, x_1, x_2) \in [0, T] \times \mathbb{R} \times \mathbb{R}$ one has the representation:

$$u(t, x_1, x_2) = \mathbb{E}\left\{ e^{-\int_t^T r(s, \mathbf{S}(s))ds} f(S(T)) \middle| \, \mathbf{S}(0) = (x_1, x_2) \right\} \tag{12}$$

Proof. This result is a classical example of the representation of solutions of parabolic PDE's as expectations over diffusion processes. Even though pure semi-group proofs can be provided, the most general ones rely on the Ito's calculus and the Feynman-Kac formula. We refer the reader interested in a detailed proof in the context of financial applications to [30]. ∎

In the case of interest to us (recall formula (8) giving the price of the spread option) we shall assume that the interest rate is constant $r(t, (x_1, x_2)) \equiv r$, and we shall use the function $f(x_1, x_2) = (x_2 - x_1 - K)^+$ for terminal condition.

4.3. **Samuelson's Model and the Black-Scholes Framework.** The system (10) is a reasonably general set-up for the pricing of the spread options. Indeed, most of the abstract theory (see for example [30]) can be applied. Unfortunately, this set-up is too general for explicit computations, and especially the derivation of pricing formulae in closed forms. so we shall often restrict ourselves to more tractable specific cases. The most natural one is presumably the model obtained by assuming that the coefficients μ_i, σ_i and ρ are constants independent of time and the underlying indexes S_1 and S_2. Setting $W_1(t) = \rho\tilde{W}_1(t) + \sqrt{1-\rho^2}\tilde{W}_2(t)$ and $W_2(t) = \tilde{W}_2(t)$, we have that:

$$\frac{dS_i(t)}{S_i(t)} = \mu_i dt + \sigma_i dW_i(t), \qquad i = 1, 2 \tag{13}$$

where $\{W_1(t)\}_t$ and $\{W_2(t)\}_t$ are two Wiener processes (Brownian motions) with correlation ρ. The two equations can be solved separately. Indeed, they are coupled only through the statistical correlation of the two driving Wiener processes. The solutions are given by:

$$S_i(t) = S_i(0) e^{(\mu_i - \sigma_i^2/2)t + \sigma_i W_i(t)}, \qquad i = 1, 2 \tag{14}$$

Defined in this way each index process $\{S_i(t)\}_t$ is a geometric Brownian motion. The unexpected $-(\sigma_i^2/2)t$ appearing in the deterministic part of the exponent is due to the idiosyncracies of the Ito's stochastic calculus. It is called the Ito's correction. If the initial conditions $S_i(0) = x_i$ are assumed to be deterministic, then the distribution of the values $S_i(t)$ of the indexes are log-normal, and we can explicitly compute their densities. This log-normality of the distribution was first advocated by Samuelson, but it is often known under the names of Black and Scholes because it is in this framework that these last two authors have derived their famous pricing formula for European call and put options on a single stock. Dynamics given by the stochastic differential equations of the form (13) are at the core of the analyses reviewed in this paper.

4.4. **Numerics.** This subsection is devoted to the discussion of the most commonly used numerical methods which are used to price and hedge financial instruments in the absence of explicit formulae in closed forms. We restrict ourselves to the Markovian models described above, and we review methods which are prevalent in the industry by illustrating their implementations on the spread valuation problem. Proposition 1 states that at least in the Markovian case, the price of the spread option is the solution of a partial differential equation. Consequently, valuing a spread option can be done by solving a PDE, and the first two of the four subsections below describe possible implementations of this general idea.

Using PDE Solvers. As explained earlier, the Feynman-Kac representation given in Proposition 1, suggests the use of a PDE solver to get a numerical value for the price of a spread option. Because of the special form of the stochastic dynamics of the underlying indexes, the coefficients in the second order terms of the PDE (11) can vanish and it appears as if the PDE is degenerate. For this reason, the change of variables $(x_1, x_2) = (\log u_1, \log u_2)$ is often used to reduce (11) to a non-degenerate parabolic equation. This PDE is three dimensional, one time dimension and two space dimensions. There is an extensive literature on the stability properties of the various numerical algorithms capable of solving these PDE's, and we shall refrain from going into these technicalities. We shall only concentrate on one very special tree based numerical scheme. This explicit finite difference method was made popular by Hull in his book [25] in the case of a single underlying index. We present the details of the generalization necessary for the implementation in the case of cross-commodity spreads.

Trinomial Trees. We refer the reader interested in the use of a classical trinomial tree for the pricing of an option on a single underlying to Hull's book [25]. We only give the details of the generalization of this classical one dimensional approach to the present two dimensional setting. (This part is directly inspired by [11].) Since we have two underlying processes, we need a tree spanning in two directions. More precisely, even though we keep the terminology of trinomial tree, each node leads to nine new nodes at the next time step. Since the computations are local, we can assume without any loss of generality that all the coefficients of the diffusion equations are constant. So solving the above stochastic differential equations we get:

$$S_i(t) = S_i(0) \exp\left[\mu_i t - \frac{1}{2}\sigma_i^2 t + \sigma_i \tilde{W}_i(t)\right]$$

where the two new Brownian motions $\tilde{W}_1$ and $\tilde{W}_2$ satisfy $\mathbb{E}\left\{\tilde{W}_1(t)\tilde{W}_2(t)\right\} = \rho t$. The basic idea behind the tree's construction is to discretize the mean zero Gaussian vector $(\sigma_1\tilde{W}_1(t), \sigma_2\tilde{W}_2(t))$ with covariance matrix Σt where

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}.$$

We first diagonalize this covariance matrix and write it as

$$\Sigma = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^T$$

where

$$\lambda_1 = \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2 + \sqrt{(\sigma_1^2 + \sigma_2^2)^2 - 4(1-\rho^2)\sigma_1^2\sigma_2^2}\right)$$
$$\lambda_2 = \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2 - \sqrt{(\sigma_1^2 + \sigma_2^2)^2 - 4(1-\rho^2)\sigma_1^2\sigma_2^2}\right)$$
$$\theta = \arctan\left(\frac{\lambda_1 - \sigma_1^2}{\rho\sigma_1\sigma_2}\right).$$

We can now write our Gaussian vector as follows:

$$\begin{pmatrix} \sigma_1 W_1(t) \\ \sigma_2 W_2(t) \end{pmatrix} = \begin{pmatrix} \cos\theta\sqrt{\lambda_1}B_1(t) - \sin\theta\sqrt{\lambda_2}B_2(t) \\ \sin\theta\sqrt{\lambda_1}B_1(t) + \cos\theta\sqrt{\lambda_2}B_2(t) \end{pmatrix}$$

where (B_1, B_2) is a two-dimensional standard Brownian motion. The discretization of our diffusion is then equivalent to the discretization of two independent standard Brownian motions. Let X_1 and X_2 be two independent and identically distributed random variables taking three values $(-h, 0, h)$ with probabilities $(p, 1-2p, p)$ respectively. The idea is to use (X_1, X_2) to approximate the increments $(B_1(t+dt) - B_1(t), B_2(t+dt) - B(t))$. We take $p = 1/6$ and $h = \sqrt{3dt}$ so that X_i and $B_i(t+dt) - B_i(t)$ have the same first four moments. Thanks to the independence of X_1 and X_2 the probabilities for getting to each one of the nine new nodes follow. We organize them in the matrix:

$$\begin{pmatrix} 1/36 & 1/9 & 1/36 \\ 1/9 & 4/9 & 1/9 \\ 1/36 & 1/9 & 1/36 \end{pmatrix}$$

From this point on, the approximation of the price is done is exactly the same fashion as in the one-dimensional tree. We simulate the diffusion up to the terminal date and we compute the price of the option at this time (which is just the payoff of the option in the different states of the world). By backward induction we compute the price at the root of the tree.

Monte Carlo Computations. Most often, a good way to compute an expectation is to use traditional Monte-Carlo methods. The idea is to generate a large number of sample paths of the process $\mathbf{S} = (S_1, S_2)$ over the interval $[0, T]$, for each of these sample paths to compute the value of the function of the path whose expectation we evaluate, and then to average these values over the sample paths. The principle of the method is simple, the computation of the sample average is usually quite straightforward, the only difficulty is to quantify and control the error. Various methods of random sampling have been proposed (stratification being one of them) and variance reduction (for example importance sampling, use of antithetic variables, ...) are used to improve the reliability of the results. Notice that, even when the function to integrate is only a function of the terminal value $\mathbf{S}(T)$ of the process $\mathbf{S}$, one generally has to generate samples from the entire path $\mathbf{S}(t)$ for $0 \le t \le T$, just because one does not know the distribution of $\mathbf{S}(T)$, simulation is often the only way to get at it. Random simulation requires the choice of a time discretization step Δt and the generation of discrete time samples $\mathbf{S}(t_0 + j\Delta t)$ for $j = 0, 1, \cdots, N$ with $t_0 = 0$ and $t_0 + n\Delta t = T$, and these steps should be taken with great care to make sure that the (stochastic) numerical scheme used to generate these discrete samples produce reasonable approximations.

The situation is much simpler when we assume that all the coefficients are deterministic. Indeed, in such a case the joint distribution of the underlying indexes at the terminal time can be computed explicitly, and there is no need to simulate entire sample paths: one can simulate samples from the terminal distribution directly. This distribution is well-known. As explained earlier, it is a joint log-normal distribution. The simulation of samples from the underlying index values at maturity is very easy. Indeed, if the coefficients are constant, the couple $(S_1(t), S_2(t))$ of indexes at maturity can be written in the form:

$$\begin{aligned} S_1(T) &= S_1(0)\exp\left[\left(\mu_1 - \sigma_1^2/2\right)T + \sigma_1\rho\sqrt{T}U + \sigma_2\sqrt{1-\rho^2}\sqrt{T}V\right] \\ S_2(T) &= S_2(0)\exp\left[\left(\mu_2 - \sigma_2^2/2\right)T + \sigma_2\sqrt{T}U\right] \end{aligned}$$

where U and V are two independent standard Gaussian random variables. The simulation of samples of (U, V) is quite easy.

Note that the same conclusion holds true when the coefficients are still deterministic but are possibly varying with time. Indeed, one can still derive the terminal joint distribution of two underlying indexes, the parameters being now functions (typically time averages) of the time dependent coefficients. We shall present the details of these computations several times in the sequel, so we refrain from dwelling on them at this stage.

Approximation via Fourier Transform. Fourier transform has become quite a popular tool in option pricing when the coefficients of the dynamical equations are deterministic and constant. This is partially due to the contributions of Heston [23], Carr and Madan [6] and to many others ones who followed in their footsteps. In [12], Dempster and Hong use the Fast-Fourier-Transform to approximate numerically the two-dimensional integral introduced earlier in our discussion of the spread valuation by multiple integrals. We refer the reader interested in Fourier methods to this paper and to the references therein.

Concluding Remarks. We conclude this section about numerical methods by pointing out some of their (well-known) flaws and in so doing we advocate the cause of analytical approximations. Although Monte-Carlo methods can give good approximations for the prices but unfortunately, they not say anything about the different sensitivities of the prices with respect to the different parameters, the Greeks as we defined them earlier. Among these sensitivities, those partial derivatives with respect to the current values of the underlying index values are of crucial importance since they give the weights used to build self-financing replicating portfolios to hedge the risk associated with the options. To compute numerically these partial derivatives, one should in principle re-compute the price of the option with a slightly different value for the underlying index, and this should be done many time which makes the use of Monte Carlo methods unreasonable. To be more specific on the reasons of this claim, the first derivative is typically approximated by computing the limit as $\epsilon \searrow 0$ of expression of the form:

$$\Delta_1(\epsilon) \approx \frac{1}{\epsilon}(p(x_1+\epsilon) - p(x_1)).$$

So we need to compute $\Delta_1(\epsilon)$ for each ϵ in a sequence going to zero, and in order to compute each single $\Delta_1(\epsilon)$ we need to redo the Monte Carlo simulation twice: once with Monte Carlo samples starting from x_1 and a second time with Monte Carlo samples starting from $x_1 + \epsilon$. This is extremely costly from the point of view of computing time. Moreover, even when the convergence of the numerical algorithm is good for the price, it may be poor for the approximations of the partial derivatives, and becoming poorer and poorer as the order of the derivative increases. This is also the source of undesirable instabilities. Most of the computational algorithms based on closed form formulae do not have these problems since it is in general not too difficult to also derive closed form formulae for the Greeks as well.

It is fair to say that a possible solution to this problem has been recently proposed in a set of two very elegant papers based on the formula of integration by parts of the Malliavin caluclus. See [17] and [18]. Even though the prescriptions seems very attractive, and despite the fact that the gain over the brute force Monte Carlo approach described above seem to be significant, we believe that they are still extremely involved and less efficient than the methods reviews later in this survey.

It is fair to notice that contrarily to the Monte Carlo method, the trinomial tree method allows to compute the partial derivatives along with the price. But its main shortcoming remains its slow rate of convergence: so precision in the approximation is traded for reasonable computing times. The other major problem with the trinomial tree method is the fact that it *blows up* exponentially with the dimension. It is still feasible with two assets as we are considering here, but it is very unlikely to succeed in any higher dimension.

Finally, none of these methods allow to efficiently compute implied volatilities or implied correlations from a set of market prices. An implied parameter (whether it is a volatility, a correlation, . . .) is the value of the parameter which reproduce best the prices actually quoted on the market. So in order to compute these implied parameters, one needs to be able to invert the pricing algorithm and recover an input, say the value of the volatility parameter for example, from a value of the output, i.e. the market quote. None of the numerical methods described above can do that, while most of the numerical methods based on the evaluation of closed form formulae can provide values for the implied parameters. The latter have a great appeal to the traders and other market makers, and being able to produce them is a very desirable property of a computational method.

5. THE BACHELIER'S MODEL

In most applications to the equity markets, the underlying indexes are modeled by means of log-normal distributions as prescribed by the Samuelson's model. As we already mentioned, this model is motivated in part by the desire to reproduce the inherent positivity of the indexes. But the positivity restriction does not apply to the spreads themselves, since the latter can be negative as differences of positive quantities. Indeed, computing histograms of historical spread values shows that the marginal distribution of a spread at a given time extends on both tails, and surprisingly enough, that the normal distribution can give a reasonable fit. This simple remark is the starting point of a series of papers proposing to use arithmetic Brownian motion (as opposed to the geometric Brownian motion leading to the log-normal distribution of the indexes) for the dynamics of spreads. In so doing, prices of options can be derived by computing Gaussian integrals leading to simple closed form formulae. This approach bears to the approach taken later in this paper, the same relationship that Bachelier's original model bears to the Samuelson model for the dynamics of the price of a single stock price. It was originally advocated by Shimko in the early nineties. See [38] for a detailed exposé of this method. For the sake of completeness we devote this section to a review of this approach, and we quantify numerically the departures of the results from the results provided by the log-normal model studied later in the paper.

In this section, we assume that the risk-neutral dynamics of the spread $S(t)$ is given by a stochastic differential equation of the form:

$$dS(t) = \mu S(t)dt + \sigma dW(t) \tag{15}$$

for some standard Brownian motion $\{W(t)\}_{t\geq 0}$ and some positive constant σ. Here and in the following, μ stands for the short interest rate r, or $r-\delta$ where δ denotes the continuous rate of dividend payments, or the cost of carry, or the convenience yield. In any case, μ is assumed to be a deterministic constant. Equation (15) is appropriate when the spread is defined as $S(t) = \alpha_2 S_2(t) - \alpha_1 S_1(t)$ for some coefficients α_1 and α_2, and when the dynamics of the individual component indexes $S_1(t)$ and $S_2(t)$ are given by stochastic differential equations of the form:

$$\begin{aligned} dS_1(t) &= \mu S_1(t)dt + \sigma_1 dW_1(t) \\ dS_2(t) &= \mu S_2(t)dt + \sigma_2 dW_2(t) \end{aligned}$$

with positive constants σ_1 and σ_2 and two Brownian motions W_1 and W_2 with correlation ρ. As usual, the initial values of the indexes will be denoted by $S_1(0) = x_1$ and $S_2(0) = x_2$. Indeed, choosing:

$$\sigma = \sqrt{\alpha_1^2\sigma_1^2 + \alpha_2^2\sigma_2^2 - 2\rho\alpha_1\alpha_2\sigma_1\sigma_2} \tag{16}$$

and:

$$W(t) = \frac{\alpha_2\sigma_2}{\sigma}W_2(t) - \frac{\alpha_1\sigma_1}{\sigma}W_1(t)$$

gives the dynamics (15) for S. The main interest of the arithmetic Brownian motion model is that it leads to a closed form formula akin the Black-Scholes formula for the price of the call spread option.

5.1. **Pricing Formulae.** However, this point of view is not directly adopted by practitioners because the dynamics for S_1 and S_2 is totally unrealistic since their marginal distributions are Gaussian and can therefore be negative with positive probability. Instead, they assume that the dynamics for S_1 and S_2 are given by given by geometric Brownian motions and that the dynamics of the spread can be approximated by an arithmetic Brownian motion. (these two assumptions are of course unreconcilable.) Let us postulate

$$dS_1(t) = \mu S_1(t)dt + \sigma_1 S_1(t)dW_1(t) \tag{17}$$

$$dS_2(t) = \mu S_2(t)dt + \sigma_2 S_2(t)dW_2(t). \tag{18}$$

Proposition 2. *If the value of the spread at maturity is assumed to have the Gaussian distribution, the price p of the call spread option with maturity T and strike K is given by:*

$$p = \left(m(T) - Ke^{-rT}\right)\Phi\left(\frac{m(T)-Ke^{-rT}}{s(T)}\right) + s(T)\varphi\left(\frac{m(T)-Ke^{-rT}}{s(T)}\right) \tag{19}$$

where we used the notation:

$$\begin{aligned} m(T) &= (x_2 - x_1)e^{(\mu-r)T} \\ s^2(T) &= e^{2(\mu-r)T}\left(x_1^2\left(e^{\sigma_1^2T}-1\right) - 2x_1x_2\left(e^{\rho\sigma_1\sigma_2T}-1\right) + x_2^2\left(e^{\sigma_2^2T}-1\right)\right) \end{aligned}$$

Notice that here and throughout the paper, we use the notation $\varphi(x)$ and $\Phi(x)$ for the density and the cumulative distribution function of the standard normal $N(0,1)$ distribution, *i.e.*,

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad \text{and} \qquad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du.$$

Proof. The dynamics (17-18) can be explicitly solved.

$$S_i(T) = S_i(0) \exp\left[\mu T - \frac{1}{2}\sigma_i^2 T + \sigma_i W_i(T)\right]$$

If we approximate the distribution of $S(T) = S_2(T) - S_1(T)$ by the Gaussian distribution, the least we can ask is that they match their first two moments. Therefore:

$$S(T) \sim Gsn(\mathbb{E}\{S_2(T) - S_1(T)\}, \text{var}\{S_2(T) - S_1(T)\})$$

and classical computations give $\mathbb{E}\{S_2(T) - S_1(T)\} = (x_2 - x_1)e^{\mu T}$ and $\text{var}\{S_2(T) - S_1(T)\} = s^2(T)e^{2rT}$. Consequently, the price p at time T of the option is given by:

$$\begin{aligned} p &= e^{-rT}\mathbb{E}\{(S(T) - K)^+\} \\ &= \mathbb{E}\{(m(T) - Ke^{-rT} + s(T)\xi)^+\} \end{aligned}$$

for some $N(0,1)$ random variable ξ. Consequently:

$$p = \frac{1}{\sqrt{2\pi}} \int_{m/s}^{\infty} (m + su) e^{-u^2/2} du$$

from which we easily get the desired result. ∎

Equation (15) can be solved explicitly, and the solution is given by:

$$S(t) = e^{\mu t} S(0) + \sigma \int_0^t e^{\mu(t-u)} dW_u$$

from which we see that $S(t)$ is a Gaussian (normal) random variable with mean $e^{\mu t}S(0)$ and variance:

$$\sigma^2 \int_0^t e^{2\mu u} du = \sigma^2 \frac{e^{2\mu t} - 1}{2\mu}.$$

We see that the practitioners' approximation is still compatible with dynamics like in (15) in case we allow for a time dependent volatility σ_t. In that case

$$dS(t) = \mu S(t) dt + \sigma_t dW(t) \tag{20}$$

and

$$S(t) = e^{\mu t} S(0) + \int_0^t \sigma_u e^{\mu(t-u)} dW_u \tag{21}$$

and the variance is

$$\int_0^t \sigma_u^2 e^{2\mu(t-u)} du.$$

In order for this last quantity to be equal to the quantity $s^2(t)$ given in Proposition 2, we must take

$$\sigma_t = \sqrt{e^{2\mu t} \frac{d}{dt} \left(e^{-2\mu t} s^2(t)\right)}.$$

5.2. **Numerical Performance of the Arithmetic Brownian Motion Model for Spreads.** Despite the suspicious naïveté of the model, and the extreme simplicity of the derivations, the pricing formulae obtained in this section can be surprisingly accurate for specific ranges of the parameters, even when the dynamics of the underlying indexes are not given by *arithmetic Brownian motions*. We illustrate this pretty *anti-climatic* fact by comparing the results obtained by this approach to the *true* values when the underlying indexes evolve like geometric Brownian motions.

The Parameters of the Experiment. For the purpose of this numerical experiment, we consider the case of a spark spread option with efficiency parameter $H_{eff} = 7.5$, and $x_1 = 2.7$ and $x_2 = 28$ for the current values of the gas and electricity contracts. We assume that their (annualized) volatilities are $\sigma_1 = 30\%$ and $\sigma_2 = 50\%$ respectively, and $\mu = 0$ since we are dealing with futures whose dividend rate is the short rate of interest. We make several runs to compare the effects of the remaining parameters. For fixed time-to-maturity T, we compare the exact price when the dynamics of the underlying indexes are given by geometric Brownian motions, and the approximation given by the Bachelier's model. We compute these two prices on a 21×41 grid of values of the couple (K, ρ). The strike K varies from $K = -5$ to $K = +5$ by increments of $.5$, while the correlation coefficient ρ varies from $\rho = -1$ to $\rho = +1$ by increments of $.05$.

The first comparison uses an option maturing in 60 days, *i.e.*, $\tau = T - t = 60/252$ yr. The results are given graphically in the left panel of Figure 7. The next comparison is done for options maturing in $\tau = T - t = 1.5$ yr. The results are given graphically in the right panel of this same figure.

The final comparison still uses options maturing in $\tau = T - t = 1.5$ yr, but we increase the volatility of the electricity price to $\sigma_2 = 80\%$. The results are shown in Figure 8.

Experimental Results. One can see from the left panel of Figure 7 that the agreement is remarkably good when the time-to-maturity is small, independently of the correlation of the underlying indexes as long as the strike is very negative. From all the experiments we made, it seems that the normal approximation underestimate the value of the option, and that the error increases with the strike, and decreases with the correlation. Comparing the two surface plots of Figure 7, one sees that increasing the time-to-maturity increased dramatically the error when the option is out of the money for large strikes. Notice the difference in scales on the vertical axes of the two plots. Both plots of Figure 8 seem to confirm this fact, with a significant deterioration of the performance of the normal approximation for more volatile log-normal indexes. We shall come back later to this model and compare it with other models in terms of hedging rather than in terms of pricing.

5.3. **Consistency with the Forward Curve.** This section is concerned with a practice intended to correct We present these ideas in the case of the model (20) and its solution (21) but we shall use them over and over. They have been made popular in the fixed income markets where the most tractable short interest models fail to be compatible with the observed yield curves. We adapt the present framework.

We now suppose that, on the day t when we value the spread option, we have information on other instruments derived from the underlying assets S_1 and S_2. Let us assume for example that we have a finite set of future dates $T_1, T_2, \cdots, T_n$, and that we have the prices at time t, say $f_1, f_2, \cdots, f_n$, of instruments maturing at these dates. In such a situation, it is very likely that if we use the model at hand to price these instruments, then we would find prices different from $f_1, f_2, \cdots, f_n$. This fact alone is enough for us to loose confidence in the model, and we may not want to price the spread

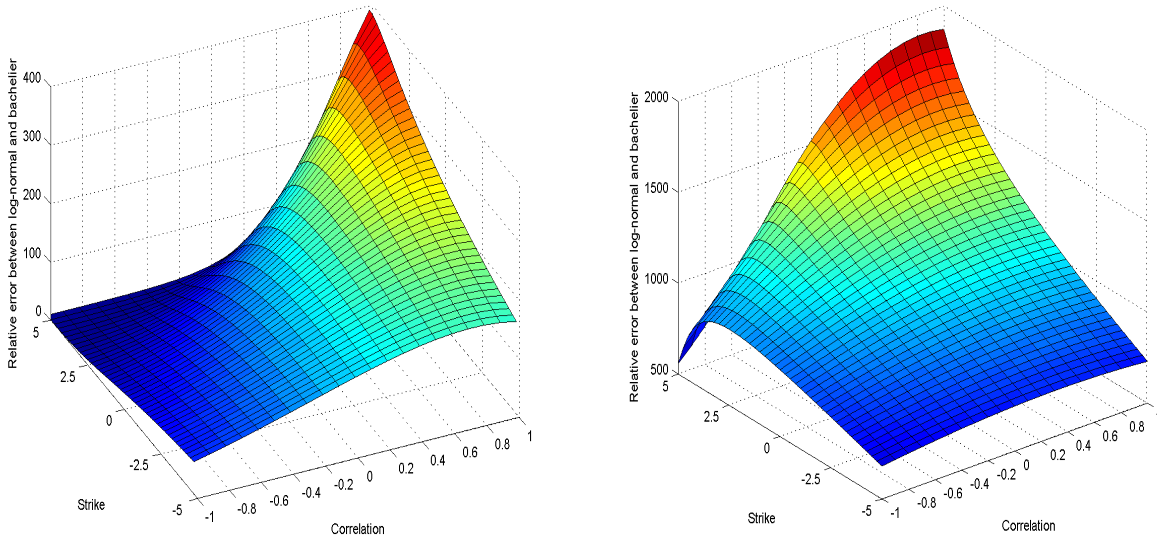

FIGURE 7. Surface plots of the ratio of the exact price divided by the Bachelier's approximation. In both cases, the parameters of the log-normal model were $x_1 = 2.7$, $x_2 = 28$, $H_{eff} = 7.5$, $\sigma_1 = 30\%$, $\sigma_2 = 50\%$, and the time-to-maturity $\tau = T - t$ was chosen to be $\tau = 60/252$ year for the computations leading to the surface on the left and $\tau = 1.5$ for the surface on the right. Ratios are given in basis points ($1bp = 10^{-4}$.)

using this model. In order to reconciliate our model with these observed price data, one usually add parameters to the model, and use these extra degrees of freedom to calibrate to the data: in other words, one chooses the extra parameters in order to replicate the prices quoted on the market.

As we already pointed out, this practice is used in the fixed income markets where the prices f_i are the prices of bonds, swaps, swaptions, ... and other liquid options whose prices are available through various financial services and brokerage houses. These prices are used to infer a curve $T \hookrightarrow f(T)$ giving the term structure of interest rates, whether it is given by all the future values of the instantaneous yield, or of the instantaneous forward rates, or even by the mere future discount curve. This initial term structure curve is then use to calibrate the model.

Because of the crucial importance of the spread options in the commodity markets, we illustrate this calibration philosophy in the case of these markets. On day t, we usually have access to the prices $f_1(t, T_1)$, $f_1(t, T_2)$, $\cdots$, $f_1(t, T_n)$ of forward contracts on the first commodity, and the prices $f_1(t, T_1)$, $f_1(t, T_2)$, $\cdots$, $f_1(t, T_n)$ of forward contracts on the second commodity. Notice that it is quite possible that the maturity dates T_1, T_2, $\cdots$, T_n may not be the same for both commodities. This is typically the case for the spark spread on power and gas. Indeed, as we already pointed out, the structure of the maturity dates of these contract differ wildly. In this case, massaging the data appropriately may easily get us where we want to be: One can indeed interpolate or smooth the values of the available forward contract prices to obtain a continuous curve which is then easy to

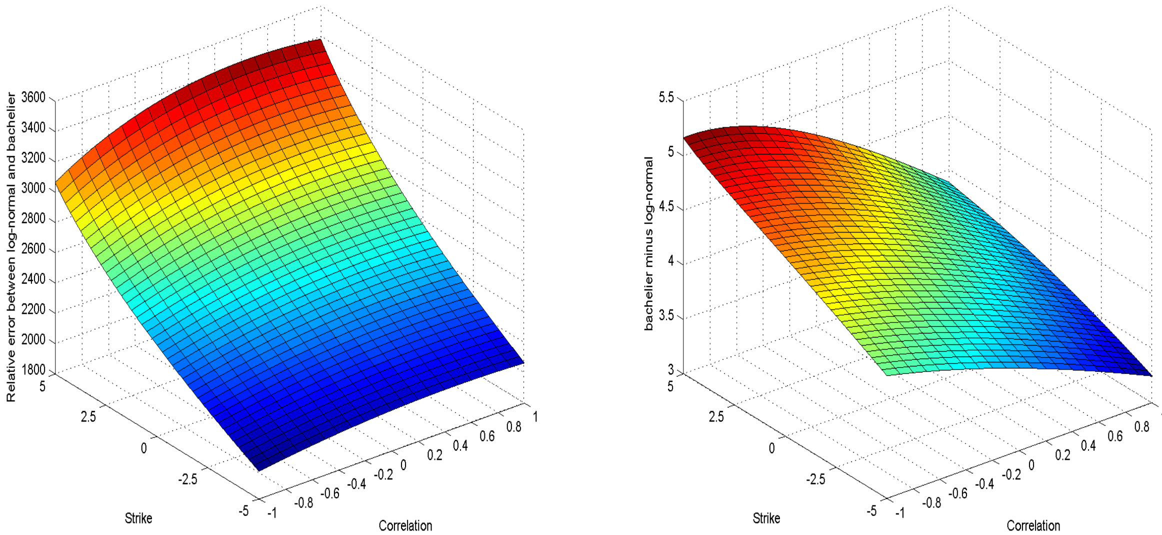

FIGURE 8. Surface plots of the ratio of the exact price divided by the Bachelier's approximation (left) and of the difference of the exact price minus the Bachelier's approximation (right). The parameters of the log-normal model were $x_1 = 2.7$, $x_2 = 28$, $H_{eff} = 7.5$, $\sigma_1 = 30\%$, $\sigma_2 = 80\%$, $r = 8\%$, and $T = 1.5$.

sample at the desired points T_1, T_2, $\cdots$, T_n in time. From this point on, it is easy to see that, one can get a set of prices for the differences (or the appropriate linear combinations in the case of more general commodity spreads.) See Figure 9 for an example of a spark spread forward curve. In this case, the first thing we need to do is to check that the pricing model used above is consistent with the forward curve at hand. In order to understand the consistency issue, we need first to identify the kind of forward curves implied by the assumptions of the model.

We now come back to our mathematical model, and we try to find out what kind of forward curves are supported by the model. Notice that, because of the independence of the increments of the Brownian motion, the stochastic integral $\int_t^T \sigma_s e^{\mu(T-s)} dW_s$ is independent of the values of the Brownian motion before time t, and hence of the values of S before time t. Consequently, recall formula (21) and the fact that we have taken the interest rate to be constant, the price $F(t,T)$ of the forward contract with date-of-maturity T is given by:

$$\begin{aligned} F(t,T) = \mathbb{E}_t\{S(T)\} &= \mathbb{E}_t\left\{e^{\mu(T-t)}S(t) + \int_t^T \sigma_s e^{\mu(T-s)} dW_s\right\} \\ &= e^{\mu(T-t)}S(t) \end{aligned}$$

where we used the notation $\mathbb{E}_t\{\ \cdot\ \}$ for the risk-neutral conditional expectation given the past up to time t. This shows that the forward curve ought to be an exponential starting from the current value of the spread. This is highly unrealistic as one can see from Figure 9. The fact that we did not consider stochastic interest rates does not change anything to this conclusion.

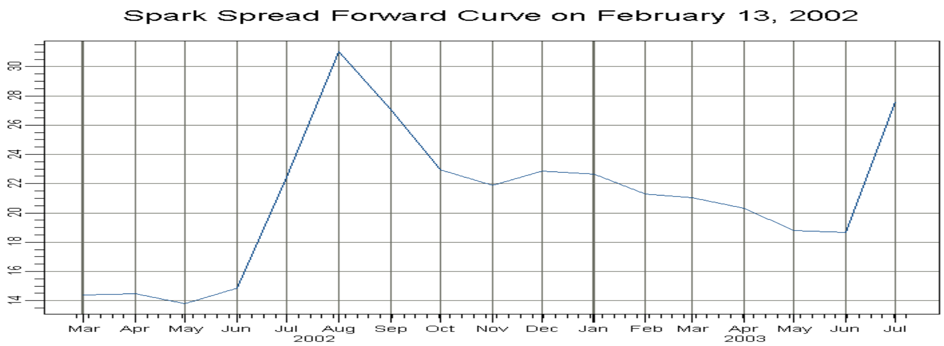

FIGURE 9. Forward curve for the spark spread on February 13, 2002. We used the California Oregon Border forward price curve for the electricity and the Henry Hub natural gas forward curve on the same day, and we used an efficiency factor of 2.5 for the sake of definiteness.

Since the class of forward curves implied by the model (20) is too small, we attempt to extend the model in order to allow for forward curves which can be observed in real life. So, for the rest of this subsection, we assume that we are given a forward curve $T \hookrightarrow F_0(T)$. One can think of F_0 as the result of the smoothing of a set of observed sample forward values of the spread as they appear in Figure 9 for example. We look for a stochastic differential equation similar to (15) which would imply a forward curve $T \hookrightarrow F(0,T) = \mathbb{E}_0\{S(T)\}$ equal to F_0. We shall assume that the equation is of the same form as (15), we merely assume that the rate of growth μ is now time dependent, *i.e.*, $\mu = \mu_t$ while still remaining deterministic. In this case, the solution formula (21) becomes:

$$S(t) = e^{\int_0^t \mu_s ds} S(0) + \int_0^t \sigma_s e^{\int_s^t \mu_u du} dW_s \tag{22}$$

The same argument as above gives that (recall that the interest rates are deterministic):

$$F(0,T) = \mathbb{E}\{S(T)\} = S(0) e^{\int_0^T \mu_u du}$$

which shows that, if we want the model to be consistent with the forward curve observed today at time $t = 0$ we should choose:

$$\mu_T = \frac{d}{dT} \log F_0(T)$$

In other words, by choosing the time dependent drift coefficient as the logarithmic derivative of the observed forward curve, the model becomes consistent in the sense that the current forward curve computed out of this model is exactly the current market curve.

This implies new expressions for the mean and variance of $S_2(T) - S_1(T)$:

$$\begin{aligned} m(T) &= (x_2 - x_1) e^{\int_0^T (\mu_u - r) du} \\ s^2(T) &= e^{2\int_0^T (\mu_u - r) du} \left(x_1^2 \left(e^{\sigma_1^2 T} - 1 \right) - 2 x_1 x_2 \left(e^{\rho \sigma_1 \sigma_2 T} - 1 \right) + x_2^2 \left(e^{\sigma_2^2 T} - 1 \right) \right) \end{aligned}$$

To be also consistent with our pricing formula (19) we therefore need take

$$\sigma_t = \sqrt{e^{2\int_0^t \mu_u du} \frac{d}{dt}\left(e^{-2\int_0^t \mu_u du} s^2(t)\right)}$$

as we have already derived.

The price p of the spread option will still be given by formula (19) as long as we use the above expressions for the constants $m(T)$ and $s(T)$.

6. THE CASE $K = 0$, OR THE OPTION TO EXCHANGE

The textbook treatment of spread options is usually restricted to the special case $K = 0$ because when the distributions of the underlying indexes S_1 and S_2 are log-normal, this is the only case for which one has a solution in a closed form *à la* Black-Scholes for the price of the spread option with zero strike. This formula was first derived by Margrabe in [33] as early as in 1978, and it bears his name. We present this derivation in full for the sake of completeness, and because of its importance, we elaborate on some of its consequences.

6.1. **Exchange of one Asset for Another.** The case $K = 0$ corresponds to an *exchange* since the pay-off $(S_2(T) - S_1(T))^+$ provides the holder of the option with the difference $S_2(T) - S_1(T)$ at time T whenever $S_2(T) > S_1(T)$.

In order to illustrate this fact, let us assume that our business requires the purchase of one of the two products at time T, that from a business point of view it does not make any difference which one, but that we have to decide at time $t = 0$ which one we will buy at time $t = T$. So let us choose the product whose price is given by the process $\{S_2(t)\}_t$ and let us purchase the spread option at time $t = 0$. If the product we chose ends up being cheaper at time T, we buy it at that time and we do not exercise the option, wasting the premium (i.e. the price we had to pay for the spread option.) On the other hand, if the product we chose ends up being more expensive at time T, we buy it but we get the price difference by exercising the option. So the spread option guarantees that pricewise, we will do as well as if we had chosen the cheapest one of the two at time T. So having to cover the premium at time $t = 0$, may just be worth it.

6.2. **Margrabe Formula.** In this subsection we assume that the risk neutral dynamics of the two underlying indexes are given by geometric Brownian motions, i.e. by stochastic differential equations of the following form:

$$\frac{dS_i(t)}{S_i(t)} = rdt + \sigma_i dW_i(t) \quad \text{for} \quad i = 1, 2. \tag{23}$$

where the discount rate r and the volatilities σ_i's are constant. We assume that the two indexes are correlated through the driving Brownian motions. To be more specific we assume that $\mathbb{E}\{dW_1(t)dW_2(t)\} = \rho dt$. In other words, ρ is the parameter controlling the correlation between the two indexes.

Proposition 3. *The price p of a spread option with strike $K = 0$ and maturity T, is given by:*

$$p = x_2\Phi(d_1) - x_1\Phi(d_0) \tag{24}$$

where:

$$d_1 = \frac{\ln(x_2/x_1)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T} \qquad d_0 = \frac{\ln(x_2/x_1)}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T} \tag{25}$$

and:

$$x_1 = S_1(0), \;\; x_2 = S_2(0), \qquad \sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2 \tag{26}$$

Proof. In order to prove formula (24), we define two new independent Brownian motions $\{\hat{W}_1(t)\}_t$ and $\{\hat{W}_2(t)\}_t$ by

$$\begin{aligned} dW_2(t) &= \rho d\hat{W}_1(t) + \rho' d\hat{W}_2(t) \\ dW_1(t) &= d\hat{W}_1(t) \end{aligned}$$

where $\rho'^2 + \rho^2 = 1$. $dW_1(t)$ and $dW_2(t)$ are well-defined as soon as $|\rho| < 1$. The risk-neutral valuation rule gives us the price of a zero-exercise price spread option as follows

$$p = e^{-rT}\mathbb{E}_{\mathbb{Q}}\{\max(S_2(T) - S_1(T), 0)\} \tag{27}$$

$$= e^{-rT}\mathbb{E}_{\mathbb{Q}}\left\{\max\left(\frac{S_2(T)}{S_1(T)} - 1, 0\right) S_1(T)\right\} \tag{28}$$

The price of S_2 expressed in the numéraire S_1 (i.e. in units of S_1) remains a geometric Brownian motion since:

$$\begin{aligned} \frac{d(S_2(t)/S_1(t))}{S_2(t)/S_1(t)} &= \frac{dS_2(t)}{S_2(t)} - \frac{dS_1(t)}{S_1(t)} - cov\left(\frac{dS_2(t)}{S_2(t)} - \frac{dS_1(t)}{S_1(t)}; \frac{dS_1(t)}{S_1(t)}\right) \\ &= (\rho\sigma_2 - \sigma_1)d\hat{W}_1(t) + \rho'\sigma_2 d\hat{W}_2(t) + \sigma_1(\sigma_1 - \rho\sigma_2)dt \end{aligned}$$

Using Girsanov's theorem, we define a new probability measure $\mathbb{P}$ which Radon-Nikodym derivative with respect to $\mathbb{Q}$ given on the σ-algebra $\mathcal{F}_T$ by:

$$\left.\frac{d\mathbb{P}}{d\mathbb{Q}}\right|_{\mathcal{F}_T} = \exp\left(-\frac{1}{2}\sigma_1^2 T + \sigma_1\hat{W}_1(T)\right) \tag{29}$$

Under $\mathbb{P}$, $\hat{W}_1(t) - \sigma_1 t$ and $\hat{W}_2(t)$ are Brownian motions. So:

$$\begin{aligned} p &= e^{-rT}S_1(0)\mathbb{E}_{\mathbb{P}}\left\{\max\left(\frac{S_2(T)}{S_1(T)} - 1, 0\right)\exp\left(\frac{1}{2}\sigma_1^2 T - \sigma_1\hat{W}_1(T) + \left(r - \frac{1}{2}\sigma_1^2\right)T + \sigma_1\hat{W}_1(T)\right)\right\} \\ &= S_1(0)\mathbb{E}_{\mathbb{P}}\left\{\max\left(\frac{S_2(T)}{S_1(T)} - 1, 0\right)\right\} \end{aligned}$$

where S_2/S_1 is geometric Brownian motion under $\mathbb{P}$ with volatility

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2 \tag{30}$$

Using this fact, the last expression can be viewed as the price of a vanilla call option with no interest rate, strike 1 and volatility σ^2. The value is thus given by the classical Black-Scholes formula. ∎

Remarks

1. The above result easily extends to allow for dividend payments at a constant rate. Suppose that instead of (23), we have

$$\frac{dS_i(t)}{S_i(t)} = (r - q_i)dt + \sigma_i dW_i(t) \quad \text{for} \quad i = 1, 2. \tag{31}$$

The method remains the same as long as we use $S_1(t)e^{q_1 t}$ and $S_2(t)e^{q_2 t}$ in lieu of $S_1(t)$ and $S_2(t)$. In that case, the price of a zero strike spread option is given by

$$p = x_2 e^{-q_2 T}\Phi(d_1) - S_1(0)e^{-q_1 T}\Phi(d_0) \tag{32}$$

$$d_1 = \frac{\ln(x_2/x_1) - (q_2 - q_1)T}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T} \tag{33}$$

$$d_0 = \frac{\ln(x_2/x_1) - (q_2 - q_1)T}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T} \tag{34}$$

2. As [25] points out, it is interesting to note that these formulae are independent of the risk-free rate r. This is because after risk adjustment (some would say *in a risk-neutral world*), both underlying indexes increase at the same discount rate, offsetting each other in the computation of the difference appearing in the definition of the spread.

3. The above closed form formulae are very nice, but unfortunately, the case where $K \neq 0$ cannot be treated with the same success. Reviewing the approach outlined in Section 4 where we discussed the use of multiple integrals, we can first condition by $W_1(T)$ in (27). This gives:

$$p = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left\{\mathbb{E}_{\mathbb{Q}}\{(S_2(T) - S_1(T) - K)^+ | W_1(T)\}\right\} \tag{35}$$

The inner expectation can again be evaluated using the Black-Scholes formula with strike $S_1(T) + K$ and a slightly modified spot. But the outer expectation (the integration of the Black-Scholes formula) cannot be done explicitly.

7. Pricing Options on the Spread of Geometric Brownian Motions

We continue our analysis of the valuation of spread options with non-zero strike prices. The results of the previous section apply when the dynamics of the underlying indexes (or the prices of the forward contracts on these indexes) are given by arithmetic Brownian motions. Now we consider the case of the Samuelson's model according to which these dynamics are given by geometric Brownian motions. Since we are only considering options with European exercises in this paper, we need only to consider the distribution of the underlying indexes at a fixed time, and according to this model, the distribution of the spread is the distribution of the difference of two log-normal distributions. Unfortunately, except in the case of the exchange option (*i.e.*, the spread option with strike $K = 0$) considered in Section 6, the price of the spread option cannot be given by a formula in closed form, and as we already explained, this was the motivation for the introduction of the arithmetic Brownian motion model. We now tackle the problem of the geometric Brownian motion.

Throughout this section we assume that, besides a riskless bank account with constant interest rate r, our arbitrage-free market consists of two assets whose risk-neutral price dynamics are given by the following stochastic differential equations:

$$\begin{aligned} dS_1(t) &= S_1(t)[(r-q_1)dt + \sigma_1 dW_1(t)] \\ dS_2(t) &= S_2(t)[(r-q_2)dt + \sigma_2 dW_2(t)] \end{aligned}$$

where q_1 and q_2 are the instantaneous dividend yields, the volatilities σ_1 and σ_2 are positive constants and W_1 and W_2 are two Brownian motions with correlation ρ. The initial conditions will be denoted by $S_1(0) = x_1$ and $S_2(0) = x_2$. The discussion of this section focuses on the pricing of spread options on two *stocks*. The case of spread options on two futures contracts follows immediately by taking $q_1 = q_2 = r$.

The price p of formula (8) can be rewritten in the form:

$$p = e^{-rT}\mathbb{E}\left\{\left(x_2 e^{(r-q_2-\sigma_2^2/2)T+\sigma_2 W_2(T)} - x_1 e^{(r-q_1-\sigma_1^2/2)T+\sigma_1 W_1(T)} - K\right)^+\right\}$$

which shows that the price p is given by the integral of a function of two variables with respect to a bivariate Gaussian distribution, namely the joint distribution of $W_1(T)$ and $W_2(T)$. This expectation is of the form:

$$\Pi = \Pi(\alpha,\beta,\gamma,\delta,\kappa,\rho) = \mathbb{E}\left\{\left(\alpha e^{\beta X_1 - \beta^2/2} - \gamma e^{\delta X_2 - \delta^2/2} - \kappa\right)^+\right\} \tag{36}$$

where α, β, γ, δ and κ are real constants and X_1 and X_2 are jointly Gaussian $N(0,1)$ random variables with correlation ρ. These expectations were studied in [5]. We can therefore apply the results of [5] to approximate the price of a spread option provided we set

$$\alpha = x_2 e^{-q_2 T} \quad \beta = \sigma_2\sqrt{T} \quad \gamma = x_1 e^{-q_1 T} \quad \delta = \sigma_1\sqrt{T} \quad \text{and} \quad \kappa = K e^{-rT}.$$

7.1. **A Pricing Formula.** The analysis of [5] is based on simple properties of the bivariate normal distribution and elementary convexity inequalities. Combining the two, the authors derived a family of upper and lower bounds for the price p. Among other things, they show that the supremum $\hat{p}$ of their lower bounds provides a very precise approximation to the exact price p. Before we can state the main result of [5], we introduce the notation θ^* for the solution of the equation:

$$\begin{aligned} &\frac{1}{\delta\cos\theta}\ln\left(-\frac{\beta\kappa\sin(\theta+\phi)}{\gamma[\beta\sin(\theta+\phi)-\delta\sin\theta]}\right) - \frac{\delta\cos\theta}{2} \\ &= \frac{1}{\beta\cos(\theta+\phi)}\ln\left(-\frac{\delta\kappa\sin\theta}{\alpha[\beta\sin(\theta+\phi)-\delta\sin\theta]}\right) - \frac{\beta\cos(\theta+\phi)}{2} \end{aligned} \tag{37}$$

where the angle ϕ is defined by setting $\rho = \cos\phi$. The following proposition gives the closed form formula derived in [5] for the approximate price $\hat{p}$.

Proposition 4. *Let us set:*

$$d^* = \frac{1}{\sigma\cos(\theta^*-\psi)\sqrt{T}}\ln\left(\frac{x_2 e^{-q_2 T}\sigma_2\sin(\theta^*+\phi)}{x_1 e^{-q_1 T}\sigma_1\sin\theta^*}\right) - \frac{1}{2}(\sigma_2\cos(\theta^*+\phi) + \sigma_1\cos\theta^*)\sqrt{T}$$

where the angles ϕ and ψ are chosen so that:

$$\cos\phi = \rho \qquad \textit{and} \qquad \cos\psi = \frac{\sigma_1 - \rho\sigma_2}{\sigma}.$$

Then

$$\hat{p} = x_2 e^{-q_2 T}\Phi\left(d^* + \sigma_2\cos(\theta^*+\phi)\sqrt{T}\right) - x_1 e^{-q_1 T}\Phi\left(d^* + \sigma_1 \sin\theta^* \sqrt{T}\right) - Ke^{-rT}\Phi(d^*) \tag{38}$$

Note that this formula is as close to the Black-Scholes' formula as we could hope for. Moreover, as documented in [5] it provided with extremely precise an approximation of the exact price of the spread option. One of the possible reasons for its accuracy is the fact that it gives exactly all known particular cases. Indeed, it has the following properties.

Proposition 5. *The approximation $\hat{p}$ is equal to the true price p when $K = 0$, or $x_1 = 0$, or $x_2 = 0$, or $\rho = -1$. In particular, $\hat{p}$ is given by Margrabe's formula when $K = 0$, and by the classical Black-Scholes' formula when $x_1 x_2 = 0$.*

Proof. We refer to [5] for a complete proof. For the sake of completeness, we show how we recover Margrabe's formula in the case $K = 0$. First we notice that θ^* is given by:

$$\theta^* = \pi + \psi = \pi + \arccos\left(\frac{\sigma_1 - \rho\sigma_2}{\sigma}\right).$$

with:

$$\sigma = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$$

which implies that

$$\sigma_2 \sin(\theta^* + \phi) = \sigma_1 \sin\theta^*$$

and

$$d^* = \frac{1}{\sigma\sqrt{T}} \ln\left(\frac{x_2 e^{-q_2 T}}{x_1 e^{-q_1 T}}\right) - \frac{1}{2}(\sigma_2\cos(\theta^*+\phi) + \sigma_1\cos\theta^*)\sqrt{T}.$$

Consequently:

$$\sigma_2\cos(\theta^*+\phi) - \sigma_1\cos\theta^* = \sigma$$

and:

$$\begin{aligned} d^* + \sigma_2\cos(\theta^*+\phi)\sqrt{T} &= \frac{1}{\sigma\sqrt{T}} \ln\left(\frac{x_2 e^{-q_2 T}}{x_1 e^{-q_1 T}}\right) + \frac{\sigma\sqrt{T}}{2} \\ d^* + \sigma_1\cos\theta^*\sqrt{T} &= \frac{1}{\sigma\sqrt{T}} \ln\left(\frac{x_2 e^{-q_2 T}}{x_1 e^{-q_1 T}}\right) - \frac{\sigma\sqrt{T}}{2}. \end{aligned}$$

hold true. Finally, we have:

$$\hat{p} = x_2 e^{-q_2 T}\Phi\left(\frac{1}{\sigma\sqrt{T}} \ln\left(\frac{x_2 e^{-q_2 T}}{x_1 e^{-q_1 T}}\right) + \frac{\sigma\sqrt{T}}{2}\right) - x_1 e^{-q_1 T}\Phi\left(\frac{1}{\sigma\sqrt{T}} \ln\left(\frac{x_2 e^{-q_2 T}}{x_1 e^{-q_1 T}}\right) - \frac{\sigma\sqrt{T}}{2}\right)$$

which is exactly Margrabe's formula. ∎

7.2. **Hedging and the Computation of the Greeks.** In Subsection 4.4 we discussed some of the shortcomings of the existing numerical approximations to the price of a spread option. There we emphasized the importance of hedging, and we explained that most of the pricing algorithms did not address this issue, failing to provide efficient methods to evaluate the so-called Greeks. Following [5], we show that hedging strategies can be computed and implemented in a very efficient way using formula (38) giving $\hat{p}$.

We first consider the replication issue. The derivation of formula (38) cannot provide an exact replication of the pay-off of the spread option. Because it is based on lower bounds, it gives a sub-hedge for the option.

Proposition 6. [5]*The portfolio comprising at each time $t \leq T$*

$$\Delta_1 = -e^{-q_1 T}\Phi\left(d^* + \sigma_1 \cos\theta^* \sqrt{T}\right)$$

and

$$\Delta_2 = e^{-q_2 T}\Phi\left(d^* + \sigma_2 \cos(\theta^* + \phi)\sqrt{T}\right)$$

units of the underlying assets is a sub-hedge for the option in the sense that its value at the maturity of the option is almost surely a lower bound for the pay-off.

But as we mentioned earlier, beyond the first partial derivatives with respect to the initial values of the underlying assets which give the portfolio of the previous proposition, all the sensitivities of the price with respect to the various parameters are of crucial importance. The closed form formula derived for $\hat{p}$ can be used to compute explicitly the other partial derivatives of the price. These are the so-called Greeks of the financial literature. We give some of them in the following proposition.

Proposition 7. [5] *Let ϑ_1 and ϑ_2 denote the sensitivities of the price functional (38) with respect to the volatilities of each asset, χ be the sensitivity with respect to their correlation parameter ρ, κ be the sensitivity with respect to the strike price K and Θ be that with respect to the maturity time T.*

$$\begin{aligned}
\vartheta_1 &= x_1 e^{-q_1 T}\varphi\left(d^* + \sigma_1 \cos\theta^* \sqrt{T}\right)\cos\theta^* \sqrt{T} \\
\vartheta_2 &= -x_2 e^{-q_2 T}\varphi\left(d^* + \sigma_2 \cos(\theta^* + \phi) \sqrt{T}\right)\cos(\theta^* + \phi) \sqrt{T} \\
\chi &= -x_1 e^{-q_1 T}\varphi\left(d^* + \sigma_1 \cos\theta^* \sqrt{T}\right)\sigma_1 \frac{\sin\theta^*}{\sin\phi}\sqrt{T} \\
\kappa &= -\Phi\left(d^*\right) e^{-rT} \\
\Theta &= \frac{\sigma_1\vartheta_1 + \sigma_2\vartheta_2}{2T} - q_1 x_1 \Delta_1 - q_2 x_2 \Delta_2 - rK\kappa
\end{aligned}$$

These formulae are of great practical value. The price of an option is determining factor for a buyer and a seller to get into a contract. But by indicating in which direction the price should change when some of the parameters change, the partial derivatives are monitored by the investors, the traders and the risk managers throughout the life of the option, i.e. up until maturity. The second order partial derivatives can also be approximated in the same way. We refrain from reproducing the results here and we refer the interested reader to [5].

7.3. **Comparison of the Three Approximations.** We conclude this section with a short comparative analysis of the three approximation methods which we recommend for the actual pricing and hedging of spread options.

• Let us first compare the results of the Bachelier's model with the results obtained with the approximation $\hat{p}$ given by formula (8). Numerical experiments show that Bachelier's price is *always* smaller than $\hat{p}$, and this strongly supports the use of $\hat{p}$. Given the simplicity of the formulae provided by the Bachelier's approximation, the overhead caused by the used of formula (8) needs to be justified. This is quite easy since the latter only requires the numerical computation of the zero of a given function. This is done very efficiently by a Newton-Raphson method. The computation time is very small.

• Let us now compare the approximation given by $\hat{p}$ with the results obtained using Kirk's approximation. As we have already pointed out, the case in favor of the use of $\hat{p}$ is mostly based on the easy computations of the Greeks which in turn give sensible hedging portfolios. Since Kirk's formula also leads to two *delta hedges*, one can wonder how the performances of the two hedging strategies compare along a given scenario.

To illustrate this point we reproduce a simulation study done in [5]. For a given scenario representing a possible sample realization of the time evolution of the underlying assets $S_1(t)$ and $S_2(t)$, we compare the payoff of the spread option at maturity with the terminal value of the portfolio obtained by n hedging operations throughout the life of the option. So for each scenario, we obtain three tracking errors computed at maturity by trying to replicate the payoff of the option by the value of portfolios obtained by re-balancing the portfolio n times using the *delta hedge* prescriptions given by the Bachelier's method, Kirk's formula and Proposition 6. To avoid an artificial dependence with respect to a particular scenario, we repeat this operation for a large number of scenarios and we actually compare the standard deviations of the tracking errors. The plots of Figure 10 where obtained by varying n from 1 to 1000 and with two different sets of parameters, but the results are pretty generic. Indeed, we found that in most cases, the hedges provided by Proposition 6 and Kirk's model are equally very good, with in some cases a significant advantage for the former which seems to perform much better in many cases. In any cases, these two models clearly out-perform the Bachelier's model. The reasons for this poor hedging performance of the Bachelier's model are twofold. Firstly, the Bachelier's model is based on a distribution assumption which stands the test of the pricing formula, but which cannot survive differentiation. Secondly, it is intrinsically a one-factor model trying to directly model the distribution of the difference $S_2(T) - S_1(T)$, and as such, it cannot capture the subtle structure of the true nature of the two-factor model log-normal model we are dealing with.

7.4. **Extension to Jump-Diffusion Models.** The inclusion of a jump term in the stochastic differential equations giving the dynamics of the underlying assets was proposed by Merton [35] and Cox and Ross [9] almost thirty years ago. Nevertheless these jump diffusion models remain at the level of a scientific curiosity, mostly because of the difficulties associated with their statistical calibration. A renewal of interest in these models was prompted by the consequences of several severe market crashes and the extremely volatile behavior of the prices of new instruments such as electric power for example.

The approximation formula for the price of a spread option which we presented in this section can easily be adjusted to apply to underlying asset price dynamics with jumps. At the risk of been swept by the tidal wave of jump-diffusion processes introduced in the last ten years, we venture outside

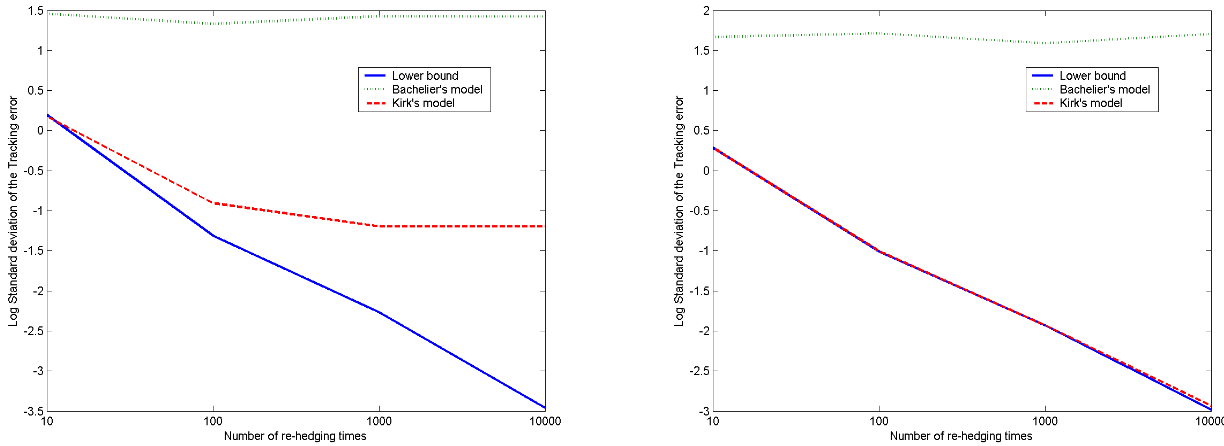

FIGURE 10. Behavior of the tracking error as the number of re-hedging times increases. The model data are $x_1 = 100$, $x_2 = 110$, $\sigma_1 = 10\%$, $\sigma_2 = 15\%$ and $T = 1$. $\rho = 0.9$, $K = 30$ (left) and $\rho = 0.6$, $K = 20$ (right).

the class of continuous diffusion processes to prove our claim. The class of jump processes we are considering is often referred to as the Merton's jump model in the financial literature. So, we allow for jumps in the risk-neutral dynamics of the underlying assets S_1 and S_2 by assuming that they are given by stochastic equations of the form:

$$\frac{dS_i(t)}{S_i(t-)} = (r - q_i - \lambda_i \mu_i)dt + \sigma_i dW_i(t) + (e^{J_i(t)} - 1)dN_i(t) \tag{39}$$

where N_1 and N_2 are two independent Poisson processes of intensity λ_1 and λ_2. They are also assumed to be independent of W_2 and W_1. Finally, $(J_1(t))_{t \geq 0}$ and $(J_1(t))_{t \geq 0}$ are assumed to be independent sequences of independent Gaussian random variables $N(m_i, s_i^2)$. In the same way the stochastic differential equation 13) giving the Samuelson's dynamics could be solved to give the explicit expressions (14) for the underlying assets, equation (39) can also be solved by a simple use of the extension of Itô's formula for processes with jumps. We get the integrated price dynamics

$$S_i(T) = x_i \exp\left((r - q_i - \sigma_i^2/2 - \lambda_i \mu_i)T + \sigma_i W_i(T) + \sum_{k=1}^{N_i(T)} J_i(k) \right)$$

The expectation giving the price p of a spread option can be computed by first conditioning with respect to the Poisson processes, and in so doing, we reduce the problem to pricing spread options for underlying assets with log-normal distributions. Indeed, given $N_1(T)$ and $N_2(T)$, the random variables $S_1(T)$ and $S_2(T)$ still have a log-normal distribution and the lower bound can be used. This leads to the following result.

Proposition 8. [5] *If we set* $\mu_i = e^{m_i + s_i^2/2} - 1$ *for* $i = 1, 2$ *and if we denote by* $\hat{p}(x_1, x_2, \sigma_1, \sigma_2, \rho)$ *the price approximation given by (8), then the price of the spread option in the Merton's model can* *be approximated by the quantity:*

$$\hat{p}^{jumps} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} e^{-(\lambda_1+\lambda_2)T}\frac{(\lambda_1 T)^i(\lambda_2 T)^j}{i!j!}\hat{p}\left(\tilde{x}_1, \tilde{x}_2, \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\rho}\right) \tag{40}$$

with

$$\begin{aligned}
\tilde{x}_1 &= x_1 e^{-\lambda_1\mu_1 T + i\left(m_1+s_1^2/2\right)} \\
\tilde{x}_2 &= x_2 e^{-\lambda_2\mu_2 T + j\left(m_2+s_2^2/2\right)} \\
\tilde{\sigma}_1 &= \sqrt{\sigma_1^2 + i s_1^2/T} \\
\tilde{\sigma}_2 &= \sqrt{\sigma_2^2 + j s_2^2/T} \\
\tilde{\rho} &= \frac{\rho\sigma_1\sigma_2}{\sqrt{\sigma_1^2 + i s_1^2/T}\sqrt{\sigma_2^2 + j s_2^2/T}}
\end{aligned}$$

The above formula involves the summation of an infinite series. Its rate of convergence can be estimated from a simple upper bound on $\hat{p}$, and given any tolerance level, one easily determines how many terms we need in (40) to satisfy the prescribe error bound.

8. EXTENSIONS AND GENERALIZATIONS TO ENERGY SPOT PRICES MODELS

As in the case of the fixed income markets, commodity markets mathematical models come in two varieties: either in the form of (finite) factor models, or models for the dynamics of the entire forward curve, as those we shall review in Section 9. This section is devoted to a short review of the simplest factor models used in the industry, together with a discussion of the consequences of the assumptions of these models on the price of spread options.

The simplest finite factor models are the one-factor models where the factor is chosen to be the commodity spot price. So for the purpose of the present section, we can assume that $S_1(t)$ and $S_2(t)$ are the spot prices at time t of two commodities. It is difficult to say whether or not the models used in the previous sections are good for these indexes $S_1(t)$ and $S_2(t)$. Indeed, the stochastic differential equations we used for the dynamics of the underlying indexes under a risk neutral measure. Indeed, these risk adjusted dynamics cannot be observed directly, and the best one can do is to calibrate the coefficients of the models to the observed prices, which we did consistently when making sure that our models could reproduce the observed forward curves.

In order to justify, or at least to motivate, the assumptions of the subsequent models, we take a short excursion in the real world of historical prices and of stochastic models for the historical dynamics of spot prices. It is important to keep in mind that in most cases, Girsanov's theory implies that the stochastic differential equations giving the historical dynamics differ from the risk adjusted models only through the drift part.

The geometric Brownian motion models proposed by Samuelson for the time evolution of equity prices are appropriate for the dynamics of many assets, but they fail to capture one of the main characteristic feature of interest rates and physical commodity prices, *mean reversion.* This feature is included in the historical models by assuming that the dynamics of the underlying indexes $S_i(t)$

are given by geometric Ornstein-Uhlenbeck processes instead of geometric Brownian motions. These processes can be defined as the solutions of stochastic differential equations of the form:

$$dS_i(t) = S_i(t)[-\lambda_i(\log S_i(t) - \alpha_i)dt + \sigma_i dW_i(t)] \tag{41}$$

where the constants λ_i are positive. These constants are called the *mean reversion* coefficients. They can easily estimated from historical data. A simple linear regression can be used to do just that. See for example [7] for details and examples. Notice that the other parameters σ_i and α_i can also be estimated very easily from historical data. Obviously σ_i is the volatility, and it can be estimated empirically from the variance of the increments of the logarithms, while α_i is a simple function of the of the asymptotic mean reversion level. Historical models are useful for risk management, and nowadays for an increasing number of pricing algorithms based on replication arguments and expected utility maximization.

But the major issue is not the statistical estimation of the parameters of the historical model, this is relatively easy. What we need in order to use the results of the theory presented in the first part of this paper, is a risk adjusted model. If we do not have any a-priori information on the risk premium, the drift of the risk adjusted dynamics can be absolutely anything. Indeed, it is easy to cook up a risk premium process so that Girsanov's theorem will turn the historical drift $S_i(t)[-\lambda_i(\log S_i(t) - \alpha_i)dt$ into any prescribed drift. For this reason, it is common practice to specify the risk neutral dynamics directly without trying to derive them from a Girsanov transformation on a model of the historical dynamics appropriately fitted from empirical data. Then, it can be argued that since we want to be working under the risk-neutral measure, the drift has to be equal to the risk free rate and changing it is pure non-sense. However, the risk-neutral drift determines the discounting factor used to determine present values of future cash flows, and as such it can be different from the short interest rate. For example, we included earlier the dividend rate in the risk neutral drift of dividend paying stocks. So in the case of physical commodities, especially those commodities which cannot be stored like electric power, the risk adjusted drift should not only encompass the dividend yield besides the risk free rate but also some form of stochastic convenience yield (*i.e.*, a stochastic dividend yield). Several recent studies have given strong empirical evidence of the presence of a term structure of convenience yield for physical commodities. In particular, a form of *spot convenience yield* is often used as a factor joining the commodity spot price in the list of factors driving the dynamics of the forward prices. See for example [36].

For the rest of this section we shall assume that the risk-neutral dynamics are specified by a mean reverting equation of the form (41). This practice can only be justified by assuming that the risk premium is conspiring to preserve the mean reverting characteristic of the historical drift. Short of mathematical convenience, we do not see on which ground to justify this assumption. Despite this obvious lack of rationale, this assumption is nevertheless now widely accepted as reasonable in the mathematical analysis of the dynamics of the commodity spot prices. See for example [7].

8.1. **Adding Mean Reversion to the Models.** We now assume that the risk neutral dynamics of the underlying indexes are given by stochastic differential equations of the form

$$dS_i(t) = S_i(t)[-\lambda_i(\log S_i(t) - \mu_i)dt + \sigma_i dW_i(t), \qquad i = 1, 2 \tag{42}$$

where as before the volatilities σ_1 and σ_2 are positive constants, where W_1 and W_2 are two Brownian motions with correlation ρ, where $\lambda_1 > 0$, $\lambda_2 > 0$, and where μ_1 and μ_2 are real constants. The positive constants λ_i give the rates of mean reversion. Indeed, as we are about to see, indexes satisfying these dynamical equations tend to revert toward the levels $e^{\mu_i^\infty}$ if we set $\mu_i^\infty = \mu_i - \sigma_i^2/2\lambda_i$. Various forms of equations (42) have been used as models for asset dynamics. For example, Schwartz [41] introduced them to derive closed form formulae for commodity contract prices.

Equations (42) can be best understood after a simple transformation leading to the dynamics of the logarithms of the underlying indexes. Setting $X_i(t) = \log S_i(t)$, a simple application of Itô's formula gives:

$$dX_i(t) = -\lambda[X_i(t) - \mu_i^\infty]dt + \sigma_i dW_i(t)], \qquad i = 1,2 \tag{43}$$

which shows that the logarithms of the indexes are nothing but classical Ornstein-Uhlenbeck processes whose mean reverting properties are well-known.

Even though equations (42) are more involved than the equations giving the dynamics of the geometric Brownian motions used in the previous sections, we can still derive explicit formulae for the indexes $S_i(T)$ in terms of exponentials of correlated Gaussian variables. Indeed, $S_i(T) = e^{X_i(T)}$ with:

$$X_i(T) = \mu_i^\infty + e^{-\lambda_i(T-t)}[X_i(t) - \mu_i^\infty] + \sigma_i \int_t^T e^{-\lambda_i(T-s)} dW_i(s) \tag{44}$$

Hence:

$$S_i(T) = e^{\mu_i^\infty + e^{-\lambda_i T}(x_i - \mu_i^\infty) + \sigma_{i,T}\xi_i} \tag{45}$$

where

$$\sigma_{i,T} = \sigma_i \sqrt{\frac{1 - e^{-2\lambda_i T}}{2\lambda_i}}, \qquad i = 1,2$$

and ξ_1 and ξ_2 are $N(0,1)$ random variables with correlation coefficient $\tilde\rho$ given by:

$$\begin{aligned}
\tilde\rho &= \frac{1}{\sigma_{1,T}\sigma_{2,T}} \mathbb{E}\{\xi_1\xi_2\} \\
&= \frac{1}{\sigma_{1,T}\sigma_{2,T}} \mathbb{E}\left\{\sigma_1 \int_0^T e^{-\lambda_1(T-s)} dW_1(s) \sigma_2 \int_0^T e^{-\lambda_2(T-s)} dW_2(s)\right\} \\
&= \frac{\rho\sigma_1\sigma_2}{\sigma_{1,T}\sigma_{2,T}} \frac{1 - e^{-(\lambda_1+\lambda_2)T}}{\lambda_1 + \lambda_2} \\
&= \rho \frac{\sqrt{\lambda_1\lambda_2}}{(\lambda_1+\lambda_2)/2} \frac{1 - e^{-(\lambda_1+\lambda_2)T}}{\sqrt{1 - e^{-2\lambda_1 T}}\sqrt{1 - e^{-2\lambda_2 T}}}
\end{aligned}$$

Consequently, the price p of a spread with strike K and maturity T on the difference between the underlying indexes S_1 and S_2 whose dynamics are given by (42) is given by the formula:

$$p = e^{-rT}\mathbb{E}\{(S_2(T) - S_1(T) - K)^+\} = \Pi(\alpha, \beta, \gamma, \delta, \kappa, \tilde\rho)$$

with

$$\alpha = e^{-rT + \mu_2^\infty + e^{-\lambda_2 T}(x_2 - \mu_2^\infty) - \sigma_{2,T}^2/2} \qquad \text{and} \qquad \beta = \sigma_{2,T} = \sigma_2 \sqrt{\frac{1 - e^{-2\lambda_2 T}}{2\lambda_2}}$$

$$\gamma = e^{-rT+\mu_1^\infty + e^{-\lambda_1 T}(x_1-\mu_1^\infty)-\sigma_{1,T}^2/2} \qquad \text{and} \qquad \delta = \sigma_{1,T} = \sigma_1\sqrt{\frac{1-e^{-2\lambda_1 T}}{2\lambda_1}}$$

with $\tilde\rho$ defined as above, and $\kappa = Ke^{-rT}$.

In particular, we can use our formula to find an excellent approximation to this price and an excellent hedging portfolio.

Calibration and Consistency with the Forward Curves. As in the case of the Bachelier model, we can try to generalize the above mean reverting models in order to make it consistent with observed forward curves. At least in principle, this should be done in exactly the same way. But as we are about to see, technical difficulties arise and the computations are more involved. Indeed, if we replace the dynamics of the spot prices by the same mean reverting equations with time dependent coefficients, we get the following stochastic differential equations (we drop the subscript $i = 1, 2$ since it is irrelevant in the present discussion)

$$dS(t) = S(t)[-\lambda(\log S(t) - \mu_t)dt + \sigma_t dW_t]$$

and the logarithm $X(t) = \log S(t)$ of the underlying index satisfies the equation:

$$dX(t) = -\lambda[X(t) - (\mu_t - \frac{\sigma_t^2}{2\lambda})]dt + \sigma_t dW_t$$

which can be explicitly solved, giving:

$$X_t = \mu_t - \frac{\sigma_t^2}{2\lambda} + e^{-\lambda t}[X_0 - \mu_0 + \frac{\sigma_0^2}{2\lambda}] - \int_0^t e^{-\lambda(t-s)}[\mu_s - \frac{\sigma_s^2}{2\lambda}]ds + \int_0^t e^{-\lambda(t-s)}\sigma_s dW_s$$

and consequently, since as before:

$$F(0,T) = \mathbb{E}_0\{S(T)\} = \mathbb{E}_0\{e^{X(T)}\}$$

we get the formula:

$$\begin{aligned} F(0,T) \quad &= \quad \exp\left[\mu_T - \frac{\sigma_T^2}{2\lambda} + e^{-\lambda T}\left(X_0 - \mu_0 + \frac{\sigma_0^2}{2\lambda}\right) - \int_0^T e^{-\lambda(T-s)}\left(\mu_s - \frac{\sigma_s^2}{2\lambda}\right)ds \right. \\ &\qquad \left. + \frac{1}{2}\int_0^T e^{-2\lambda(T-s)}\sigma_s^2 ds\right] \end{aligned}$$

The problem is now to find, for each given forward curve $T \hookrightarrow F(0,T)$, a function $t \hookrightarrow \mu_t$ and/or a function $t \hookrightarrow \sigma_t$ satisfying the above equality.

8.2. **Introducing Jumps.** If the models of the previous subsection can easily be argued to be appropriate for most commodity spot prices, it will still fall short of adequate for electricity spot prices. Indeed the latter exhibit extreme volatility and introducing jumps in the model may appear as the best way to include sudden and extreme departures from the mean reverting level. See nevertheless [2] for an alternative way to achieve the same thing with a continuous diffusion without jumps. At the risk of promoting the confusion with the historical and the risk adjusted dynamics, we refer the reader to the right panel of Figure 1 where one can see that the sudden and extreme departures from the mean do appear in the historical data. This figure illustrate that fact in the case of the Palo Verde electric power spot prices. In order to accommodate the case of spread options involving electric power (spark spreads are the typical examples), we allow for jumps in the risk-neutral dynamics of S_2. We assume that:

$$dS_2(t) = S_2(t-)[(r - \lambda\mu)dt + \sigma_2 dW_2(t) + (e^{J_t} - 1)dN(t)$$

where N is a Poisson process of intensity λ independent of W_2 and W_1. $(J_t)_{t\geq 0}$ is a sequence of independent Gaussian random variables (m, s^2) and μ is function of m and s^2. This case is even simpler than the one studied before. Pricing formula have therefore already been obtained.

8.3. **Consistency with the Observed Forward Curves.** As before in the case of the simpler geometric Brownian motion model, we first compute the analytic form of the forward curves, and we find ways to adjust the coefficients of the model to match the theoretical curves carried by the model to the empirical curves.

9. Modeling the Dynamics of the Forward Curves

The spot price models considered in the previous sections are extremely popular because of their intuitive appeal and because of their mathematical tractability. Indeed, modelling the dynamics of the spot prices seems like a reasonable thing to do, and the log-normal distribution is amenable to closed form formulae for many of the simplest single commodity derivatives. Unfortunately, these models are not always satisfactory. Indeed, given the spot price process $\{S_t\}_{t\geq 0}$ of a commodity, the no-arbitrage price $F(t, T)$ at time t of a forward contract with expiry T should be given by the formula:

$$F(t, T) = \mathbb{E}_t\{S_T\} \tag{46}$$

where as before, we use the notation $\mathbb{E}_t\{\ \cdot\ \}$ for the conditional expectation with respect to all the information available up to time t. The log-normal models and their generalizations of the previous sections are simple enough to allow for explicit formulae for the conditional expectations appearing in the right hand side of (46), but unfortunately, the forward curves $T \hookrightarrow F(t, T)$ produced in this way are very rarely consistent with the actual forward curves observed in practice, and the various fixes which we proposed in the previous sections are not satisfactory since they force us to re-calibrate the model every day! This major shortcoming is at the core of the search for more sophisticated models which could account for the observable features of the forward curves. Some of these models are considered in this section. The main departure from the previous approach is to model the dynamics of the entire forward curve instead of modeling only the dynamics of its leftmost point. So we state a stochastic differential equation (in fact a system of a continuum of such equations) for the dynamics of the entire forward curve. In analogy with the HJM models of the fixed income markets we assume that for each maturity date T, the dynamics of the forward curves are given by:

$$\frac{dF(t, T)}{F(t, T)} = \mu(t, T)dt + \sum_{k=1}^{n} \sigma_k(t, T)dW_k(t) \qquad t \leq T \tag{47}$$

where $\boldsymbol{W} = (W_1, \ldots, W_n)$ is a n-dimensional standard Brownian motion, and where the n volatilities σ_k are deterministic functions of the current date t and the time-of-maturity T. Such a model is called an n-factor forward curve model. It has been described as desirable in the technical literature (see, *e.g.*, Chapter 8 of the book [7]) though implementations in the commercial software packages available for energy risk management are still at a rather primitive stage.

Whenever we work on pricing, hedging, or asset valuation (power plant, gas storage, . . .), we need to assume that the dynamics have been adjusted for risk, *i.e.*, we need to use risk-neutral probabilities. In that case we will need to set $\mu(t,T) \equiv r$, since it guarantees that $t \hookrightarrow F(t,T)$ is a martingale for each fixed T when discounted. But when we work on risk management, the drift $\mu(t,T)$ will need to be calibrated to historical data.

An Explicit Solution. The coefficients μ and σ_k being assumed deterministic, the dynamical equation (47) can be solved explicitly and we get:

$$F(t,T) = F(0,T)\exp\left[\int_0^t\left[\mu(s,T) - \frac{1}{2}\sum_{k=1}^n \sigma_k(s,T)^2\right]ds + \sum_{k=1}^n\int_0^t \sigma_k(s,T)dW_k(s)\right] \tag{48}$$

which shows that the forward prices have a log-normal distribution. More precisely, we have:

$$F(t,T) = \alpha e^{\beta X - \beta^2/2} \tag{49}$$

for $X \sim N(0,1)$ and

$$\alpha = F(0,T)\exp\left[\int_0^t \mu(s,T)ds\right], \qquad \text{and} \qquad \beta = \sqrt{\sum_{k=1}^n \sigma_k(s,T)^2} \tag{50}$$

Dynamics of the Spot Price. Since the spot price is the left hand point of the forward curve (*i.e.*, $S(t) = F(t,t)$) we can derive an explicit expression for the spot price from equation (48) above.

$$S(t) = F(0,t)\exp\left[\int_0^t [\mu(s,t) - \frac{1}{2}\sum_{k=1}^n \sigma_k(s,t)^2]ds + \sum_{k=1}^n\int_0^t \sigma_k(s,t)dW_k(s)\right] \tag{51}$$

and differentiating both sides we get:

$$\begin{aligned} dS(t) &= S(t)\left[\left(\frac{1}{F(0,t)}\frac{\partial F(0,t)}{\partial t} + \mu(t,t) + \int_0^t \frac{\partial \mu(s,t)}{\partial t}ds - \frac{1}{2}\sigma_S(t)^2\right.\right. \\ &\qquad \left.\left. - \sum_{k=1}^n\int_0^t \sigma_k(s,t)\frac{\partial\sigma_k(s,t)}{\partial t}ds + \sum_{k=1}^n\int_0^t \frac{\partial\sigma_k(s,t)}{\partial t}dW_k(s)\right)dt + \sum_{k=1}^n \sigma_k(t,t)dW_k(t)\right] \end{aligned}$$

where we set:

$$\sigma_S(t)^2 = \sum_{k=1}^n \sigma_k(t,t)^2. \tag{52}$$

One deduce from equation (53) the fact that the instantaneous volatility of the spot price is given by $\sigma_S(t)$. Consequently we have:

$$\frac{dS(t)}{S(t)} = \left[\frac{\partial \log F(0,t)}{\partial t} + D(t)\right]dt + \sum_{k=1}^n \sigma_k(t,t)dW_k(t) \tag{53}$$

where the drift term $D(t)$ is given by:

$$D(t) = \mu(t,t) - \frac{1}{2}\sigma_S(t)^2 + \int_0^t \frac{\partial\mu(s,t)}{\partial t}ds - \sum_{k=1}^n\int_0^t \sigma_k(s,t)\frac{\partial\sigma_k(s,t)}{\partial t}ds + \sum_{k=1}^n\int_0^t \frac{\partial\sigma_k(s,t)}{\partial t}dW_k(s)$$

Remarks:
1. In a risk-neutral setting, the drift appearing in formula (53) has a nice interpretation. The first term, *i.e.*, the logarithmic derivative of the forward, can be interpreted as a discount rate (*i.e.*, the running interest rate) while the term $D(t)$ can be interpreted as a convenience yield which many researchers in the field tried to model directly in order to find no-arbitrage models consistent with the data.
2. As emphasized in [7], this drift is most of the time not Markovian because of the presence of the stochastic integral which typically depends upon the entire past of the forward curve evolution. An exception is provided by the risk-neutral (*i.e.*, $\mu(t,T) \equiv r$) one factor model (*i.e.*, $n = 1$) with volatility $\sigma_1(t,T) = \sigma e^{-\lambda(T-t)}$. Indeed, in this special case, the dynamics of the spot prices are given by (53) with drift:

$$D(t) = \lambda[\log F(0,t) - \log S(t)] + \frac{\sigma^2}{4}(1 - e^{-2\lambda t})$$

which shows that the spot price dynamics are those of an Ornstein-Uhlenbeck process with time dependent mean reverting level defined by:

$$\frac{dS(t)}{S(t)} = [\mu(t) - \lambda \log S(t)]dt + \sigma dW(t) \tag{54}$$

which is exactly the generalization considered in Subsection 8.1 to accommodate mean reversion. The point of this remark is to stress the fact that mean reversion of the spot price is closely related to the exponential decay of the forward volatility for large times to maturity.

Changing Variables. During our discussion of the format of the data available to analysts, we made a case for the switch to the time-to-maturity variable τ as an alternative to the time-of-maturity T. This change of variable could deceivingly appear as a mere change of notation without much effect on the analytic expression giving the dynamics of the forward curves. This is not the case, for computing derivatives with respect to t of the function $t \hookrightarrow F(t, t+\tau)$ for τ fixed, involves partial derivatives of F with respect to both of its variables since t appears in both places. This is in contrast with computing the derivative with respect to t of $t \hookrightarrow F(t,T)$ for T fixed which obviously only involves the first partial derivative of F. To be more specific, if we set:

$$\tilde{F}(t,\tau) = F(t, t+\tau), \qquad \tilde{\mu}(t,\tau) = \mu(t, t+\tau), \quad \text{and} \quad \tilde{\sigma}_k(t,\tau) = \sigma_k(t, t+\tau), \tag{55}$$

then the dynamics (47) of the forward curve rewrites:

$$d\tilde{F}(t,\tau) = \tilde{F}(t,\tau)\left[\left(\tilde{\mu}(t,\tau) + \frac{\partial}{\partial \tau}\log \tilde{F}(t,\tau)\right)dt + \sum_{k=1}^{n} \tilde{\sigma}_k(t,\tau) dW_k(t)\right], \qquad \tau \geq 0. \tag{56}$$

9.1. **Calibration by Principal Component Analysis.** We now present an application of a standard data analysis technique to the calibration of forward energy curve models. Our goal is twofold: we try to justify the assumptions behind the model (47) chosen for the dynamics of the forward curves, and at the same time we show how to identify and estimate the σ_k from actual historical market data.

Fundamental Assumption. The main assumption of this subsection concerns the actual form of the volatility functions σ_k. We shall assume that the latter are of the form:

$$\sigma_k(t,T) = \sigma(t)\sigma_k(T-t) = \sigma(t)\sigma_k(\tau) \tag{57}$$

where the function $\sigma(t)$ of the single variable t will be determined shortly. Revisiting formula (52) giving the instantaneous volatility of the spot price in light of this new assumption we get:

$$\sigma_S(t) = \tilde{\sigma}(0)\sigma(t) \tag{58}$$

provided we set:

$$\tilde{\sigma}(\tau) = \sqrt{\sum_{k=1}^{n} \sigma_k(\tau)^2}.$$

This shows that our assumption (57) implies that the function $t \hookrightarrow \sigma(t)$ is up to a constant multiplicative factor, necessarily equal to the instantaneous volatility of the spot price. We shall use this remark to estimate $\sigma(t)$ from the data, for example by computing the standard deviation of the spot price in a sliding window of 30 days. The plot of the instantaneous standard deviation of the Henry Hub natural gas spot price computed in this way is given in Figure 11.

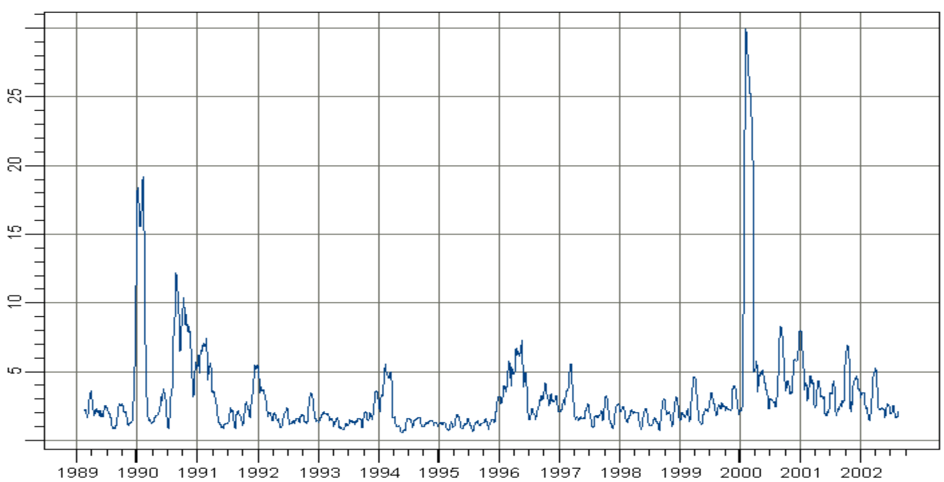

FIGURE 11. Henry Hub natural gas instantaneous standard deviation computed in a sliding window of length 30 days.

The Rationale for the Use of PCA. For the purpose of the present discussion we fix times to maturity $\tau_1, \tau_2, \cdots, \tau_N$ and we assume that on each given day t, we have quoted forward prices for each of the times to maturity $T_1 = t + \tau_1, T_2 = t + \tau_2, \cdots, T_n = t + \tau_N$. This assumption may require a little massaging of the data which we explain in detail below. In any case, it is a reasonable starting point for the present discussion. According to the stochastic dynamics derived in (56), we have:

$$\frac{dF(t,\tau_i)}{F(t,\tau_i)} = \left(\tilde{\mu}(t,\tau_i) + \frac{\partial}{\partial \tau} \log \tilde{F}(t,\tau_i)\right) dt + \sigma(t) \sum_{k=1}^{n} \sigma_k(\tau_i) dW_k(t) \qquad i = 1, \ldots, N$$

We now define the $N \times n$ deterministic matrix F by $F = [\sigma_k(\tau_i)]_{i=1,\cdots,N,\ k=1,\cdots,n}$. We can assume without any loss of generality that the n column vectors $\sigma_k(\,\cdot\,)$ for $k = 1, \dots, n$ are orthonormal vectors of $\mathbb{R}^N$. The goal of this subsection is to explain how we estimate these vectors from historical data. First, we derive the dynamics of the logarithm of the forward prices applying Itô's formula:
(59)

$$d \log \tilde{F}(t,\tau_i) = \left(\tilde{\mu}(t,\tau_i) + \frac{\partial}{\partial \tau_i} \log \tilde{F}(t,\tau_i) - \frac{1}{2}\sigma(t)^2 \tilde{\sigma}(\tau_i)^2 \right) dt + \sigma(t) \sum_{k=1}^{n} \sigma_k(\tau_i) dW_k(t), \quad \tau \geq 0.$$

Next we compute the instantaneous variance/covariance matrix $\{M(t); t \geq 0\}$ defined by:

$$d[\log(F(\,\cdot\,,\tau_i)), \log(F(\,\cdot\,,\tau_j))]_t = M_{i,j}(t)dt$$

From (59) we see that:

$$M(t) = \sigma(t)^2 \left(\sum_{k=1}^{n} \sigma_k(\tau_i)\sigma_k(\tau_j) \right)$$

or equivalently that

$$M(t) = \sigma(t)^2 F F^T.$$

Our interest in the above computation is the fact that it gives a clear strategy to extract the components of the matrix F from historical data. Indeed, after estimating the instantaneous spot volatility $\sigma(t)$ in a rolling window as we explained earlier, we can estimate the matrix FF^T from historical data as the empirical autocovariance of $\ln(F(t,\cdot) - \ln(F(t-1,\cdot))$ after normalization by $\sigma(t)$. Diagonalizing this empirical variance/covariance matrix, identifying the principal components among the orthonormal basis of eigenvectors to get the columns of F is the purpose of classical principal component analysis.

So the gist of this derivation is that under the assumption (57), if one is willing to *normalize* the log returns of the forward contract prices by the instantaneous volatility of the spot price, then the instantaneous autocovariance structure of the entire forward curve becomes time independent and hence, it can be estimated from the data. Moreover, diagonalizing this autocovariance matrix will provide the function $\sigma_k(\tau)$, and from their relative sizes, we will be able to decide how many do contribute significantly to the dynamics, effectively choosing the order n of the model. This singular value decomposition of the covariance matrix and its interpretation are known as Principal Component Analysis (PCA for short) and its relevance to the historical modeling of the dynamics of the forward curves has been recognized in the fixed income markets first, before being adopted in the analysis of the energy markets. See for example [32] and [7].

Remark: PCA is based on the estimation of a covariance matrix, and as long as statistical estimation goes, some form of stationarity is needed to be able to base the estimates on time averages. In the approach described above, the reduction to stationarity was done the transformation of the data provided by the division by $\sigma(t)$. Introducing the instantaneous spot volatility as a normalizing factor to capture the seasonality of the data is not the only way to use the PCA as a calibration tool. Another approach has been advocated by the scientists of Financial Engineering Associates (an energy software and consulting provider). They propose to bin the forward curves into 12 groups according to the month of the date t, and then to perform the PCA in each of these bins. This *seasonal principal component analysis* as they call it more likely to capture the seasonality of the forward curves when present. But at the same time, it reduces drastically the size of the data sets from which the covariance matrices are estimated. Not only does it restrict the length of the forward curves modeled in this way (typically to 12 months) but it also creates serious difficulties in the assessment of the confidence in the results. Indeed, the PCA involves the inversion and the diagonalization of the covariance matrix estimate, and these operations are notoriously erratic and non-robust, especially when the matrix is poorly conditioned, which is the expected situation.

9.2. **Pricing Calendar Spreads in this Framework.** As we already explained, a calendar spread is the simplest forms of spread option because it involves only one underlying commodity, the two underlying indexes being the prices of two forward contracts with different maturities, say T_1 and T_2. So using the same notation as before we have:

$$S_1(t) = F(t, T_1) \qquad \text{and} \qquad S_2(t) = F(t, T_2)$$

and since we consider only deterministic factors, these forward prices are log-normally distributed and we can use our pricing formula with the coefficients α, β, γ, δ and ρ given by

$$\alpha = F(0, T_1), \quad \beta = \sqrt{\sum_{k=1}^{n} \sigma_k(s, T_1)^2} \quad \gamma = F(0, T_2), \quad \text{and} \qquad \delta = \sqrt{\sum_{k=1}^{n} \sigma_k(s, T_2)^2}$$

according to (50). Recall that $\mu \equiv 0$ since we are using the risk-neutral dynamics of the forward curves for pricing purposes. On the other hand the correlation coefficient is given by:

$$\rho = \frac{1}{T} \sum_{k=1}^{n} \int_0^T \sigma_k(s, T_1) \sigma_k(s, T_2)\, ds$$

9.3. **Pricing Spark Spreads in this Framework.** The methodology of this subsection can be used for all the cross-commodity spreads discussed earlier in the text. For the purpose of illustration, we discuss the specific case of the spark spread options, for they provide a challenging example of cross-commodity instrument. Their pricing is of great importance both for risk management and asset valuation purposes. We proceed to price them in the framework introduced in this section. Because we are now dealing with two commodities, we need to adjust the notation: we choose to use the subscript *e* for the forward prices, times to maturity, volatility functions, . . . pertaining to the electric power, and the subscript *g* for the quantities pertaining to natural gas.

Description of a Spark Spread Option. Let $F_e(t, T_e)$ and $F_g(t, T_g)$ be the prices at time t of a forward electricity contract with time-of-maturity T_e and a forward natural gas contract with time-of-maturity T_g. For the purpose of this subsection, a crack spread option with maturity date T is a contingent claim maturing at time T which pays the amount

$$\left(F_e(T, \tau_e) - H F_g(T, \tau_g) - K\right)^+ .$$

where the efficiency H is a fixed conversion factor and K is the strike. Obviously we assume that $T < \min\{T_1, T_2\}$. The buyer of such an option may be the owner of a power plant that transforms gas into electricity and may want to protect himself against too low electricity prices and too high gas prices.

Joint Dynamics of the Commodities. In complete analogy with (47) we assume that the joint dynamics of the forward prices of the two commodities are given by equations of the form:

$$\left\{ \begin{array}{lcl} dF_e(t,T) &=& F_e(t,T)[\mu_e(t,T)dt + \sum_{k=1}^n \sigma_{e,k}(t,T)dW_k(t) \\ dF_g(t,T) &=& F_g(t,T)[\mu_g(t,T)dt + \sum_{k=1}^n \sigma_{g,k}(t,T)dW_k(t) \end{array} \right. \tag{60}$$

Notice that each commodity has its own volatility factors, and that the correlation between the two dynamics is built into the fact that they share the same driving Brownian motion.

Fitting the Join Cross-Commodity Model. Without going into all the gory details of the implementation, we give the general components of the fitting procedure.

We assume that on any given day t we have electricity forward contract prices for $N^{(e)}$ times to maturity $\tau_1^{(e)} < \tau_2^{(e)}, \cdots < \tau_{N^{(e)}}^{(e)}$, and respectively that we have natural gas forward contract prices for $N^{(g)}$ times to maturity $\tau_1^{(g)} < \tau_2^{(g)}, \cdots < \tau_{N^{(g)}}^{(g)}$. As explained in our discussion of the empirical data issues, typically we have $N^{(e)} = 12$ and $N^{(g)} = 36$ and even $N^{(g)} = 84$ more recently. Assuming (57) for both the electricity and the natural gas forward volatilities and estimating the instantaneous volatilities $\sigma^{(e)}(t)$ and $\sigma^{(g)}(t)$ of the electricity and gas spot prices in a rolling window of 30 days, we can consider, for each day t, the $N = N^{(e)} + N^{(g)}$ dimensional random vector $\mathbf{X}(t)$ defined by:

$$\mathbf{X}(t) = \left[\begin{array}{c} \left(\frac{\log F_e(t+1,\tau_j^{(e)}) - \log F_e(t,\tau_j^{(e)})}{\sigma^{(e)}(t)} \right)_{j=1,\cdots,N^{(e)}} \\ \left(\frac{\log F_g(t+1,\tau_j^{(g)}) - \log F_g(t,\tau_j^{(g)})}{\sigma^{(g)}(t)} \right)_{j=1,\cdots,N^{(g)}} \end{array} \right]$$

Running the PCA algorithm on the historical samples of this random vector $\mathbf{X}(t)$ will provide a small number n of significant factors, and for $k = 1, \cdots, n$, the first $N^{(e)}$ coordinates of these factors will give the electricity volatilities $\tau \hookrightarrow \sigma_k^{(e)}(\tau)$ for $k = 1, \cdots, n$, while the remaining $N^{(g)}$ coordinates will give the gas volatilities $\tau \hookrightarrow \sigma_k^{(g)}(\tau)$.

Pricing a Spark Spread Option. Risk neutral pricing arguments imply that the price of the spark-spread option at time t is given by:

$$p_t = e^{-r(T-t)} \mathbb{E}_t \left\{ (F_e(T,T_e) - HF_g(T,T_g) - K)^+ \right\} \tag{61}$$

This pricing formula is handled with the pricing algorithms developed in this paper because the distributions of $F_e(T,\tau_e)$ and $F_g(T,\tau_g)$ will be log-normal under the pricing measure. Indeed, the prices at time T are given by our PCA-based model. So under a risk-neutral measure the prices become

$$F_e(T,T_e) = F_e(0,T_e) \exp \left[-\frac{1}{2} \sum_{k=1}^n \int_0^T \sigma_{e,k}(s,T_e)^2 ds + \sum_{k=1}^n \int_0^T \sigma_{e,k}(s,T_e) dW_k(s) \right]$$

and:

$$F_g(T,T_g) = F_g(0,T_g) \exp \left[-\frac{1}{2} \sum_{k=1}^n \int_0^T \sigma_{g,k}(s,T_g)^2 ds + \sum_{k=1}^n \int_0^T \sigma_{g,k}(s,T_g) dW_k(s) \right]$$

As explained above, the correlation between the two markets in built into the random driving factors because we use the same Brownian motions to drive the stochastic differential equations (60) both for electricity and gas. Consequently we can use our pricing algorithm with $S_1(t) = HF_g(t, T_g)$ and $S_2(t) = F_e(t, T_e)$ so that we need only to choose:

$$\alpha = F_e(t, T_e), \qquad \text{and} \qquad \beta = \sqrt{\frac{1}{T}\sum_{k=1}^{n}\int_0^T \sigma_{e,k}(s,T)^2\, ds}$$

for the first log-normal distribution,

$$\gamma = HF_g(t, T_e), \qquad \text{and} \qquad \delta = \sqrt{\frac{1}{T}\sum_{k=1}^{n}\int_0^T \sigma_{e,k}(s,T)^2\, ds}$$

for the second one and

$$\rho = \frac{1}{T}\int_0^T \sum_{k=1}^{n} \sigma_{e,k}(s, T_e)\sigma_{g,k}(s, T_g) ds$$

for their correlation.

10. CONCLUSION

This review concentrates on the mathematics of the pricing and hedging of spread options. This choice was motivated by the utmost importance of these financial instruments, and the rich variety of mathematical tools which have already been used in their analysis. We reviewed the results published on this subject in the economics, financial, business and mathematical literature, and we tried to shed some light on the major issue of hedging and computing accuracy in approximation schemes. Our goal is to make the applied mathematical community aware of the slue of problems remaining to be solved.

In so doing, we devoted a good part of our efforts to the discussion of a new pricing paradigm introduced recently by the authors. While giving an exhaustive review of the existing literature, we compared the performance both from a numerical and an analytical points of view, of our algorithm against all the existing methods known to us.

We chose to illustrate the concepts and the numerical methods reviewed in this paper with examples from the energy markets. In doing so, we ended up providing a rather thorough review of the main technical challenges of these markets. Indeed, we reviewed extensively a) the important features of the data available to the traders and analysts, b) the statistical models used to describe them and to build risk management systems, and c) the pricing models used to value the complex instruments of the energy markets. The latter are a fertile ground for mathematical investigations and we believed that applied mathematicians will be well advised to pay more attention to their analysis.

The paper tried to prepare the reader for some of the challenges facing that part of the applied mathematics community interested in the practical applications of financial mathematics. Natural extensions of some of the results reviewed in this paper could lead to useful developments. For example, some of the numerical approximations presented here can clearly be generalized to the case of an option on linear combinations of assets (basket options, rainbow options, ...) or any linear combination of prices of a single asset at different times (discrete-time average Asian option for example.) Such extensions would provide efficient algorithms to compute prices and hedges for these options; they are sorely needed. On another front, it is clear that some of the explicit formulae reviewed in the text can be inverted. This allows for efficient computations of implied quantities such as volatilities and correlations. From these implied volatilities and correlations, one should be able to build more complex models (stochastic volatility and/or stochastic correlation, jumps, . . .) which should better fit the market reality. This would be a welcome development for it is still unclear what these implied quantities should be in the framework of spread options.

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