HEDGING IN PARTIALLY OBSERVABLE MARKETS

SIMON BRENDLE AND RENÉ CARMONA

Abstract. This paper deals with optimal control problems appearing in pricing and hedging theories when financial market models are incomplete. We use exponential, power and logarithmic utility functions and in each case, we derive probabilistic representation formulae for the value function of the classical Merton problem whether or not the portfolio includes a contingent claim written on some of the non-tradable factors, value functions crucial in indifference pricing. These formulae become inequalities when the contingent claims also depend on the tradable instruments. We work in the full generality of non-Markovian dynamics, and since we cannot rely on the derivation and the analyzes of Hamilton-Jacobi-Bellman equations, we rely exclusively on probabilistic arguments. The importance of the generalization to the non-anticipative case is illustrated with the discussion of partially observable markets. We show that their analyzes can be reduced to the analyzes of fully observable non-Markovian market models, and we use the results of the first part of the paper to compute the value functions in some conditionally Gaussian models.

In each of the market models considered in the paper, we analyze the mean-variance hedging problem with the tools developed to compute the values functions.

1. Introduction

Merton's problem is a classic of the quantitative theory of financial markets. Its solution highlighted the importance of stochastic control in financial applications. Recently, a renewal of interest in this problem was created by progress made in the understanding of incomplete market models, and especially the developments in the theory of indifference pricing first introcuded by Hodges and Neuberger in [10] (see for example [6, 14]), and entire research programs are devoted to computations of the value functions of all sorts of optimal portfolio problems. The reader interested in an exhaustive overview of indifference pricing is referred to [2]. This paper is concerned with the computation of the value functions of stochastic control problems with partial observations. The motivation is twofold. Firstly, we want to gain a better understanding of pricing and hedging when some of the economic factors cannot be observed. See for example [4] for an example from the energy markets. Secondly, we try to understand the robustness of economic models with possible misspecifications, like those discussed in the recent works of Hansen and collaborators on robustness. See for example [11] and the references therein.

Most results in this area rely on the analysis of Hamilton-Jacobi-Bellman equations derived from the dynamic programming principle [1, 7]. These non-linear partial differential equations do not always have solutions in the classical sense. The theory of viscosity solutions (see for example [7]) was invented in part to overcome this difficulty, but its technical nature is

Date: June 24, 2004.

regarded as an *unnecessary overkill* by many financial engineers and mathematicians. Besides this issue of taste or background, one of the disadvantages of this approach is the fact that it requires the factor dynamics to be Markovian. This is especially constraining in the case of partially observed models. Indeed, in order to remain in the Markovian setting, one needs to take a filtering point of view and replace the unobserved factors by their conditional distributions (the so-called filters) and rely on the Zakai and Kushner equations to preserve the Markov property. So in this approach, the existence of unobservable factors transforms the original finite dimensional optimal control problem into an infinite dimensional one. In particular, the Hamilton-Jacobi-Bellman equation is a non-linear partial differential equation in infinite dimensions! See for example [12] and [4].

We show in Section 3 that a simple conditioning argument can be used to replace the original finite dimensional control problem with partial observations by a control problem with full observations and the same dimension. However, the coefficients of this new control problem depend on the past of the observed factors, even if the original dynamics were assumed to be Markovian. This prompted us to extend the results of the partial differential approach to the more general case of non-anticipative dynamics. We use ideas from the duality approach successfully implemented in [5] and thoroughly reviewed in [15] to identify the candidates for optimal portfolios and changes of measures, and we use direct probabilistic computations to derive formulae for the value functions. It is known from [14, 9, 3] that these formulae ought to be exact when the terminal wealth include contingent claims written on the non-tradable factors, but that only bounds can be proven when these claims also depend upon the tradable factors [16]. We extend these results to the general non-anticipative setting, and we prove that value function probabilistic representations exist beyond the Markovian framework.

Because mean-variance hedging [8] is very much in the same spirit as the hedging procedures exhibited in this paper, for each of the market models considered in the paper, we analyze the mean-variance hedging problem with the tools developed in this paper to compute the values functions.

After the completion of this work, we learned that M. Tehranchi proved independently some of the results of Section 2 using different methods [17].

2. Hedging in Incomplete Markets with Full Observations

We consider a pair of stochastic process $\{Y(t) : t \geq 0\}$ and $\{S(t) : t \geq 0\}$ such that

$$dY(t) = g(t)\,dt + h(t)\,dW(t) \tag{1}$$

and

$$\frac{dS(t)}{S(t)} = \mu(t)\,dt + \sqrt{1-\rho^2}\,\sigma(t)\,dW^{\perp}(t) + \rho\,\frac{\sigma(t)}{h(t)}\,dY(t), \tag{2}$$

where $\{W(t) : t \geq 0\}$ and $\{W^{\perp}(t) : t \geq 0\}$ are independent Wiener processes. We denote by $\mathbb{F}^Y = \{\mathcal{F}^Y(t) : t \geq 0\}$ and $\mathbb{F}^{Y,S} = \{\mathcal{F}^{S,Y}(t) : t \geq 0\}$ the natural filtrations of the processes $\{Y(t) : t \geq 0\}$ and $\{(S(t), Y(t)) : t \geq 0\}$ respectively. Equation (2) can be rewritten in the form

$$\frac{dS(t)}{S(t)} = \sigma(t)\,(\lambda(t)\,dt + dW^S(t)), \tag{3}$$

provided we set

$$W^S(t) = \rho\, W(t) + \sqrt{1-\rho^2}\, W^{\perp}(t) \quad \text{and} \quad \lambda(t) = \frac{\mu(t)}{\sigma(t)} + \rho\,\frac{g(t)}{h(t)}. \tag{4}$$

We assume that the processes $\{g(t) : t \geq 0\}$, $\{h(t) : t \geq 0\}$, $\{\lambda(t) : t \geq 0\}$ are adapted to the filtration $\{\mathcal{F}^Y(t) : t \geq 0\}$. For example, we could assume that $g(t)$, $h(t)$, $\lambda(t)$ are deterministic functions of $Y(t)$ as we shall do to deal with the stochastic control problem with partial observations which is studied in the second part of this paper. We also assume that the asset $\{S(t) : t \geq 0\}$ is traded while $\{Y(t) : t \geq 0\}$ is not. We consider agents investing in the traded asset only. The wealth process $\{X(t) : t \geq 0\}$ satisfies the stochastic differential equation

$$dX(t) = \theta(t)\,\frac{1}{\sigma(t)}\,\frac{dS(t)}{S(t)} = \theta(t)\,\lambda(t)\,dt + \theta(t)\,dW^S(t), \tag{5}$$

where $\theta(t)/\sigma(t)$ denotes the amount of wealth invested in the risky asset. The portfolio processes admissible considered admissible in this paper will be square integrable in the sense that $\mathbb{E}\{\int_0^T |\theta(t)|^2\,dt\} < \infty$. Alternatively, we may write

$$dX(t) = X(t)\,\pi(t)\,\frac{1}{\sigma(t)}\,\frac{dS(t)}{S(t)} = X(t)\,\pi(t)\,\lambda(t)\,dt + X(t)\,\pi(t)\,dW^S(t),$$

where $\pi(t)/\sigma(t)$ denotes the fraction of wealth invested in the risky asset.

2.1. **The Optimal Control Problem.** Throughout this section, we consider a contingent claim written on the non-traded asset Y. Its payoff is modelled by a random variable ξ which we assume to be $\mathcal{F}^Y(T)$-measurable. Apart from European-style contingent claims, this includes contingent claims with path-dependent payoff, such as Asian options or barrier options. We assume that the investor's risk preferences are described by a utility function. We consider three cases which we treat separately: exponential, power and logarithmic utility functions. In each case, we derive an explicit formula when the claim is underlied by the non-traded asset Y. When the claim pay-off depends also upon the tradable asset, we derive an upper bound for the optimal utility of the stochastic control problem.

Constant Absolute Risk Aversion. We first assume that the investor's risk preferences are modelled by the exponential utility function $U(x) = -\exp(-\gamma\, x)$. According to the martingale representation theorem, there exists a process $\{\theta^*(t) : 0 \leq t \leq T\}$ adapted to the filtration $\{\mathcal{F}^Y(t) : 0 \le t \le T\}$ and such that

$$\exp\Big(-\frac{1}{\rho}\int_0^T \Big(\lambda(t) - \gamma\,(1-\rho^2)\,\theta^*(t)\Big)\,dW(t) - \frac{1}{2\rho^2}\int_0^T \Big(\lambda(t) - \gamma\,(1-\rho^2)\,\theta^*(t)\Big)^2\,dt\Big)$$
$$= \frac{\exp\Big(-\gamma\,(1-\rho^2)\,\xi - \rho\int_0^T \lambda(t)\,dW(t) \;-\frac{1}{2}\int_0^T \lambda(t)^2\,dt\Big)}{\mathbb{E}\Big\{\exp\Big(-\gamma\,(1-\rho^2)\,\xi - \rho\int_0^T \lambda(t)\,dW(t) - \frac{1}{2}\int_0^T \lambda(t)^2\,dt\Big)\Big\}}.$$

Then, we define a process $\{Z(t) : 0 \le t \le T\}$ by

$$Z(t) = \exp\Big(-\frac{1}{\rho}\int_0^t \lambda(u)\,dW(u) + \gamma\,\frac{\sqrt{1-\rho^2}}{\rho}\int_0^t \theta^*(u)\,dW^{S\perp}(u)$$
$$-\frac{1}{2}\,\frac{1}{\rho^2}\int_0^t \lambda(u)^2\,du + \gamma\,\frac{1-\rho^2}{\rho^2}\int_0^t \theta^*(u)\,\lambda(u)\,du - \frac{1}{2}\,\gamma^2\,\frac{1-\rho^2}{\rho^2}\int_0^t \theta^*(u)^2\,du\Big) \tag{6}$$

for all $0 \le t \le T$, where we set:

$$W^{S\perp}(t) = \sqrt{1-\rho^2}\,W(t) - \rho\,W^\perp(t).$$

Notice that, as defined, the process $\{W^{S\perp}(t) : 0 \le t \le T\}$ is a Wiener process independent of $\{W^S(t) : 0 \le t \le T\}$. A straightforward computation gives:

Lemma 1. *Let $\{X^*(t) : 0 \le t \le T\}$ be the wealth process associated to the portfolio process $\{\theta^*(t) : 0 \le t \le T\}$. Then we have*

$$X^*(T) + \xi = x - \frac{1}{\gamma}\,\log Z(T) - \frac{1}{\gamma}\,\frac{1}{1-\rho^2}\,\log\mathbb{E}\Big\{\exp\Big(-\gamma\,(1-\rho^2)\,\xi$$
$$-\rho\int_0^T \lambda(t)\,dW(t) - \frac{1}{2}\int_t^T \lambda(t)^2\,dt\Big)\Big\}. \tag{7}$$

Since $\mathbb{E}\{Z(T)\,X^*(T)\} = x$, this implies

$$\mathbb{E}\{Z(T)\,\log Z(T)\} + \gamma\,\mathbb{E}\{Z(T)\,\xi\} \tag{8}$$
$$= -\frac{1}{1-\rho^2}\,\log\mathbb{E}\Big\{\exp\Big(-\gamma\,(1-\rho^2)\,\xi - \rho\int_0^T \lambda(t)\,dW(t) - \frac{1}{2}\int_0^T \lambda(t)^2\,dt\Big)\Big\}.$$

Whenever $\{\theta(t) : 0 \le t \le T\}$ is an admissible portfolio process, the corresponding wealth process $\{X(t) : 0 \le t \le T\}$ satisfies

$$\mathbb{E}\big\{\exp(-\gamma\,(X(T)+\xi))\big|\mathcal{F}^{S,Y}(t)\big\} \ge \exp\big(-\gamma\,X(t) - \mathbb{E}\{Z(T)\,\log Z(T)\} - \gamma\,\mathbb{E}\{Z(T)\,\xi\}\big).$$

Indeed, using the identity $\mathbb{E}\{Z(T)\,X(T)\} = X(t)$, and Jensen's inequality, we obtain

$$\begin{aligned}\mathbb{E}\big\{\exp(-\gamma\,(X(T)+\xi))\big\} &= \mathbb{E}\Big\{Z(T)\,\exp\big(-\gamma\,X(T) - \log Z(T) - \gamma\,\xi\big)\Big\}\\ &\ge \exp\big(-\gamma\,\mathbb{E}\{Z(T)\,X(T)\} - \mathbb{E}\{Z(T)\,\log Z(T)\} - \gamma\,\mathbb{E}\{Z(T)\,\xi\}\big)\\ &= \exp\big(-\gamma\,x - \mathbb{E}\{Z(T)\,\log Z(T)\} - \gamma\,\mathbb{E}\{Z(T)\,\xi\}\big). \end{aligned}\tag{9}$$

Furthermore, if we put $\theta(t) = \theta^*(t)$, then the equality sign holds.

The value for the optimal control problem considered in this subsection is

$$V(x) = \sup \mathbb{E}\big\{ -\exp(-\gamma\,(X(T)+\xi))\big\}, \tag{10}$$

where x denotes the initial capital. Recalling the lower bound (9) together with the equality holding when $\theta(t) = \theta^*(t)$, we have proved the following result

Proposition 1. *The value function $V(x)$ defined in (10) is given by*

$$V(x) = -\exp\big(-\gamma\, x - \mathbb{E}\{Z(T)\,\log Z(T)\} - \gamma\,\mathbb{E}\{Z(T)\,\xi\}\big) \tag{11}$$

or, equivalently by

$$V(x) = -\exp(-\gamma\, x)\mathbb{E}\Big\{ \exp\Big(-\gamma\,(1-\rho^2)\,\xi - \rho \int_0^T \lambda(t)\, dW(t) - \frac{1}{2}\int_t^T \lambda(t)^2\, dt\Big)\Big\}^{\frac{1}{1-\rho^2}}. \tag{12}$$

In particular, for $\xi = 0$, we obtain

$$V(x) = -\exp\,(-\gamma\, x)\,\mathbb{E}\Big\{ \exp\Big(-\rho \int_0^T \lambda(t)\, dW(t) - \frac{1}{2}\int_0^T \lambda(t)^2\, dt\Big)\Big\}^{\frac{1}{1-\rho^2}}, \tag{13}$$

which together with (12) proves that the indifference price of ξ is given by:

$$\frac{1}{\gamma\,(1-\rho^2)}\Big(\mathbb{E}\Big\{ \exp\Big(-\rho \int_0^T \lambda(t)\, dW(t) - \frac{1}{2}\int_0^T \lambda(t)^2\, dt\Big)\Big\} - \mathbb{E}\Big\{ \exp\Big(-\gamma\,(1-\rho^2)\,\xi - \rho \int_0^T \lambda(t)\, dW(t) - \frac{1}{2}\int_0^T \lambda(t)^2\, dt\Big)\Big\}\Big).$$

This formula extend to the non-Markovian case the results obtained in [14, 9, 3] with various degrees of generality.

Suppose now that the contingent claim is now assumed to depend upon both the traded and the non-traded asset. This means that we assume that the random variable ξ is $\mathcal{F}^{S,Y}(T)$-measurable instead of $\mathcal{F}^{Y}(T)$-measurable. In this case, we do not have an explicit formula for the value function, but in the spirit of the bounds derived in [16], we can still derive an upper bound.

Lemma 2. *Let $\{X(t) : 0 \le t \le T\}$ be the wealth process associated to an admissible portfolio $\{\theta(t) : 0 \le t \le T\}$. We have*

$$\mathbb{E}\big\{ \exp(-\gamma\,(X(T)+\xi)\big\} \ge \exp\big(-\gamma\, x - \mathbb{E}\{Z(T)\,\log Z(T)\} - \gamma\,\mathbb{E}\{Z(T)\,\xi\}\big). \tag{14}$$

Proof. Using the martingale identity $\mathbb{E}\{Z(T)\,X(T)\} = x$, and Jensen's inequality, we obtain

$$\begin{aligned}\mathbb{E}\big\{ \exp(-\gamma\,(X(T)+\xi))\big\} &= \mathbb{E}\Big[\{Z(T)\,\exp\big(-\gamma\, X(T) - \log Z(T) - \gamma\,\xi\big)\Big\} \\ &\ge \exp\big(-\gamma\,\mathbb{E}\{Z(T)\,X(T)\} - \mathbb{E}\{Z(T)\,\log Z(T)\} - \gamma\,\mathbb{E}\{Z(T)\,\xi\}\big) \\ &= \exp\big(-\gamma\, x - \mathbb{E}\{Z(T)\,\log Z(T)\} - \gamma\,\mathbb{E}\{Z(T)\,\xi\}\big).\end{aligned}$$

This proves the assertion. ∎

Therefore, the maximal expected utility satisfies the upper bound

$$V(x) \leq \exp\big(- \gamma\, x - \mathbb{E}\{Z(T)\, \log Z(T)\} - \gamma\, \mathbb{E}\{Z(T)\, \xi\}\big). \tag{15}$$

Constant Relative Risk Aversion. We now assume that the investor's risk preferences are given by a power-law utility function of the form $U(x) = x^\gamma$ for some $0 < \gamma < 1$. By virtue of the martingale representation theorem, we can find a process $\{\pi^*(t) : 0 \leq t \leq T\}$ such that $\{\pi^*(t) : 0 \leq t \leq T\}$ is adapted to the filtration $\{\mathcal{F}^Y(t) : 0 \leq t \leq T\}$ and

$$\begin{aligned} &\exp\bigg(- \frac{1}{\rho} \int_0^T \Big(\lambda(t) - (1-\gamma+\gamma\,\rho^2)\,\pi^*(t)\Big)\, dW(t) - \frac{1}{2}\,\frac{1}{\rho^2} \int_0^T \Big(\lambda(t) - (1-\gamma+\gamma\,\rho^2)\,\pi^*(t)\Big)^2\, du\bigg) \\ &\quad = \frac{\exp\Big(\frac{\gamma\,\rho}{1-\gamma} \int_0^T \lambda(t)\, dW(t) + \frac{1}{2}\,\frac{\gamma}{1-\gamma} \int_0^T \lambda(t)^2\, dt\Big)}{\mathbb{E}\Big\{\exp\Big(\frac{\gamma\,\rho}{1-\gamma} \int_0^T \lambda(t)\, dW(t) + \frac{1}{2}\,\frac{\gamma}{1-\gamma} \int_0^T \lambda(t)^2\, dt\Big)\Big\}}. \end{aligned}$$

Furthermore, we define a process $\{Z(t) : 0 \leq t \leq T\}$ by

$$\begin{aligned} Z(t) = \exp\bigg(&- \frac{1}{\rho} \int_0^t \lambda(u)\, dW(u) + (1-\gamma)\,\frac{\sqrt{1-\rho^2}}{\rho} \int_0^t \pi^*(u)\, dW^{S\perp}(u) \\ &- \frac{1}{2}\,\frac{1}{\rho^2} \int_0^t \lambda(u)^2\, du + (1-\gamma)\,\frac{1-\rho^2}{\rho^2} \int_0^t \pi^*(u)\,\lambda(u)\, du \\ &- \frac{1}{2}\,(1-\gamma)^2\,\frac{1-\rho^2}{\rho^2} \int_0^t \pi^*(u)^2\, du\bigg) \qquad 0 \leq t \leq T. \end{aligned}$$

A straightforward computation shows that:

Lemma 3. *Let $\{X^*(t) : 0 \leq t \leq T\}$ be the wealth process corresponding to the portfolio process $\{\pi^*(t) : 0 \leq t \leq T\}$. Then we have*

$$X^*(T) = x\, Z(T)^{-\frac{1}{1-\gamma}}\, \mathbb{E}\bigg\{\exp\bigg(\frac{\gamma\,\rho}{1-\gamma} \int_0^T \lambda(t)\, dW(t) + \frac{1}{2}\,\frac{\gamma}{1-\gamma} \int_0^T \lambda(t)^2\, dt\bigg)\bigg\}^{-\frac{1}{1-\gamma+\gamma\,\rho^2}}.$$

In particular, since $\mathbb{E}\{Z(T)\, X^*(T)\} = x$, this implies

$$\mathbb{E}\{Z(T)^{-\frac{\gamma}{1-\gamma}}\} = \mathbb{E}\bigg\{\exp\bigg(\frac{\gamma\,\rho}{1-\gamma} \int_0^T \lambda(t)\, dW(t) + \frac{1}{2}\,\frac{\gamma}{1-\gamma} \int_0^T \lambda(t)^2\, dt\bigg)\bigg\}^{\frac{1}{1-\gamma+\gamma\,\rho^2}}.$$

Note that, by the martingale identity $\mathbb{E}\{Z(T)\, X(T)\} = x$ and Hölder's inequality, we have

$$\begin{aligned} \mathbb{E}\{X(T)^\gamma\} &\leq \mathbb{E}\{Z(T)\, X(T)\}^\gamma\, \mathbb{E}\{Z(T)^{-\frac{\gamma}{1-\gamma}}\}^{1-\gamma} \\ &= X(t)^\gamma\, \mathbb{E}\{Z(T)^{-\frac{\gamma}{1-\gamma}}\}^{1-\gamma}, \end{aligned}$$

with equality when $\pi(t) = \pi^*(t)$. Hence, if we define the value function $V(x)$ by

$$V(x) = \sup \mathbb{E}\{X(T)^\gamma\},$$

the above computations proved:

Proposition 2. *The optimal utility $V(x)$ is given by*

$$V(x) = x^\gamma\, \mathbb{E}\{Z(T)^{-\frac{\gamma}{1-\gamma}}\}^{1-\gamma}$$

or, equivalently,

$$V(x) = x^\gamma\, \mathbb{E}\bigg\{ \exp\bigg(\frac{\gamma\,\rho}{1-\gamma} \int_0^T \lambda(t)\, dW(t) + \frac{1}{2}\,\frac{\gamma}{1-\gamma} \int_0^T \lambda(t)^2\, dt \bigg) \bigg\}^{\frac{1-\gamma}{1-\gamma+\gamma\,\rho^2}}.$$

Remark: Case of a more general claim.

As before, we can also consider the case of a contingent claim whose payoff is a non-negative $\mathcal{F}^{S,Y}(T)$-random variable ξ. As in the case of the exponential utility function, we derive an upper bound similar to (15). Indeed, using the martingale identity $\mathbb{E}\{Z(T)\,X(T)\} = x$ and Hölder's inequality, we obtain

$$\begin{aligned}\mathbb{E}\{(X(T)+\eta)^\gamma\} &\le \mathbb{E}\{Z(T)\,(X(T)+\eta)\}^\gamma\, \mathbb{E}\{Z(T)^{-\frac{\gamma}{1-\gamma}}\}^{1-\gamma} \\ &= \big(X(t) + \mathbb{E}\{Z(T)\,\eta\}\big)^\gamma\, \mathbb{E}\{Z(T)^{-\frac{\gamma}{1-\gamma}}\}^{1-\gamma},\end{aligned}$$

which proves that the value function satisfies the upper bound

$$V(x) \le \big(x + \mathbb{E}\{Z(T)\,\xi\}\big)^\gamma\, \mathbb{E}\{Z(T)^{-\frac{\gamma}{1-\gamma}}\}^{1-\gamma}.$$

Logarithmic Utility Function. If the investor's risk preferences are modelled by the utility function $U(x) = \log x$, we define the portfolio process $\{\pi^*(t) : 0 \le t \le T\}$ by $\pi^*(t) = \lambda(t)$ and the process Z by

$$Z(t) = \exp\bigg(-\int_0^t \lambda(u)\,[\rho\, dW(u) + \sqrt{1-\rho^2}\, dW^\perp(u)] - \frac{1}{2}\int_0^t \lambda(u)^2\, du \bigg) \qquad 0 \le t \le T$$

Direct computations give

Lemma 4. *Let $\{X^*(t) : 0 \le t \le T\}$ be the wealth process corresponding to the portfolio process $\{\pi^*(t) : 0 \le t \le T\}$. Then we have*

$$X^*(T) = x\, Z(T)^{-1},$$

and hence

$$\mathbb{E}\{\log X^*(T)\} = \log x - \mathbb{E}\{\log Z(T)\}.$$

If $\{X(t) : 0 \le t \le T\}$ is the wealth process associated to a generic admissible portfolio $\{\pi(t) : 0 \le t \le T\}$, using the martingale identity $\mathbb{E}\{Z(T)\,X(T)\} = x$ and Jensen's inequality, we obtain

$$\begin{aligned}\mathbb{E}[\log X(T)] &= \mathbb{E}[\log(Z(T)\,X(T))] - \mathbb{E}[\log Z(T)] \\ &\le \log \mathbb{E}[Z(T)\,X(T)] - \mathbb{E}[\log Z(T)] \\ &= \log X(t) - \mathbb{E}[\log Z(T)],\end{aligned}$$

with equality when $\pi(t) = \pi^*(t)$. So, if we define the value function $V(x)$ by

$$V(x) = \sup \mathbb{E}\{\log X(T)\},$$

then we proved

Proposition 3. *The value function $V(x)$ is given by*

$$V(x) = \log x - \mathbb{E}\{\log Z(T)\} = \log x + \frac{1}{2}\,\mathbb{E}\bigg\{\int_0^T \lambda(t)^2\,dt\bigg\}.$$

Remark. As before, we can prove a one-sided inequality when the contingent claim is only assumed to be $\mathcal{F}^{S,Y}(T)$-measurable. Indeed, using the martingale property of $\{Z(t)\,X(t) : 0 \le t \le T\}$ and Jensen's inequality, we obtain

$$\begin{aligned}\mathbb{E}\{\log(X(T)+\eta)\} &= \mathbb{E}\{\log(Z(T)\,(X(T)+\eta))\} - \mathbb{E}\{\log Z(T)\} \\ &\le \log \mathbb{E}\{Z(T)\,(X(T)+\eta)\} - \mathbb{E}\{\log Z(T)\} \\ &= \log(X(t) + \mathbb{E}\{Z(T)\,\eta\}) - \mathbb{E}\{\log Z(T)\},\end{aligned}$$

which implies that the value function satisfies the upper bound

$$V(x) \le \log(x + \mathbb{E}\{Z(T)\,\xi\}) - \mathbb{E}\{\log Z(T)\}. \tag{16}$$

2.2. **Mean-Variance Hedging.** We now assume that the investor's risk preferences are modeled by the mean-variance principle. Hence, the investor aims to minimize the variance

$$\mathbb{E}\{X(T)^2\} - \mathbb{E}\{X(T)\}^2$$

subject to the constraint $\mathbb{E}\{X(T)\} = y$ where y is a fixed positive real number.

By virtue of the martingale representation theorem, we can find a process $\{\pi^*(t) : 0 \le t \le T\}$ such that $\{\pi^*(t) : 0 \le t \le T\}$ is adapted to the filtration $\{\mathcal{F}^Y(t) : 0 \le t \le T\}$ and

$$\begin{aligned}&\exp\bigg(-\frac{1}{\rho}\int_0^T \Big(\lambda(t) + (1-2\rho^2)\,\pi^*(t)\Big)\,dW(t) - \frac{1}{2}\,\frac{1}{\rho^2}\int_0^T \Big(\lambda(t) + (1-2\rho^2)\,\pi^*(t)\Big)^2\,dt\bigg) \\ &\quad= \frac{\exp\Big(-2\rho\int_0^T \lambda(t)\,dW(t) - \int_0^T \lambda(t)^2\,dt\Big)}{\mathbb{E}\Big\{\exp\Big(-2\rho\int_0^T \lambda(t)\,dW(t) - \int_0^T \lambda(t)^2\,dt\Big)\Big\}}.\end{aligned}$$

Furthermore, we define a process $\{Z(t) : 0 \le t \le T\}$ by

$$\begin{aligned}Z(t) = \exp\bigg(&-\frac{1}{\rho}\int_0^t \lambda(u)\,dW(u) - \frac{\sqrt{1-\rho^2}}{\rho}\int_0^t \pi^*(u)\,dW^{S\perp}(u) \\ &- \frac{1}{2}\,\frac{1}{\rho^2}\int_0^t \lambda(u)^2\,du - \frac{1-\rho^2}{\rho^2}\int_0^t \pi^*(u)\,\lambda(u)\,du - \frac{1}{2}\,\frac{1-\rho^2}{\rho^2}\int_0^t \pi^*(u)^2\,du\bigg) \qquad 0 \le t \le T.\end{aligned}$$

Lemma 5. *If the wealth process $\{X(t) : 0 \le t \le T\}$ associated to an admissible portfolio $\{\pi(t) : 0 \le t \le T\}$ satisfies $\mathbb{E}\{X(T)\} = y$, then we have*

$$\mathbb{E}\{X(T)^2\} - \mathbb{E}\{X(T)\}^2 \ge \frac{(y-x)^2}{\mathbb{E}\{Z(T)^2\} - 1}.$$

Proof. Using the martingale identity $\mathbb{E}\{Z(T)\,X(T)\} = x$ and the constraint $\mathbb{E}\{X(T)\} = y$, we obtain

$$\begin{aligned}\mathbb{E}\{X(T)^2\} - \mathbb{E}\{X(T)\}^2 &\geq 2\,\mathbb{E}\Big\{\Big(\alpha + (x-\alpha)\,Z(T)\,\mathbb{E}[Z(T)^2]^{-1}\Big)\,X(T)\Big\}\\ &\quad - \mathbb{E}\Big\{\Big(\alpha + (x-\alpha)\,Z(T)\,\mathbb{E}[Z(T)^2]^{-1}\Big)^2\Big\} - y^2\\ &= 2\,\alpha\,y + 2\,(x-\alpha)\,x\,\mathbb{E}\{Z(T)^2\}^{-1} - \alpha^2 - (x^2-\alpha^2)\,\mathbb{E}\{Z(T)^2\}^{-1} - y^2\\ &= (x-\alpha)^2\,\mathbb{E}\{Z(T)^2\}^{-1} - (y-\alpha)^2\end{aligned}$$

for all real numbers α. Hence, if we choose

$$\alpha = \frac{y\,\mathbb{E}\{Z(T)^2\} - x}{\mathbb{E}\{Z(T)^2\} - 1},$$

we obtain

$$\mathbb{E}\{X(T)^2\} - \mathbb{E}\{X(T)\}^2 \geq \frac{(y-x)^2}{\mathbb{E}\{Z(T)^2\} - 1}. \quad \blacksquare$$

Direct computations prove

Lemma 6. *Let α be a real number, and let $\{X^*(t) : 0 \leq t \leq T\}$ be the solution of the stochastic differential equation*

$$dX^*(t) = (X^*(t) - \alpha)\,\pi^*(t)\,\frac{1}{\sigma(t)}\,\frac{dS(t)}{S(t)}$$

with the initial condition $X^*(0) = x$. Then we have

$$X^*(T) = \alpha + (x-\alpha)\,Z(T)\,\mathbb{E}\Big\{\exp\Big(-2\rho\int_0^T \lambda(t)\,dW(t) - \int_0^T \lambda(t)^2\,dt\Big)\Big\}^{\frac{1}{1-2\rho^2}}.$$

In particular, since $\mathbb{E}\{X^*(T)\} = x$, we must have

$$\mathbb{E}\{Z(T)^2\} = \mathbb{E}\Big\{\exp\Big(-2\rho\int_0^T \lambda(t)\,dW(t) - \int_0^T \lambda(t)^2\,dt\Big)\Big\}^{-\frac{1}{1-2\rho^2}}.$$

Lemma 7. *Let α be a real number, and let $\{X^*(t) : 0 \leq t \leq T\}$ be the solution of the stochastic differential equation*

$$dX^*(t) = (X^*(t) - \alpha)\,\pi^*(t)\,\frac{1}{\sigma(t)}\,\frac{dS(t)}{S(t)}$$

with the initial condition $X^*(0) = x$. Then we have

$$X^*(T) = \alpha + (x-\alpha)\,Z(T)\,\mathbb{E}\{Z(T)^2\}^{-1}.$$

In particular, we have

$$\mathbb{E}\{X^*(T)\} = \alpha + (x-\alpha)\,\mathbb{E}\{Z(T)^2\}^{-1}$$

and

$$\mathbb{E}\{X^*(T)^2\} - \mathbb{E}\{X^*(T)\}^2 = (x-\alpha)^2\,\mathbb{E}\{Z(T)^2\}^{-1} - (x-\alpha)^2\,\mathbb{E}\{Z(T)^2\}^{-2}.$$

Proof. Using the relations

$$X^*(T) = \alpha + (x - \alpha)\, Z(T)\mathbb{E}\bigg\{ \exp\bigg(-2\rho \int_0^T \lambda(t)\, dW(t) - \int_0^T \lambda(t)^2\, dt \bigg) \bigg\}^{\frac{1}{1-2\rho^2}}$$

and

$$\mathbb{E}\{Z(T)^2\} = \mathbb{E}\bigg\{ \exp\bigg(-2\rho \int_0^T \lambda(t)\, dW(t) - \int_0^T \lambda(t)^2\, dt \bigg) \bigg\}^{-\frac{1}{1-2\rho^2}},$$

we obtain

$$X^*(T) = \alpha + (x - \alpha)\, Z(T)\, \mathbb{E}\{Z(T)^2\}^{-1}.$$

From this it follows that

$$\mathbb{E}\{X^*(T)\} = \alpha + (x - \alpha)\, \mathbb{E}\{Z(T)^2\}^{-1}$$

and

$$\mathbb{E}\{X^*(T)^2\} - \mathbb{E}\{X^*(T)\}^2 = (x - \alpha)^2\, \mathbb{E}\{Z(T)^2\}^{-1} - (x - \alpha)^2\, \mathbb{E}\{Z(T)^2\}^{-2}. \blacksquare$$

Hence, if we choose

$$\alpha = \frac{y\, \mathbb{E}\{Z(T)^2\} - x}{\mathbb{E}\{Z(T)^2\} - 1},$$

then we obtain

$$\mathbb{E}\{X^*(T)\} = \alpha + (x - \alpha)\, \mathbb{E}\{Z(T)^2\}^{-1} = y$$

and

$$\mathbb{E}\{X^*(T)^2\} - \mathbb{E}\{X^*(T)\}^2 = (x - \alpha)^2\, \mathbb{E}\{Z(T)^2\}^{-1} - (x - \alpha)^2\, \mathbb{E}\{Z(T)^2\}^{-2} = \frac{(y - x)^2}{\mathbb{E}\{Z(T)^2\} - 1}.$$

We now define the mean-variance value function by

$$V(x, y) = \inf \mathbb{E}\{X(T)^2\} - \mathbb{E}\{X(T)\}^2,$$

where the infimum is taken over all portfolio processes with initial capital x and prescribed expected return $\mathbb{E}\{X(T)\} = y$. Then we obtain the following result:

Proposition 4. *The mean-variance value function is given by*

$$V(x, y) = \frac{(y - x)^2}{\mathbb{E}\{Z(T)^2\} - 1}$$

or, equivalently,

$$V(x, y) = (y - x)^2 \bigg(\mathbb{E}\bigg\{ \exp\bigg(-2\rho \int_0^T \lambda(t)\, dW(t) - \int_0^T \lambda(t)^2\, dt \bigg) \bigg\}^{-\frac{1}{1-2\rho^2}} - 1 \bigg)^{-1}.$$

We now consider a contingent claim with payoff given by a $\mathcal{F}^{S,Y}(T)$-measurable random variable η. For every real number x, we define the minimal replication error by

$$R_\eta(x) = \inf \mathbb{E}\{(X(T) - \eta)^2\},$$

where the infimum is taken over all admissible portfolio processes $\{\theta(t) : t \geq 0\}$ with initial capital x.

Proposition 5. *The minimal replication error $R_\eta(x)$ is smallest for $x = \mathbb{E}\{Z(T)\,\eta\}$.*

Proof. This is a special case of a general result of M. Schweizer. In this special situation, we can give a direct proof based on Lemma 7. Let us start with

$$dX(t) = \theta(t)\,\frac{1}{\sigma(t)}\,\frac{dS(t)}{S(t)}$$

with the initial condition $X(0) = x$. Furthermore, let $\{X^*(t) : 0 \le t \le T\}$ the solution of the stochastic differential equation

$$dX^*(t) = X^*(t)\,\pi^*(t)\,\frac{1}{\sigma(t)}\,\frac{dS(t)}{S(t)}$$

with the initial condition $X^*(0) = \mathbb{E}\{Z(T)\,\eta\} - x$, so that

$$X^*(T) = \frac{\mathbb{E}\{Z(T)\,\eta\} - x}{\mathbb{E}\{Z(T)^2\}}\,Z(T).$$

Then the sum $\{X(t) + X^*(t) : 0 \le t \le T\}$ satisfies the stochastic differential equation

$$d(X(t) + X^*(t)) = (\theta(t) + X^*(t)\,\pi^*(t))\,\frac{1}{\sigma(t)}\,\frac{dS(t)}{S(t)}$$

with $X(0) + X^*(0) = \mathbb{E}\{Z(T)\,\eta\}$. From this it follows that

$$\begin{aligned}
\mathbb{E}\{(X(T) - \eta)^2\} &= \mathbb{E}\{(X(T) + X^*(T) - \eta)^2\} - 2\,\mathbb{E}\{X^*(T)\,(X(T) + X^*(T) - \eta)\} + \mathbb{E}\{X^*(T)\}^2 \\
&= \mathbb{E}\{(X(T) + X^*(T) - \eta)^2\} + \frac{(\mathbb{E}\{Z(T)\,\eta\} - x)^2}{\mathbb{E}\{Z(T)^2\}} \\
&- 2\,\frac{\mathbb{E}\{Z(T)\,\eta\} - x}{\mathbb{E}\{Z(T)^2\}}\,\mathbb{E}\{Z(T)\,(X(T) + X^*(T) - \eta)\} \\
&= \mathbb{E}\{(X(T) + X^*(T) - \eta)^2\} + \frac{(\mathbb{E}\{Z(T)\,\eta\} - x)^2}{\mathbb{E}\{Z(T)^2\}}.
\end{aligned}$$

Hence, if we take the infimum over all portfolio processes $\{\theta(t) : 0 \le t \le T\}$, then we obtain

$$R_\eta(x) = R_\eta(\mathbb{E}\{Z(T)\,\eta\}) + \frac{(\mathbb{E}\{Z(T)\,\eta\} - x)^2}{\mathbb{E}\{Z(T)^2\}}.$$

Note that we do not need to assume that the infimum is attained. From this the assertion follows immediately. ∎

3. Optimal Hedging in Partially Observed Markets

We now consider the case of dynamics driven in part by factors which cannot be observed.

3.1. **The Model.** As before, we consider a pair of stochastic process $\{Y(t) : t \geq 0\}$ and $\{S(t) : t \geq 0\}$ whose dynamics are given by (1) - (4), but we now assume a specific form for the coefficients of the stochastic differential equations which could depend on the whole past in the previous section. We assume that the coefficients of (1) are of the form

$$g(t) = g(Y(t), t) + \varphi(Y(t), t)\,\zeta(t), \quad \text{and} \quad h(t) = h(Y(t), t),$$

those of (2) of the form

$$\mu(t) = \mu(Y(t), t) + \varphi(Y(t), t)\,\zeta(t), \quad \text{and} \quad \sigma(t) = \sigma(Y(t), t),$$

and the term $\lambda(t)$ of (3) is of the form

$$\lambda(t) = \lambda(Y(t), t)\,dt + \rho\,\frac{\varphi(Y(t), t)}{h(Y(t), t)}\,\zeta(t),$$

where as before, $\sigma > 0$ and that $h \geq \overline{h}$ for some strictly positive constant $\overline{h}$. The process $\{\zeta(t) : t \geq 0\}$ is assumed to satisfy a stochastic differential equation of the form

$$d\zeta(t) = p(Y(t), \zeta(t), t)\,dt + q(Y(t), \zeta(t), t)\,dB(t)$$

and $\{B(t) : t \geq 0\}$ is a Wiener process independent of $\{W(t) : t \geq 0\}$ and $\{W^{\perp}(t) : t \geq 0\}$.

The thrust of this section is the fact that the process $\{\zeta(t) : t \geq 0\}$ is unobservable. Therefore, the investor's decisions must be based solely on the past observations of the processes $\{S(t) : t \geq 0\}$ and $\{Y(t) : t \geq 0\}$. This means that the portfolio processes $\{\theta(t) : t \geq 0\}$ and $\{\pi(t) : t \geq 0\}$ must be adapted to the filtration $\{\mathcal{F}^{S,Y}(t) : t \geq 0\}$.

Since the process $\{\zeta(t) : t \geq 0\}$ is not necessarily adapted to the filtration $\{\mathcal{F}^{S,Y}(t) : t \geq 0\}$, it is natural to introduce the conditional expectation of $\delta(t)$ conditional on the σ-algebra $\mathcal{F}^{S,Y}(t)$, i.e.

$$\hat{\zeta}(t) = \mathbb{E}\{\zeta(t) | \mathcal{F}^{S,Y}(t)\}.$$

By construction, the process $\{\hat{\zeta}(t) : t \geq 0\}$ is adapted to the filtration $\{\mathcal{F}^{S,Y}(t) : t \geq 0\}$. In the present situation, it is not hard to see that the process $\{\hat{\zeta}(t) : t \geq 0\}$ is adapted to the filtration $\{\mathcal{F}^{Y}(t) : t \geq 0\}$. In other words, the best guess for $\delta(t)$ depends only upon the past observations $\{Y(u),\, 0 \leq u \leq t\}$. The past observations $\{S(u),\, 0 \leq u \leq t\}$ are of no use for the filtering problem. To see this, we use the identity

$$\mathcal{F}^{S,Y}(t) = \sigma\{Y(u), W^{\perp}(u); 0 \leq u \leq t\}.$$

Since the σ-algebra $\sigma\{\zeta(u), Y(u);\, 0 \leq u \leq t\}$ is independent of the σ-algebra $\sigma\{W^{\perp}(u);\, 0 \leq u \leq t\}$, we obtain

$$\begin{aligned}
\hat{\zeta}(t) &= \mathbb{E}\{\zeta(t) | \mathcal{F}^{S,Y}(t)\} \\
&= \mathbb{E}\{\zeta(t) | \sigma\{Y(u), W^{\perp}(u);\, 0 \leq u \leq t\}\} \\
&= \mathbb{E}\{\zeta(t) | \sigma\{Y(u) : 0 \leq u \leq t\}\} \\
&= \mathbb{E}\{\zeta(t) | \mathcal{F}^{Y}(t)\},
\end{aligned}$$

which proves our claim.

3.2. **Reduction to the Full Observation Case.** Let us define a probability measure $\hat{\mathbb{P}}$ by

$$\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} = \exp\Big(-\int_0^T \frac{\varphi(Y(t),t)}{h(Y(t),t)} (\zeta(t)-\hat{\zeta}(t))\, dW(t) - \frac{1}{2}\int_0^T \frac{\varphi(Y(t),t)^2}{h(Y(t),t)^2} (\zeta(t)-\hat{\zeta}(t))^2\, dt\Big).$$

Furthermore, we define the process $\{\tilde{W}(t) : 0 \le t \le T\}$ by

$$\hat{W}(t) = W(t) + \int_0^t \frac{\varphi(Y(u),u)}{h(Y(u),u)} (\zeta(u)-\hat{\zeta}(u))\, du.$$

It follows from Girsanov's theorem that $\{\hat{W}(t) : 0 \le t \le T\}$ is a Wiener process relative to the probability measure $\hat{\mathbb{P}}$. Moreover, this same theorem tells us that the dynamics of $\{Y(t) : 0 \le t \le T\}$ and $\{S(t) : 0 \le t \le T\}$ under the probability measure $\hat{\mathbb{P}}$ are given by

$$dY(t) = (g(Y(t),t) + \varphi(Y(t),t)\,\hat{\zeta}(t))\, dt + h(Y(t),t)\, d\hat{W}(t)$$

and

$$\frac{dS(t)}{S(t)} = \sigma(Y(t),t)\Big(\lambda(Y(t),t) + \rho\,\frac{\varphi(Y(t),t)}{h(Y(t),t)}\,\hat{\zeta}(t)\Big)\, dt + d\hat{W}^S(t)\Big) \tag{17}$$

provided we set

$$\hat{W}^S(t) = \rho\,\hat{W}(t) + \sqrt{1-\rho^2}\, W^{\perp}(t). \tag{18}$$

For the sake of convenience we introduce the notation

$$H(t) = \exp\Big(\int_0^t \frac{\varphi(Y(u),u)}{h(Y(u),u)} (\zeta(u)-\hat{\zeta}(u))\, d\hat{W}(u) - \frac{1}{2}\int_0^t \frac{\varphi(Y(u),u)^2}{h(Y(u),u)^2} (\zeta(u)-\hat{\zeta}(u))^2\, du\Big)$$

for all $0 \le t \le T$. We have

$$\begin{aligned} H(T) &= \exp\Big(\int_0^T \frac{\varphi(Y(t),t)}{h(Y(t),t)} (\zeta(t)-\hat{\zeta}(t))\, d\hat{W}(t) - \frac{1}{2}\int_0^T \frac{\varphi(Y(t),t)^2}{h(Y(t),t)^2} (\zeta(t)-\hat{\zeta}(t))^2\, dt\Big) \\ &= \exp\Big(\int_0^T \frac{\varphi(Y(t),t)}{h(Y(t),t)} (\zeta(t)-\hat{\zeta}(t))\, dW(t) + \frac{1}{2}\int_0^T \frac{\varphi(Y(t),t)^2}{h(Y(t),t)^2} (\zeta(t)-\hat{\zeta}(t))^2\, dt\Big) \\ &= \frac{d\mathbb{P}}{d\hat{\mathbb{P}}}. \end{aligned}$$

Notice that the random variable $H(T)$ is not $\mathcal{F}^{S,Y}(T)$-measurable. However,

Lemma 8. *The random variable $H(T)$ satisfies*

$$\hat{\mathbb{E}}\{H(t)|\mathcal{F}^{S,Y}(T)\} = 1, \qquad a.s. \tag{19}$$

for all $0 \le t \le T$.

Proof. The process $\{H(t) : 0 \le t \le T\}$ satisfies the stochastic differential equation

$$dH(t) = H(t)\,\frac{\varphi(Y(t),t)}{h(Y(t),t)} (\zeta(u)-\hat{\zeta}(t))\, d\hat{W}(t)$$

which implies that

$$H(t) - 1 = \int_0^t H(u)\,\frac{\varphi(Y(u),u)}{h(Y(u),u)} (\zeta(u)-\hat{\zeta}(u))\, d\hat{W}(u).$$

By definition of $\hat{\zeta}(u)$, we have

$$\mathbb{E}\{\zeta(u) - \hat{\zeta}(u) | \mathcal{F}^{S,Y}(u)\} = 0,$$

and since

$$\frac{d\mathbb{P}}{d\hat{\mathbb{P}}} = H(T),$$

we obtain

$$\hat{\mathbb{E}}\{H(T)\,(\zeta(u) - \hat{\zeta}(u)) | \mathcal{F}^{S,Y}(u)\} = 0.$$

Using the identity

$$\hat{\mathbb{E}}\{H(T) | \mathcal{F}^{\zeta,S,Y}(u)\} = H(u),$$

we also get

$$\hat{\mathbb{E}}\{H(u)\,(\zeta(u) - \hat{\zeta}(u)) | \mathcal{F}^{S,Y}(u)\} = 0.$$

Since the σ-algebra $\sigma\{\hat{W}(v) - \hat{W}(u), W^{\perp}(v) - W^{\perp}(u);\ u \leq v \leq T\}$ generated by the increments is independent of $\mathcal{F}^{\zeta,S,Y}(u)$, we conclude that

$$\hat{\mathbb{E}}\{H(u)\,(\zeta(u) - \hat{\zeta}(u)) | \mathcal{F}^{S,Y}(u) \vee \sigma\{\hat{W}(v) - \hat{W}(u), W^{\perp}(v) - W^{\perp}(u);\ u \leq v \leq T\}\} = 0.$$

From this it follows that

$$\hat{\mathbb{E}}\{H(u)\,(\zeta(u) - \hat{\zeta}(u)) | \mathcal{F}^{S,Y}(T)\} = 0.$$

Therefore, we obtain

$$\hat{\mathbb{E}}\bigg\{ \int_0^t H(u)\, \frac{\varphi(Y(u),u)}{h(Y(u),u)}\, (\zeta(u) - \hat{\zeta}(u))\, d\hat{W}(u) \bigg| \mathcal{F}^{S,Y}(T) \bigg\} = 0,$$

hence

$$\hat{\mathbb{E}}\{H(t) | \mathcal{F}^{S,Y}(T)\} = 1,$$

which is the desired result.

Let's assume that the investor's risk preferences are modeled by some utility function U. Furthermore, let us also consider a contingent claim whose payoff at maturity T is given by a random variable ξ. Naturally, we assume that the payoff depends only on the observable variables. In other words, we assume that the random variable ξ is $\mathcal{F}^{S,Y}(T)$-measurable. The investor wants to find an admissible portfolio process $\{\theta(t) : 0 \leq t \leq T\}$ which maximizes the expected terminal time utility

$$\mathbb{E}\Big\{U\Big(x + \int_0^T \theta(t)\, \frac{1}{\sigma(Y(t),t)}\, \frac{dS(t)}{S(t)} + \xi\Big)\Big\},$$

where x is the initial endowment. A portfolio process $\{\theta(t) : 0 \leq t \leq T\}$ is admissible if, among other things, it is adapted to the filtration $\{\mathcal{F}^{S,Y}(t) : 0 \leq t \leq T\}$. This means that the investor's portfolio decisions must be based exclusively on the past observations of the processes $\{S(t) : 0 \leq t \leq T\}$ and $\{Y(t) : 0 \leq t \leq T\}$.

The optimal utility for this problem is defined as

$$V(x) = \sup \mathbb{E}\Big\{U\Big(x + \int_0^T \theta(t)\, \frac{1}{\sigma(Y(t),t)}\, \frac{dS(t)}{S(t)} + \xi\Big)\Big\},$$

where the supremum is taken over all admissibile portfolio processes $\{\theta(t) : 0 \le t \le T\}$. Our aim is to identify the optimal utility $V(x)$ with the optimal utility $\hat{V}(x)$ of a stochastic control problem with full observation. To this end, we define

$$\hat{V}(x) = \sup \hat{\mathbb{E}}\Big\{U\Big(x + \int_0^T \theta(t)\,\frac{1}{\sigma(Y(t),t)}\,\frac{dS(t)}{S(t)} + \xi\Big)\Big\}. \tag{20}$$

As before, the supremum is computed over all admissible portfolio processes $\{\theta(t) : 0 \le t \le T\}$. We have:

Proposition 6. *The optimal utility $V(x)$ coincides with $\hat{V}(x)$.*

Proof. Let $\{\theta(t) : 0 \le t \le T\}$ be an arbitrary admissible portfolio process . Since the process $\{\theta(t) : 0 \le t \le T\}$ is adapted to the filtration $\{\mathcal{F}^{S,Y}(t) : 0 \le t \le T\}$, the random variable

$$\int_0^T \theta(t)\,\frac{1}{\sigma(Y(t),t)}\,\frac{dS(t)}{S(t)}$$

is $\mathcal{F}^{S,Y}(T)$-measurable. Since the random variable ξ is $\mathcal{F}^{S,Y}(T)$-measurable by assumption, the random variable

$$\int_t^T \theta(t)\,\frac{1}{\sigma(Y(t),t)}\,\frac{dS(t)}{S(t)} + \xi$$

is also $\mathcal{F}^{S,Y}(T)$-measurable. From this it follows that

$$\begin{aligned}
&\mathbb{E}\Big\{U\Big(x + \int_0^T \theta(t)\,\frac{1}{\sigma(Y(t),t)}\,\frac{dS(t)}{S(t)} + \xi\Big)\Big\}\\
&\quad = \hat{\mathbb{E}}\Big\{H(T)\,U\Big(x + \int_0^T \theta(t)\,\frac{1}{\sigma(Y(t),t)}\,\frac{dS(t)}{S(t)} + \xi\Big)\Big\}\\
&\quad = \hat{\mathbb{E}}\Big\{\hat{\mathbb{E}}\{H(T)|\mathcal{F}^{S,Y}(T)\}\,U\Big(x + \int_0^T \theta(t)\,\frac{1}{\sigma(Y(t),t)}\,\frac{dS(t)}{S(t)} + \xi\Big)\Big\}\\
&\quad = \hat{\mathbb{E}}\Big\{U\Big(x + \int_0^T \theta(t)\,\frac{1}{\sigma(Y(t),t)}\,\frac{dS(t)}{S(t)} + \xi\Big)\Big\}.
\end{aligned}$$

Hence, if we take the supremum over all admissible portfolio processes $\{\theta(t) : 0 \le t \le T\}$, then the assertion follows. ∎

4. **The Conditionally Gaussian Case**

In this section, we look at a special case where the filtering problem can be reduced to a system of finitely many stochastic differential equations. To this end, we assume that the processes $\{Y(t) : 0 \le t \le T\}$ and $\{S(t) : 0 \le t \le T\}$ satisfy the stochastic differential equations

$$dY(t) = \big(g(Y(t),t) + \varphi(Y(t),t)\,\zeta(t)\big)\,dt + h(Y(t),t)\,W(t)$$

and

$$\frac{dS(t)}{S(t)} = \mu(Y(t),t)\,dt + \sqrt{1-\rho^2}\,\sigma(t)\,dW^{\perp}(t) + \rho\,\frac{\sigma(Y(t),t)}{h(Y(t),t)}\,dY(t),$$

where $\{W(t) : t \geq 0\}$ and $\{W^\perp(t) : t \geq 0\}$ are independent Wiener processes. The second equation can be written in the form

$$\frac{dS(t)}{S(t)} = \sigma(Y(t),t)\left(\lambda(Y(t),t)\,dt + \rho\,\frac{\varphi(Y(t),t)}{h(Y(t),t)}\,\zeta(t)\,dt + dW^S(t)\right), \tag{21}$$

where

$$\lambda(Y(t),t) = \frac{\mu(Y(t),t)}{\sigma(Y(t),t)} + \rho\,\frac{g(Y(t),t)}{h(Y(t),t)},$$

and where the Wiener process $\{W^S(t) : t \geq 0\}$ is defined as before.

Furthermore, we assume that the dynamics of the process $\{\zeta(t) : 0 \leq t \leq T\}$ are given by

$$d\zeta(t) = \big(a(Y(t),t) + b(Y(t),t)\,\zeta(t)\big)\,dt + q(Y(t),t)\,dB(t),$$

where $\{B(t) : 0 \leq t \leq T\}$ is a Wiener process independent of $\{W(t) : t \geq 0\}$ and $\{W^\perp(t) : t \geq 0\}$. For these dynamics, we assume that the drift is affine in $\zeta(t)$ and that the diffusion coefficient is independent of $\zeta(t)$. The latter implies that we are dealing with a conditionally Gaussian model, and the latter are amenable to Kalman filtering theory which we use next.

4.1. **Equations for the Optimal Filter.** We denote by $\hat\zeta(t)$ and $\omega(t)$ the conditional mean and variance of $\zeta(t)$, i.e.

$$\hat\zeta(t) = \mathbb{E}\{\zeta(t)|\mathcal{F}^{S,Y}(t)\}$$

and

$$\omega(t) = \mathbb{E}\{(\zeta(t) - \hat\zeta(t))^2|\mathcal{F}^{S,Y}(t)\}.$$

Following the classical presentation of the Kalman filter given in [13], one can prove that the dynamics of these first two conditional moments are given by

$$\begin{aligned} d\hat\zeta(t) &= (a(Y(t),t) + b(Y(t),t)\,\hat\zeta(t))\,dt \\ &\quad + \frac{1}{h(Y(t),t)^2}\,\varphi(Y(t),t)\,\omega(t)\,\big(dY(t) - g(Y(t),t)\,dt - \varphi(Y(t),t)\,\hat\zeta(t)\,dt\big) \end{aligned} \tag{22}$$

and

$$d\omega(t) = \left(q(Y(t),t)^2 + 2\,b(Y(t),t)\,\omega(t) - \frac{1}{h(Y(t),t)^2}\,\varphi(Y(t),t)^2\,\omega(t)^2\right)dt. \tag{23}$$

4.2. **Formulae for the Value Function.** We are now in a position to compute the optimal expected utility of terminal wealth

$$V(x) = \sup \mathbb{E}\big\{U(X(T) + \xi)\big|\mathcal{F}^{S,Y}(t)\big\},$$

for the three most commonly used utility functions U already used in the previous section.

The Case of Constant Absolute Risk Aversion. Suppose that the investor's risk preferences are given by a utility function of the form $U(x) = -\exp(-\gamma\, x)$ for some $\gamma > 0$. We consider a contingent claim whose payoff is modelled by a $\mathcal{F}^Y(T)$-measurable random variable ξ, and we compute the optimal utility

$$V(x) = \sup \mathbb{E}\big\{ -\exp(-\gamma\, (X(T) + \xi))\big\},$$

where the supremum is taken over all admissible portfolio processes $\{\theta(t) : 0 \le t \le T\}$. To this end, we replace $V(x)$ by $\hat{V}(x)$ defined by

$$\hat{V}(x) = \sup \hat{\mathbb{E}}\big\{ -\exp(-\gamma\, (X(T) + \xi))\big\}.$$

As above, the supremum runs over all admissible portfolio processes $\{\theta(t) : 0 \le t \le T\}$. This is a stochastic control problem with full observation. Its solution is given by

$$\hat{V}(x) = \exp(-\gamma\, x)\, \hat{\mathbb{E}}\Bigg\{ \exp\Bigg(-\gamma\,(1-\rho^2)\,\xi - \rho \int_0^T \Big(\lambda(Y(t),t) + \rho\, \frac{\varphi(Y(t),t)}{h(Y(t),t)}\, \hat{\zeta}(t)\Big)\, d\hat{W}(t)$$
$$- \frac{1}{2}\int_0^T \Big(\lambda(Y(t),t) + \rho\, \frac{\varphi(Y(t),t)}{h(Y(t),t)}\, \hat{\zeta}(t)\Big)^2\, dt\Bigg)\Bigg\}^{\frac{1}{1-\rho^2}}.$$

Note that, under the probability measure $\hat{\mathbb{P}}$, the dynamics of $\{S(t) : 0 \le t \le T\}$, $\{Y(t) : 0 \le t \le T\}$, and $\{\hat{\zeta}(t) : 0 \le t \le T\}$ are given by

$$\begin{aligned} dY(t) &= \big(g(Y(t),t) + \varphi(Y(t),t)\,\hat{\zeta}(t)\big)\, dt + h(Y(t),t)\, d\hat{W}(t), \\ \frac{dS(t)}{S(t)} &= \sigma(Y(t),t)\,\Big(\lambda(Y(t),t)\, dt + \rho\, \frac{\varphi(Y(t),t)}{h(Y(t),t)}\, \hat{\zeta}(t)\, dt + d\hat{W}^S(t)\Big), \end{aligned}$$

where we use the notation (18) for $\hat{W}^S$, and

$$d\hat{\zeta}(t) = (a(Y(t),t) + b(Y(t),t)\,\hat{\zeta}(t))\, dt + \frac{\varphi(Y(t),t)}{h(Y(t),t)^2}\, \omega(t)\, d\hat{W}(t).$$

To simplify this result, it is convenient to introduce the probability measure $\hat{\mathbb{P}}_1$ defined by

$$\frac{d\hat{\mathbb{P}}_1}{d\hat{\mathbb{P}}} = \exp\Bigg(-\rho \int_0^T \Big(\lambda(Y(t),t) + \rho\, \frac{\varphi(Y(t),t)}{h(Y(t),t)}\, \hat{\zeta}(t)\Big)\, d\hat{W}(t)$$
$$- \frac{1}{2}\,\rho^2 \int_0^T \Big(\lambda(Y(t),t) + \rho\, \frac{\varphi(Y(t),t)}{h(Y(t),t)}\, \hat{\zeta}(t)\Big)^2\, dt\Bigg).$$

With this definition, the optimal utility can be written as

$$\hat{V}(x) = \exp(-\gamma\, x)\, \hat{\mathbb{E}}_1\Bigg\{ \exp\Bigg(-\gamma\,(1-\rho^2)\,\xi - \frac{1}{2}\,(1-\rho^2) \int_0^T \Big(\lambda(Y(t),t) + \rho\, \frac{\varphi(Y(t),t)}{h(Y(t),t)}\, \hat{\zeta}(t)\Big)^2\, dt\Bigg)\Bigg\}^{\frac{1}{1-\rho^2}}.$$

This formula is of the Feynman-Kac type and as such it is amenable to computations, either by Monte Carlo methods or by numerical methods for partial differential equations. Notice that, under the probability measure $\hat{\mathbb{P}}_1$, the dynamics of the processes $\{S(t) : 0 \le t \le T\}$ and $\{Y(t) : 0 \le t \le T\}$ are given by

$$dY(t) = \Big(g(Y(t),t) - \rho\, h(Y(t),t)\,\lambda(Y(t),t) + (1-\rho^2)\,\varphi(Y(t),t)\,\hat{\zeta}(t)\Big)\, dt + h(Y(t),t)\, d\hat{W}_1(t)$$

and

$$\frac{dS(t)}{S(t)} = \sigma(Y(t),t)\Big((1-\rho^2)\,\lambda(Y(t),t)\,dt + \rho\,(1-\rho^2)\,\frac{\varphi(Y(t),t)}{h(Y(t),t)}\,\hat{\zeta}(t)\,dt \\ + \rho\,d\hat{W}_1(t) + \sqrt{1-\rho^2}\,dW^{\perp}(t)\Big),$$

where $\{\hat{W}_1(t) : 0 \le t \le T\}$ is a Brownian motion relative to the probability measure $\hat{\mathbb{P}}_1$. Moreover, the conditional expectation $\hat{\zeta}(t)$ satisfies the stochastic differential equation

$$d\hat{\zeta}(t) = (a(Y(t),t) + b(Y(t),t)\,\hat{\zeta}(t))\,dt \\ + \frac{\varphi(Y(t),t)}{h(Y(t),t)}\,\omega(t)\Big(d\hat{W}_1(t) - \rho\,\lambda(Y(t),t)\,dt - \rho^2\,\frac{\varphi(Y(t),t)}{h(Y(t),t)}\,\hat{\zeta}(t)\,dt\Big).$$

Constant Relative Risk Aversion. We now assume that the investor's risk preferences are described by the utility function function $U(x) = x^\gamma$ for some $0 < \gamma < 1$ and we compute the optimal utility

$$V(x) = \sup \mathbb{E}\{X(T)^\gamma\},$$

where the supremum is taken over all admissible portfolio processes $\{\pi(t) : 0 \le t \le T\}$. As before we can replace $V(x)$ by the optimal utility $\hat{V}(x)$ of a fully observable model, and the value of the latter is given by

$$\hat{V}(x) = x^\gamma\,\hat{\mathbb{E}}\Big\{\exp\Big(\frac{\gamma\,\rho}{1-\gamma}\int_0^T \Big(\lambda(Y(t),t) + \rho\,\frac{\varphi(Y(t),t)}{h(Y(t),t)}\,\hat{\zeta}(t)\Big)\,d\hat{W}(t) \\ + \frac{1}{2}\,\frac{\gamma}{1-\gamma}\int_0^T \Big(\lambda(Y(t),t) + \rho\,\frac{\varphi(Y(t),t)}{h(Y(t),t)}\,\hat{\zeta}(t)\Big)^2\,dt\Big)\Big\}^{\frac{1-\gamma}{1-\gamma+\gamma\,\rho^2}},$$

where the expectation is taken under the probability measure $\hat{\mathbb{P}}$. To simplify this result, it is convenient to define a probability measure $\hat{\mathbb{P}}_2$ by

$$\frac{d\hat{\mathbb{P}}_2}{d\hat{\mathbb{P}}} = \exp\Big(\frac{\gamma\,\rho}{1-\gamma}\int_0^T \Big(\lambda(Y(t),t) + \rho\,\frac{\varphi(Y(t),t)}{h(Y(t),t)}\,\hat{\zeta}(t)\Big)\,d\hat{W}(t) \\ - \frac{1}{2}\,\frac{\gamma^2\,\rho^2}{(1-\gamma)^2}\int_0^T \Big(\lambda(Y(t),t) + \rho\,\frac{\varphi(Y(t),t)}{h(Y(t),t)}\,\hat{\zeta}(t)\Big)^2\,dt\Big).$$

With this definition, the optimal utility can be written as

$$\hat{V}(x) = x^\gamma\,\hat{\mathbb{E}}_2\Big\{\exp\Big(\frac{1}{2}\,\frac{\gamma}{1-\gamma}\,\frac{1-\gamma+\gamma\,\rho^2}{1-\gamma} \\ \int_0^T \Big(\lambda(Y(t),t) + \rho\,\frac{\varphi(Y(t),t)}{h(Y(t),t)}\,\hat{\zeta}(t)\Big)^2\,du\Big)\Big\}^{\frac{1-\gamma}{1-\gamma+\gamma\,\rho^2}},$$

and, under the probability measure $\hat{\mathbb{P}}_2$, the dynamics of the processes $\{S(t) : 0 \le t \le T\}$ and $\{Y(t) : 0 \le t \le T\}$ are given by

$$dY(t) = \Big(g(Y(t),t) + \frac{\gamma\,\rho}{1-\gamma}\, h(Y(t),t)\,\lambda(Y(t),t) \\ + \frac{1-\gamma+\gamma\,\rho^2}{1-\gamma}\,\varphi(Y(t),t)\,\hat{\zeta}(t)\Big)\,du + h(Y(t),t)\,d\hat{W}_2(t)$$

and

$$\frac{dS(t)}{S(t)} = \sigma(Y(t),t)\,\Big(\frac{1-\gamma+\gamma\,\rho^2}{1-\gamma}\,\lambda(Y(t),t)\,dt + \rho\,\frac{1-\gamma+\gamma\,\rho^2}{1-\gamma}\,\frac{\varphi(Y(t),t)}{h(Y(t),t)}\,\hat{\zeta}(t)\,dt \\ + \rho\,d\hat{W}_2(t) + \sqrt{1-\rho^2}\,dW^{\perp}(t)\Big),$$

where $\{\hat{W}_2(t) : 0 \le t \le T\}$ is a Wiener process relative to the probability measure $\hat{\mathbb{P}}_2$. Moreover, the conditional expectation $\hat{\zeta}(t)$ satisfies the stochastic differential equation

$$d\hat{\zeta}(t) = (a(Y(t),t) + b(Y(t),t)\,\hat{\zeta}(t))\,dt \\ + \frac{\varphi(Y(t),t)}{h(Y(t),t)}\,\omega(t)\Big(d\hat{W}_2(t) + \frac{\gamma\,\rho}{1-\gamma}\,\lambda(Y(t),t)\,dt + \frac{\gamma\,\rho^2}{1-\gamma}\,\frac{\varphi(Y(t),t)}{h(Y(t),t)}\,\hat{\zeta}(t)\,dt\Big).$$

Logarithmic Utility Function. We next consider the logarithmic utility function $U(x) = \log x$. In this case, the optimal utility is

$$V(x) = \sup \mathbb{E}\{\log X(T)\},$$

where the supremum is over all admissible portfolio processes $\{\pi(t) : 0 \le t \le T\}$. In order to derive a formula for $V(x)$, we use the identity $V(x) = \hat{V}(x)$, where

$$\hat{V}(x) = \sup \hat{\mathbb{E}}\{\log X(T)\}.$$

Again, the supremum runs over all admissible portfolio processes $\{\pi(t) : 0 \le t \le T\}$. This is a stochastic control problem with full observation, the solution of which is given by

$$\hat{V}(x) = \log x + \frac{1}{2}\,\hat{\mathbb{E}}\Big\{\int_0^T \Big(\lambda(Y(t),t) + \rho\,\frac{\varphi(Y(t),t)}{h(Y(t),t)}\,\hat{\zeta}(t)\Big)^2\,dt\Big\}.$$

4.3. **Mean-Variance Hedging.** We finally assume that the investor's risk preferences are modeled by the mean-variance principle. We define the value function $V(x,y)$ by

$$V(x,y) = \inf \mathbb{E}\{X(T)^2\} - \mathbb{E}\{X(T)\}^2,$$

where the infimum is taken over all admissible portfolio processes with initial capital x and expected return $\mathbb{E}\{X(T)\} = y$.

As above, one can show that $V(x,y) = \hat{V}(x,y)$, where

$$\hat{V}(x,y) = \inf \hat{\mathbb{E}}\{X(T)^2\} - \hat{\mathbb{E}}\{X(T)\}^2,$$

where the infimum runs over all admissible portfolio processes with initial capital x and expected return $\hat{\mathbb{E}}[X(T)] = y$. Hence, we have the reduced the problem to a stochastic control problem with full observation, the solution of which is given by

$$\hat{V}(x,y) = (y-x)^2 \left(\hat{\mathbb{E}}\left\{ \exp\left(-2\rho \int_0^T \left(\lambda(Y(t),t) + \rho\, \frac{\varphi(Y(t),t)}{h(Y(t),t)}\, \hat{\zeta}(t) \right) d\hat{W}(t) \right.\right.\right.$$
$$\left.\left.\left. - \int_0^T \left(\lambda(Y(t),t) + \rho\, \frac{\varphi(Y(t),t)}{h(Y(t),t)}\, \hat{\zeta}(t) \right)^2 dt \right) \right\}^{-\frac{1}{1-2\rho^2}} - 1 \right)^{-1}.$$

To simplify this result, we define a probability measure $\hat{\mathbb{P}}_3$ by

$$\frac{d\hat{\mathbb{P}}_3}{d\hat{\mathbb{P}}} = \exp\left(-2\rho \int_0^T \left(\lambda(Y(t),t) + \rho\, \frac{\varphi(Y(t),t)}{h(Y(t),t)}\, \hat{\zeta}(t) \right) d\hat{W}(t) \right.$$
$$\left. - 2\rho^2 \int_0^T \left(\lambda(Y(t),t) + \rho\, \frac{\varphi(Y(t),t)}{h(Y(t),t)}\, \hat{\zeta}(t) \right)^2 dt \right).$$

With this definition, the optimal utility $\hat{V}(x,y)$ can be written in the form

$$\hat{V}(x,y) = (y-x)^2 \left(\hat{\mathbb{E}}_3\left\{ \exp\left(-(1-2\rho^2) \int_0^T \left(\lambda(Y(t),t) + \rho\, \frac{\varphi(Y(t),t)}{h(Y(t),t)}\, \hat{\zeta}(t) \right)^2 dt \right) \right\}^{-\frac{1}{1-2\rho^2}} - 1 \right)^{-1}.$$

Furthermore, the dynamics of $\{Y(t) : 0 \le t \le T\}$ and $\{S(t) : 0 \le t \le T\}$ is given by

$$dY(t) = \Big(g(Y(t),t) - 2\rho\, h(Y(t),t)\, \lambda(Y(t),t)$$
$$+ (1-2\rho^2)\, \varphi(Y(t),t)\, \hat{\zeta}(t) \Big)\, dt + h(Y(t),t)\, d\hat{W}_3(t)$$

and

$$\frac{dS(t)}{S(t)} = \sigma(Y(t),t) \left((1-2\rho^2)\, \lambda(Y(t),t)\, dt + \rho\, (1-2\rho^2)\, \frac{\varphi(Y(t),t)}{h(Y(t),t)}\, \hat{\zeta}(t)\, dt \right.$$
$$\left. + \rho\, d\hat{W}_3(t) + \sqrt{1-\rho^2}\, dW^\perp(t) \right),$$

where $\{\hat{W}_3(t) : 0 \le t \le T\}$ is a Wiener process under the probability measure $\hat{\mathbb{P}}_3$. Moreover, the conditional expectation $\hat{\zeta}(t)$ satisfies the stochastic differential equation

$$d\hat{\zeta}(t) = (a(Y(t),t) + b(Y(t),t)\, \hat{\zeta}(t))\, dt$$
$$+ \frac{\varphi(Y(t),t)}{h(Y(t),t)}\, \omega(t) \left(d\hat{W}_3(t) - 2\rho\, \lambda(Y(t),t)\, dt - 2\rho^2\, \frac{\varphi(Y(t),t)}{h(Y(t),t)}\, \hat{\zeta}(t)\, dt \right).$$

Consider now a contingent claim, settling at time T, written on the observable processes S and Y. Its payoff is modeled by a $\mathcal{F}^{S,Y}(T)$-measurable random variable η. For every initial capital x, we define the minimal replication error

$$R_\eta(x) = \inf \mathbb{E}\{(X(T) - \eta)^2\},$$

where the infimum is taken over all admissible portfolio processes $\{\theta(t) : 0 \le t \le T\}$, and $\{X(t) : 0 \le t \le T\}$ is the wealth process associated with $\{\theta(t) : 0 \le t \le T\}$ and the initial capital x. Argueing as above, we obtain $R_\eta(x) = \hat{R}_\eta(x)$, where

$$\hat{R}_\eta(x) = \inf \hat{\mathbb{E}}\{(X(T)-\eta)^2\},$$

where the infimum is taken over all admissible portfolio processes $\{\theta(t) : 0 \le t \le T\}$. For abbreviation, let

$$\hat{\lambda}(t) = \frac{\mu(Y(t),t)}{\sigma(Y(t),t)} + \rho\,\frac{g(Y(t),t)}{h(Y(t),t)} + \rho\,\frac{\varphi(Y(t),t)}{h(Y(t),t)}\,\hat{\zeta}(t),$$

and assume that $\{\hat{\pi}(t) : 0 \le t \le T\}$ is chosen such that

$$\begin{aligned}
\exp\Big(& -\frac{1}{\rho}\int_0^T \Big(\hat{\lambda}(t) + (1-2\rho^2)\,\hat{\pi}^*(t)\Big)\,d\hat{W}(t) \\
& \qquad - \frac{1}{2}\,\frac{1}{\rho^2}\int_0^T \Big(\hat{\lambda}(t) + (1-2\rho^2)\,\hat{\pi}^*(t)\Big)^2\,dt\Big) \\
= \hat{\mathbb{E}}\Big\{ & \exp\Big(-2\rho\int_0^T \hat{\lambda}(t)\,d\hat{W}(t) - \int_0^T \hat{\lambda}(t)^2\,dt\Big)\Big\}^{-1} \\
\cdot \exp\Big(& -2\rho\int_0^T \hat{\lambda}(t)\,d\hat{W}(t) - \int_0^T \hat{\lambda}(t)^2\,dt\Big).
\end{aligned}$$

Finally, we introduce the exponential martingale $\{\hat{Z}(t) : 0 \le t \le T\}$, where

$$\begin{aligned}
\hat{Z}(t) = \exp\Big(& -\frac{1}{\rho}\int_0^t \hat{\lambda}(u)\,d\hat{W}(u) \\
& - \frac{\sqrt{1-\rho^2}}{\rho}\int_0^t \hat{\pi}^*(u)\,[\sqrt{1-\rho^2}\,d\hat{W}(u) - \rho\,dW^\perp(u)] \\
& - \frac{1}{2}\,\frac{1}{\rho^2}\int_0^t \hat{\lambda}(u)^2\,du - \frac{1-\rho^2}{\rho^2}\int_0^t \hat{\pi}^*(u)\,\hat{\lambda}(u)\,du \\
& - \frac{1}{2}\,\frac{1-\rho^2}{\rho^2}\int_0^t \hat{\pi}^*(u)^2\,du\Big).
\end{aligned}$$

Then the minimal replication error $\hat{R}_\eta(x)$ is smallest for

$$x = \hat{\mathbb{E}}\{\hat{Z}(T)\,\eta\}.$$

This is the Föllmer-Schweizer hedging price [8] of η in our partially observable market.

References

[1] A. Bensoussan. *Stochastic Control of Partially Observable Systems*. Cambridge University Press, 1992.

[2] R. Carmona ed. *Indifference Pricing*. Princeton University Press, Princeton, NJ, 2005.

[3] R. Carmona and A. Danilova. Hedging financial instruments written on non-tradable indexes. Technical report, SIAM 50th Anniversary and Annula Meeting, (2002) Philadelphia, 2002.

[4] R. Carmona and M. Ludkowski. Convenience yield model with partial observations and exponential utility. Technical report, Princeton Universtiy, May 2004.

[5] J. Cvitanić and I. Karatzas. Convex duality in constrained portfolio optimization. *Annals of Applied Probability*, 2(4):767–818, 1992.

[6] M.H.A. Davis. Optimal hedging with basis risk, 2000.

[7] W. Fleming and M. Soner. *Controlled Markov Processes and Viscosity Solutions*. Springer-Verlag, New York, 1993.
[8] H. Follmer and C. Schweizer. Hedging of contingent claims under incomplete information. In M. H. A. Davis and R. J. Elliott, editors, *Applied Stochastic Analysis*, pages 389–414. Gordon and Breach, London, 1991.
[9] D. Hobson and V. Henderson. Real options with constant relative risk aversion. *Journal of Economic Dynamics and Control*, 27:329–355, 2002.
[10] S. Hodges and A. Neuberger. Optimal replication of contingent claims under transactions costs. *Review of Futures Markets*, 8:222–239, 1989.
[11] G. Turmuhambetova L. Hansen, T. J. Sargent and N. Williams. Robust control and model misspecification, 2004.
[12] J-M. Lasry and P-L. Lions. Contrôle stochastique avec informations partielles et applications à la finance. *C. R. Acad. Sci. Paris, Série I*, 328:1003–1010, 1999.
[13] Lipster and Shiryaev. *Statistics of Random Process (Second Edition)*. Springer Verlag, New York, NY, 2001.
[14] M. Musiela and T. Zariphopoulou. A valuation algorithm for indifference prices in incomplete markets. *Finance & Stochastics*, forthcoming, 2004.
[15] L.C.G. Rogers. Duality in constrained optimal investment and consumption problems: a synthesis. In R.Carmona et al., editor, *Paris Princeton Seminar in Mathematical Finance*, pages 101–137. Lect. Notes in Math. # **1814**, Springer Verlag, N.Y., 2003.
[16] R. Sircar and T. Zariphopoulou. Bounds and asymptotic approximations for utility prices when volatility is random. preprint, 2004.
[17] M. Tehranchi. Explicit solution of some utility maximization problems in incomplete markets. *Stochastic Proc. and Appl.*, (to appear):1568–1590, October 2004.

Department of Mathematics, Princeton University, Princeton, NJ 08544.
E-mail address: sbrendle@princeton.edu

Bendheim Center for Finance, ORFE, Princeton University, Princeton, NJ 08544
E-mail address: rcarmona@princeton.edu