Consistency of the Geometric Brownian Motion Model of Stock Prices with Asymmetric Information.

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Abstract

This work is concentrated on the microeconomic foundation of modern option pricing models. We develop a model of market agents' interactions, induced by heterogeneity of information, which is consistent with both modern option pricing models and empirical facts about stock price behavior. In particular, we focus on the connection between volatility and trading volume.

We show that the geometric Brownian motion model of asset prices is consistent with agents' learning and asymmetric information. We verify empirically the theoretical implication of our model that trading volume drives the price process: indeed, at very high frequency, the volume of trade is able to explain more then one third of the variability in asset returns.

1 Introduction

The modelling of stock prices by an exogenous stochastic process originates with Bachelier (1900) who assumed that stock prices follow a Wiener process. Since then, with the development of stochas-

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tic calculus and probability theory, this assumption has grown in depth and complexity, leaving the geometric Brownian Motion assumption as a benchmark model for asset prices.

The development of stochastic calculus, especially martingale representation and Girsanov theorems gave way to tremendous development in no arbitrage pricing and hedging of derivatives for a broad class of asset prices models. Initially, it was standard in the mathematical finance literature to assume that the asset price process is exogenous, and the underlying role of asset prices to clear the market and transmit possible private information was overlooked.

More recently, the question of consistency of asset prices models with agents' interactions has gained more attention and a theoretical justification for a broad class of stochastic diffusion models has been achieved (see, among others, Horst (2005), Follmer and Schweizer (2003), He and Leland (1993)).

However, this research does not take into account the informational role of the asset prices as well relationship between the prices and other important market processes such as number of trades and volume of trades. The aim of this work is to propose a consistent treatment of the stock price process, number of trades process and and volume of trade process. We develop a microstructure model which provides a theoretical explanation of the empirical facts about the role of volume of trades in the stock price process formation and results in a geometric Brownian Motion price process in the limit as intensity of trading goes to infinity.

The empirical research on the relationship between trading volume and volatility documents that these processes are strongly related (see, among others, Gallant, Rossi, and Touchen (1992) and Karpoff (1987)). However, the theoretical relationship between volume of trades and volatility is not yet well understood. This relationship is widely discussed in the market microstructure literature both in the *noisy equilibrium* framework (see O'Hara, 1995 for the review of this approach), which concentrates on volume of trades and in the *asymmetric information sequential trading* (see, among others, Glosten and Milgom (1985) and Andersen (1996)) setting, which concentrates on the number of trades as a proxy for volume of trades. The general theoretical view on the phenomenon is that the relation is caused by private information, which is revealed by the informed traders through their strategies, and gets reflected in the prices and volume in different ways.

Since the empirical findings on relationship between volume of trades, number of trades and volatility are based on high frequency data, the model presented in this work hinges on explaining the trade by trade evolution rather than the long run movements of the asset price. Hence, it is set in the market microstructure theory framework (see O'Hara (1995) and Hasbrouck (2004) for reviews of the subject) and uses asymmetric information as a mechanism for generating the relationship between volume of trade and volatility. The model allows us to distinguish the roles of number of trades and of trading volume in the price process formation. The results of the theoretical model and the empirical tests suggest that on the trade by trade basis the trading volume carries more information then number of trades process. However, it should be noted that this singular role of volume of trades in asset price process formation hinges on our assumption of zero transaction costs.

Another important feature of this model is that it delivers the price process as geometric Brownian motion thus showing that it is consistent not only with efficient markets but also with ones with asymmetric information and agents' learning.

The rest of the paper is organized as following: in Section 2 we present the model and derive an equilibrium market maker's ask and bid pricing functions and informed traders' optimal strategy, in Section 3 we derive the resulting equilibrium price and prove convergence result and in Section 4 we present empirical tests of the model.

2 Theoretical model and the equilibrium

Our model concentrates on the adverse selection component of the bid/ask spread. We assume that both order processing and inventory costs are zero.

In what follows it is assumed that all random variables are defined on a stochastic basis $\left(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\leq T}, \mathbb{P}\right)$.

The market considered has finite time horizon T and consists of one riskless asset with a constant growth rate r and one risky stock whose value at time T is given by e^{D_T}, where $\{D_t\}_{t\leq T}$ is the continuous log profit process of the firm which issued the stock, and where D follows the stochastic differential equation

$$\begin{aligned} dD_t &= \mu dt + \sigma dW_t \\ D_0 &= \text{const} \end{aligned}$$

The trading process is modelled as a generalization of the model of Glosten and Milgrom (1985).

There are three types of market participants: uninformed traders, informed traders and market makers, all of which have power utility function $U(x) = \frac{x^\gamma}{\gamma}$ with $\gamma \in (0,1]$. At each time t, the market maker chooses two functions $(B_t(v^-))$, the bid price and $(A_t(v^+))$, the ask price (or, for brevity, "bid" and "ask"). The argument, v, of these functions is interpreted as the order size[1]. If v is positive it is interpreted as an order to buy and if it is negative then as an order to sell. It is assumed that investors arrive one by one according to some counting stochastic process N_t (the number of arrivals by time t process) with associated stopping times $\theta_i = \inf\{t \geq 0 : N_t = i\}$. At the arrival time θ_i, the probability of the event U_i, that an uninformed trader arrives is q^u, and the probability that an informed trader arrives is $(1-q^u)$. Later we will define different information sets for these two types of traders. Denote by

$$\begin{aligned} N_t^U &= \sum_{i=1}^{\infty} 1_{\{\theta_i \leq t\} \cap U_i}, \\ \theta_i^U &= \inf\left\{t \geq 0 : N_t^U = i\right\}, \end{aligned}$$

the number of uninformed traders arrivals process and associated stopping times. Similarily, denote by

$$\begin{aligned} N_t^I &= \sum_{i=1}^{\infty} 1_{\{\theta_i \leq t\} \cap \bar{U}_i}, \\ \theta_i^I &= \inf\left\{t \geq 0 : N_t^I = i\right\}, \end{aligned}$$

the number of informed traders arrivals process and associated stopping times.

Each investor is informed of $B_{\theta_i}(\cdot)$ and $A_{\theta_i}(\cdot)$ upon arrival, and is free to trade v_i units of the risky asset (negative v is interpreted as an order to sell, positive as an order to buy and $v = 0$ means that the investor decides not to trade).

Let the process L_t be the cumulative number of trades by time t, i.e. a counting process given

[1] The assumption that bid and ask quotes depend on the order size is supported by the market data: there are often several quotes for different order sizes. The assumption that ask price depends only on positive part and bid price only on the negative part of order size is equivalent to assuming that a trader cannot submit buy and sell orders at the same time.

by

$$L_t = \sum_{i=1}^{\infty} 1_{\{\theta_i \le t\} \cap \{v_i \ne 0\}}$$

with associated stopping times $\tau_i = \inf\{t \ge 0 : L_t = i\}$, and define

$$V_t = \sum_{i=1}^{\infty} v_i 1_{\{\theta_i \le t\}} \tag{1}$$

as the cumulative order size of all trades by time t. Define also

$$\tilde{v}_i = \sum_{j=1}^{\infty} v_j 1_{\{\theta_j = \tau_i\}}$$

the order size of the i^{th} trade.[2]

The i^{th} trade occurs at the price (note that by definition $\tilde{v}_i \ne 0$)

$$\tilde{p}_i = A_{\tau_i}\left(\tilde{v}_i^+\right) 1_{\{\tilde{v}_i > 0\}} + B_{\tau_i}\left(\tilde{v}_i^-\right) 1_{\{\tilde{v}_i < 0\}}. \tag{2}$$

Define

$$P_t = \sum_{i=1}^{\infty} \tilde{p}_i 1_{\{\tau_i \le t < \tau_{i+1}\}} + \tilde{p}_0 1_{\{t < \tau_1\}} \tag{3}$$

where $\tilde{p}_0 = \exp\left\{D_0 + \left(\mu - \frac{r}{\gamma} + \frac{\sigma^2\gamma}{2}\right) T\right\}$ is the price at which the last transaction before or at t was dealt[3].

The market participants differ in their information sets, as follows.

The uninformed investors observe transaction price process, volume of trade process and receive some private signal $s_i = S_{\theta_i^U}$ at time θ_i^U and their information set at their time of arrival is

$$\mathcal{H}_i^U = \overline{\sigma\left(\mathcal{F}_{\theta_i^U}^{(P,V)} \cup \sigma(s_i)\right)}.$$

The informed traders observe the profit process D, the transaction price process and the volume of

[2]Notice that $V_t = \sum_{i=1}^{\infty} \tilde{v}_i 1_{\{\tau_i \le t\}}$.

[3]The price process should be defined for all $t \ge 0$. Since there are no trades before τ_1 we postulate that the price before τ_1 is p_0 which is the equlibrium price of the market at $t = 0$ given the utility functions of the market participants which are introduced later.

trade process their information set at their time of arrival θ_i^I is

$$\mathcal{H}_i^I = \overline{\sigma\left(\mathcal{F}_{\theta_i^I}^{(P,V,D)}\right)}.$$

The market maker observes bid, ask, transaction prices and volume of trades evolution and knows the structure of the market, so her information set at time t is

$$\mathcal{G}_t^M = \overline{\sigma\left(\mathcal{F}_t^{(P,V)}\right)}$$

and notice that number of trades process is adapted to the market maker's filtration, i.e. $\overline{\mathcal{F}_t^L} \subset \mathcal{G}_t^M$.

In the spirit of Glosten and Milgrom, traders maximize their utility change at the time of their entry to the market,[4] but instead of being risk neutral they have a power utility function. The discounted utility from holding the stock for a market participant at time θ_i^k is

$$z_i^k = \frac{1}{\gamma} e^{-r\left(T-\theta_i^k\right)} E\left[e^{\gamma D_T} | \mathcal{H}_i^k\right], \quad k = I, U$$

and $1-\gamma$ is the Arrow-Pratt relative risk aversion parameter, for $\gamma \in (0,1]$. Hence the trader arriving at time θ_i^k maximizes

$$\max_v \left(\left(v^+\right)^\gamma \left[z_i^k - \frac{A_{\theta_i^k}^\gamma \left(v^+\right)}{\gamma} \right] + \left(v^-\right)^\gamma \left[\frac{B_{\theta_i^k}^{\gamma 2} \left(v^-\right)}{\gamma} - z_i^k \right] \right) \tag{4}$$

where $k = I, U$ depending on the type of investor. Let

$$z_i = \sum_{j=1}^{\infty} \left[z_j^I 1_{\left\{\theta_i = \theta_j^I\right\}} + z_j^U 1_{\left\{\theta_i = \theta_j^U\right\}} \right],$$

i.e. z_i is the discounted utility of one stock for the investor which arrived at time θ_i.

Following Glosten and Milgrom, we assume that the market maker sets up bid and ask prices under zero utility gain constraint. This assumption consists of two conditions on the market maker's behavior. First, she does not regret the trade ex-post,[5]. Second, the competition among market

[4]Maximizing the utility change instead of the final utility allows to omit the dependence of optimal strategy on the initial wealth distribution and to concentrate on its informational content.

[5]i.e. if trader submits the order of size v the market maker's utility wouldn't decrease if she executes it at the quoted price

makers does not allow her to set up ask and bid such that executing the order will result in a utility gain. [6] Therefore, the optimal bid (for $v^- > 0$) and ask (for $v^+ > 0$) at time t should satisfy[7]

$$A_t^\gamma\left(v^+\right) = e^{-r(T-t)} \sum_{i=1}^{\infty} E\left[e^{\gamma D_T} | \tilde{\mathcal{H}}_{i-1}, \tilde{v}_i, \tau_i\right]\Big|_{\tilde{v}_i = v^+, \tau_i = t} 1_{\{L_{t-} = i-1\}} \tag{5}$$

$$B_t^\gamma\left(v^-\right) = e^{-r(T-t)} \sum_{i=1}^{\infty} E\left[e^{\gamma D_T} | \tilde{\mathcal{H}}_{i-1}, \tilde{v}_i, \tau_i\right]\Big|_{\tilde{v}_i = v^-, \tau_i = t} 1_{\{L_{t-} = i-1\}} \tag{6}$$

where $\tilde{\mathcal{H}}_i^M = \mathcal{G}_{\tau_i}^M = \sigma\left(\{\tilde{v}_j\}_{j=1}^i, \{\tau_j\}_{j=1}^i, \mathcal{N}\right)$.

In addition, in order to have a well defined problem, we search for ask and bid curve processes satisfying the following technical conditions:

C1 For fixed v, the processes $B_t\left(v^-\right)$ and $A_t\left(v^+\right)$ are left continuous with right limits.[8]

C2 For a fixed t, $A_t : \bar{\mathbb{R}}_+ \to \bar{\mathbb{R}}_+ \setminus \{0\}$ is continuous, nondecreasing and unbounded, i.e. $lim_{v\to\infty} A_t(v) = \infty$.

C3 For a fixed t, $B_t : \mathbb{R}_+ \to \bar{\mathbb{R}}_+$ is continuous, nonincreasing and $lim_{v\to\infty} B_t(v) = 0$.[9]

C4 For a fixed t, $A_t\left(0\right) \geq B_t\left(0\right)$ for all ω.

In what follows it is assumed that N_t, D_t, N_t^U and S_t satisfy the following assumptions:

A1 $dD_t = \mu dt + \sigma dW_t$ and W_t is BM wrt $\mathcal{F}_t$;

A2 $\mathcal{F}_T^W, \mathcal{F}_T^N$ and S_{θ_i} are independent given $\mathcal{H}_{i-1}$ where $\mathcal{H}_i = \overline{\sigma\left(\{z_j\}_{j=1}^i, \{\theta_j\}_{j=1}^i\right)}$ for any i;

A3 U_i is independent of $\sigma\left(\mathcal{F}_T^{N,S,D} \cup \sigma\left(\left(U_k\right)_{k\neq i}\right)\right)$;

A4 $N_T < \infty$ a.s.;

[6] This is what would happen with many market makers and perfect competition between them.

[7] It should be noted, that at the time when the market maker sets bid and ask, she doesn't know the next order size and direction (i.e. whether it will be the buy or sell order). All she can do is to derive $A_t\left(v^+\right)$ assuming that the next order will be the order to buy v^+ shares and to derive $B_t\left(v^-\right)$ assuming that the next order will be the order to sell v^- shares.

[8] Notice that by definition of $\tilde{p}_i$ in (2) the specialist can not revise the bid and ask prices at the moment of trade, however, she is free to revise prices after the trade occurrence and before the arrival of the next investor, i.e. $B_t\left(v^-\right)$ and $A_t\left(v^+\right)$ must be left continuous processes for a fixed v

[9] Since the empirical investigation by Lin, Sanger and Booth (1995) demonstrated that adverse information component in the bid-ask spread grows with the order size, it is reasonable to search for an optimal ask (respectively, bid) curve which increase (respectively, decrease) with order size.

A5 $\mathbb{P}(z_i \in C|\mathcal{H}_{i-1}, U_i^c, \theta_i) = \mathbb{P}(z_i \in C|\mathcal{H}_{i-1}, U_i, \theta_i)$ for $C \in \mathcal{B}(\mathbb{R})$ which means that trading strategy of a participant does not allow the identification of her type.

Notation 1 *In what follows all processes are considered on three different timescales - continuous time, number of trades and number of arrivals. To distinguish between them we use following notation: processes considered on continuous time are denoted by upper case letters (e.g. G_t), the same processes on the scale of number of arrivals are denoted by lower case letters (e.g. $g_i = G_{\theta_i}$) and on the scale of number of trades are denoted by lower case letters with tilde (e.g. $\tilde{g}_i = G_{\tau_i}$)*

2.1 Existence and uniqueness of the equilibrium

To derive the equilibrium price, we first characterize the traders' optimal strategy and then use it to find the optimal bid and ask curves which define the price process.

Since, as assumed above, the trader who arrives at time θ_i observes $B_{\theta_i}(\cdot)$ and $A_{\theta_i}(\cdot)$, she can solve her optimization problem (4) and the characterization of the solution is given by the following theorem:

Theorem 2 *Suppose $A_t(v^+)$ and $B_t(v^-)$ satisfy conditions **C1-C4**. Consider a trader who enters the market at time θ_i. Suppose she observes bid and ask prices as a functions of the order size $(B_{\theta_i}(v^-),\ A_{\theta_i}(v^+))$. Then*

- *if $z_i > \frac{1}{\gamma} A^{\gamma}_{\theta_i}(0)$ her optimal trade is given by $v > 0$ satisfying*

$$z_i = \frac{1}{\gamma}\left[A^{\gamma}_{\theta_i}(v) + vA^{\gamma-1}_{\theta_i}(v)\, A'_{\theta_i}(v)\right] \tag{7}$$

- *if $z_i < \frac{1}{\gamma} B^{\gamma}_{\theta_i}(0)$ her optimal trade is given by $v < 0$ satisfying*

$$z_i = \frac{1}{\gamma}\left[B^{\gamma}_{\theta_i}(-v) - vB^{\gamma-1}_{\theta_i}(-v)\, B'_{\theta_i}(-v)\right] \tag{8}$$

- *if $\frac{1}{\gamma} B^{\gamma}_{\theta_i}(0) \le z_i \le \frac{1}{\gamma} A^{\gamma}_{\theta_i}(0)$ then her optimal trade is given by $v = 0$*

Proof. Apply the first order condition to the trader's maximization problem and notice that conditions **C2** and **C3** insure existence and finiteness of the global maximum ■

Remark 3 *In Theorem 2, v is not necessarily unique. However, we can ensure uniqueness by assuming that if there are multiple optimal order sizes, then trader picks the smallest one. In what follows we assume that this is the case.*

It follows from above that if market maker's optimal ask and bid curve processes satisfy conditions **C1-C4** then the size of the order placed by a trader is in one-to-one correspondence with the discounted utility derived by the trader from holding the stock, as expressed by (7) and (8). Therefore, we have $\sigma(v_j) = \sigma(z_j)$ for any j, and hence

$$\mathcal{H}_i = \overline{\sigma\left(\{v_j\}_{j=1}^{i}, \{\theta_j\}_{j=1}^{i}\right)},$$

for any i, and assumption **A5** is equivalent to

$$\mathbb{P}\left(v_i \in C | \mathcal{H}_{i-1}, U_i^c, \theta_i\right) = \mathbb{P}\left(v_i \in C | \mathcal{H}_{i-1}, U_i, \theta_i\right).$$

Moreover, due to the homeomorphism between trader's order size and utility of the stock, existence and uniqueness of the equilibrium of this market is equivalent to the existence and uniqueness of the optimal bid and ask curve processes satisfying conditions **C1-C4** which we now proceed to show.

As the market maker observes the buy and sell orders, it follows from the above that she observes the value of the stock for the trader. Therefore, based on the optimal trader's strategy it is possible to determine the market maker's optimal bid and ask functions.

Theorem 4 *Suppose assumptions **A1- A5** are satisfied. Then there exist optimal bid and ask such that conditions **C2-C4** and equations (5), (6) are satisfied. Moreover, A_t and B_t can be expressed as*

$$A_t^{\gamma}\left(v^+\right) = X_t^M \left(1 + A\left(v^+\right)^{p\gamma}\right), \tag{9}$$

$$B_t^{\gamma}\left(v^-\right) = \begin{cases} X_t^M \left(1 - B\left(v^-\right)^{p\gamma}\right), & 1 > B\left(v^-\right)^{p\gamma}, \\ 0 & \textit{otherwise}, \end{cases} \tag{10}$$

where A and B are positive constants, $p = \frac{q^u}{1-q^u}$ and

$$X_t^M = e^{-r(T-t)} \sum_{i=1}^{\infty} \mathbb{E}\left[e^{\gamma D_T} | \tilde{\mathcal{H}}_{i-1}^M, \tau_i\right]\Big|_{\tau_i = t} 1_{\{L_{t-}=i-1\}}.$$

Proof. ¿From the definition (5) of ask $A(v^+)$, and (6) of bid $B(v^-)$ and Theorem 2 it follows (by Bayes rule) that

$$\left(A_t^{\gamma}\left(v^+\right) - X_t\left(v^+\right)\right) p\left(v^+, t, \omega\right) = v^+ A_t^{\gamma-1}\left(v^+\right) A_t^{'}\left(v^+\right), \tag{11}$$

$$\left(B_t^{\gamma}\left(v^-\right) - X_t\left(-v^-\right)\right) p\left(-v^-, t, \omega\right) = v^- B_t^{\gamma-1}\left(v^-\right) B_t^{'}\left(v^-\right), \tag{12}$$

where

$$p\left(x, t, \omega\right) = \sum_{i=1}^{\infty} \frac{\mathbb{P}\left(\tilde{U}_i | \tilde{\mathcal{H}}_{i-1}^M, \tilde{v}_i, \tau_i\right)}{\mathbb{P}\left(\tilde{U}_i^c | \tilde{\mathcal{H}}_{i-1}^M, \tilde{v}_i, \tau_i\right)}\Bigg|_{\tilde{v}_i = x, \tau_i = t} 1_{\{L_{t-}=i-1\}}, \tag{13}$$

$$X_t\left(x\right) = e^{-r(T-t)} \sum_{i=1}^{\infty} \mathbb{E}\left[e^{\gamma D_T} | \tilde{\mathcal{H}}_{i-1}^M, \tilde{v}_i, \tilde{U}_i, \tau_i\right]\Big|_{\tilde{v}_i = x, \tau_i = t} 1_{\{L_{t-}=i-1\}} \tag{14}$$

and $\tilde{U}_i = \cup_{j=1}^{\infty}\left\{U_j \cap \{\tau_i = \theta_j\}\right\}$ is the event that an uninformed trader submitted the order at time τ_i.

To simplify (13) notice that

$$\mathbb{P}\left(\tilde{U}_i | \tilde{\mathcal{H}}_{i-1}^M, \tilde{v}_i \in C, \tau_i\right) = \frac{\mathbb{P}\left(\tilde{U}_i | \tilde{\mathcal{H}}_{i-1}^M, \tau_i\right)}{\frac{\mathbb{P}\left(\tilde{v}_i \in C | \tilde{\mathcal{H}}_{i-1}^M, \tau_i\right)}{\mathbb{P}\left(\tilde{v}_i \in C | \tilde{\mathcal{H}}_{i-1}^M, \tilde{U}_i, \tau_i\right).}} \tag{15}$$

Since a trade can happen only at entry time we have $\tau_i = \theta_j$ for some j. Moreover, since on $\{\tau_i = \theta_j\}$ we have $\tilde{\mathcal{H}}_{i-1}^M \subset \mathcal{H}_{j-1}$, $\tilde{v}_i = v_j$, $\tilde{U}_i = U_j$, assumption **A5** gives

$$\mathbb{P}\left(\tilde{v}_i \in C | \tilde{\mathcal{H}}_{i-1}^M, \tau_i\right) = \mathbb{P}\left(\tilde{v}_i \in C | \tilde{\mathcal{H}}_{i-1}^M, \tilde{U}_i, \tau_i\right),$$

for any $C \in \mathcal{B}\left(\mathbb{R}\right)$. Hence (15) becomes

$$\mathbb{P}\left(\tilde{U}_i | \tilde{\mathcal{H}}_{i-1}^M, \tilde{v}_i \in C, \tau_i\right) = \mathbb{P}\left(\tilde{U}_i | \tilde{\mathcal{H}}_{i-1}^M, \tau_i\right).$$

It follows from assumption **A3** that U_j is independent of $\mathcal{H}_{j-1}$ and θ_j for any j, and $\tau_i = \theta_j$ for some j, so we have

$$\mathbb{P}\left(\tilde{U}_i|\tilde{\mathcal{H}}_i^M\right) = \mathbb{P}\left(\tilde{U}_i\right) = q^u. \tag{16}$$

Therefore

$$p = p\left(x, t, \omega\right) = \frac{q^u}{1 - q^u}.$$

To simplify (14), notice that by the tower property and **A2**,

$$\begin{aligned} X_t\left(v\right) e^{r(T-t)} &= \sum_{i=1}^{\infty} \mathbb{E}\left[\mathbb{E}\left[e^{\gamma D_T}|\tilde{\mathcal{H}}_{i-1}^M, S_{\tau_i}, \tilde{U}_i, \tau_i\right]|\tilde{\mathcal{H}}_{i-1}^M, \tilde{v}_i, \tilde{U}_i, \tau_i\right]\Big|_{\tilde{v}_i = v, \tau_i = t} 1_{\{L_{t-} = i-1\}} \\ &= \sum_{i=1}^{\infty} \mathbb{E}\left[e^{\gamma D_T}|\tilde{\mathcal{H}}_{i-1}^M, \tau_i\right]\Big|_{\tau_i = t} 1_{\{L_{t-} = i-1\}}. \end{aligned}$$

Therefore

$$X_t\left(v\right) = X_t^M,$$

i.e. $X_t\left(v\right)$ does not depend on v and (11) and (12) can be rewritten as

$$\begin{aligned} \left(A_t^\gamma\left(v^+\right) - X_t^M\right) p &= \frac{v^+}{\gamma}\left(A_t^\gamma\left(v^+\right)\right)', \\ \left(B_t^\gamma\left(v^-\right) - X_t^M\right) p &= \frac{v^-}{\gamma}\left(B_t^\gamma\left(v^-\right)\right)'. \end{aligned}$$

The solutions of these ODEs, given conditions **C2- C4**, are unique and given by (9) and (10) ■

The above theorem shows that the optimal ask and bid curves satisfying conditions **C2-C4** exist and are unique for each t. However, whether ask and bid curves satisfy condition **C1** (and therefore defined for each t) is still not proven.

Clearly, condition **C1** is a condition on the process X_t^M. Therefore showing the existence (and a.s. uniqueness) of the process X_t^M satisfying **C1** is our next goal.

To prove this we will need the following auxiliary result: under assumptions **A1-A5** and conditions **C1-C4** the process of number of arrivals coincides with the process of number of trades.

This result is based on the observation that due to the shape of the ask and bid curves, $A_t^\gamma\left(0\right) = B_t^\gamma\left(0\right) = X_t^M$ and therefore by Theorem 1 the trade at time θ_i occurs if and only if $x_i^M \neq \gamma z_i$, where $x_i^M = X_{\theta_i}^M$. Therefore, by Theorem 1, the market maker observes the value of the stock, if and only if her valuation of the stock differs from that of the trader entering the market. So, in the next lemma we only need to prove that the event of market maker's valuation being equal to trader's valuation has probability zero.

Lemma 1 *Suppose assumptions* ***A1-A5*** *and conditions* ***C1-C4*** *are satisfied. Then*

$$N_T = L_T \text{ a.s.}$$

Proof. Let $n_k = N_{\theta_k}$ and $l_k = L_{\theta_k}$. We will use induction to prove that $n_k = l_k$. Clearly for $k = 0$ we have $\theta_k = 0$ and therefore $n_0 = l_0 = 0$.

Suppose that $n_{k-1} = l_{k-1}$ a.s.. Let $\mathcal{H}_i^M = \mathcal{G}_{\theta_i}^M$ and define $(i < k)$

$$\begin{aligned} \mathcal{H}_i^k \quad : \quad &= \sigma\left(\sigma\left(\theta_k\right), \mathcal{H}_i^M\right) \cap \{n_{k-1} = l_{k-1}\} \\ &= \sigma\left(\sigma\left(\theta_k\right), \mathcal{H}_i\right) \cap \{n_{k-1} = l_{k-1}\}. \end{aligned} \tag{17}$$

Notice that $\mathcal{H}_{k-1}^k = \mathcal{G}_{\theta_k-}^M \cap \{n_{k-1} = l_{k-1}\}$. Then by Theorem 4 (and the discussion following it) and since $n_{k-1} = l_{k-1}$ a.s. we have that

$$\begin{aligned} P\left(n_k \neq l_k | \mathcal{G}_{\theta_k-}^M\right) &= P\left(n_k \neq l_k | \mathcal{H}_{k-1}^k\right) \\ &= P\left(x_k^M = \gamma z_k | \mathcal{H}_{k-1}^k\right). \end{aligned}$$

Notice that x_k^M is measurable wrt $\mathcal{H}_{k-1}^k$ since

$$x_k^M e^{r(T-t)} = \mathbb{E}\left[e^{\gamma D_T} | \mathcal{H}_{k-1}^M, \tau_k\right]\Big|_{\tau_k = \theta_k}.$$

Hence to prove that $n_k = l_k$ a.s. it is enough to show that $z_k | \mathcal{H}_{k-1}^k$ has a continuous distribution.

In fact, due to the assumption **A5** and Theorem 2 it is enough to show that $z_k | \mathcal{H}_{k-1}^k, U_k^c$ has continuous distribution. By assumption **A3**

$$\begin{aligned} P\left[z_k \leq z | \mathcal{H}_{k-1}^k, U_k^c\right] &= P\left[d_k \leq f_k\left(z\right) | \mathcal{H}_{k-1}^k, U_k^c\right] \\ &= P\left[d_k \leq f_k\left(z\right) | \mathcal{H}_{k-1}^k\right], \end{aligned}$$

where $d_k = D_{\theta_k}$ and

$$\begin{aligned} f_k(z) &= \frac{\log(\gamma z)}{\gamma} - \left(\mu_\gamma - \frac{r}{\gamma}\right)(T - \theta_k), \\ \mu_\gamma &= \mu + \frac{\sigma^2 \gamma}{2}. \end{aligned}$$

Since $f_k(z)$ is measurable with respect to $\mathcal{H}^k_{k-1}$ it is enough to show that $d_k | \mathcal{H}^k_{k-1}$ has a continuous distribution. Since $n_{k-1} = l_{k-1}$ a.s. (hence $\theta_{i-1} = \tau_{i-1}$ a.s. for any $i \leq k$) and due to **A3** we have for any $i \leq k$

$$\begin{aligned} P\left(U_{i-1} | \mathcal{H}^k_{i-1}\right) &= P\left(U_{i-1} | \mathcal{H}_{i-1}, \theta_k, n_{k-1} = l_{k-1}\right) \\ &= P\left(U_{i-1} | \mathcal{H}_{i-1}\right) = q^u \end{aligned} \tag{18}$$

a.s., and therefore for any $i \leq k$

$$\begin{aligned} P\left[d_k \leq f_k(z) | \mathcal{H}^k_{i-1}\right] &= (1 - q^u) P\left[d_k \leq f_k(z) | U^c_{i-1}, \mathcal{H}^k_{i-1}\right] \\ &\quad + q^u P\left[d_k \leq f_k(z) | U_{i-1}, \mathcal{H}^k_{i-1}\right]. \end{aligned} \tag{19}$$

Due to assumptions **A3** and **A2** we have for any $i \leq k$ a.s.

$$\begin{aligned} P\left[d_k \leq f_k(z) | U_{i-1}, \mathcal{H}^k_{i-1}\right] &= E\left[P\left[d_k \leq f_k(z) | s_{i-1}, \theta_{i-1}, U_{i-1}, \mathcal{H}^k_{i-2}\right] | U_{i-1}, \mathcal{H}^k_{i-1}\right] \\ &= P\left[d_k \leq f_k(z) | \mathcal{H}^k_{i-2}\right]. \end{aligned}$$

Due to the assumptions **A2** and **A1** we have, for any $i \leq k-1$,

$$\begin{aligned} P\left[d_k \leq f_k(z) | U^c_i, \mathcal{H}^k_i\right] &= P\left[f_i(z_i) + \varepsilon_{k,i} \leq f_k(z) | \mathcal{H}^k_i\right] \\ P\left[d_k \leq f_k(z) | U_1, \mathcal{H}^k_1\right] &= P\left[d_0 + \varepsilon_{k,0} \leq f_k(z) | \mathcal{H}^k_0\right] \end{aligned} \tag{20}$$

where $\varepsilon_{i,j} = \mu \Delta_{i,j} + \sigma \sqrt{\Delta_{i,j}} \nu_{ij}$, $\Delta_{i,j} = \theta_i - \theta_j$; $\nu_{ij} \sim N(0,1)$ and is independent of $\mathcal{H}^k_j$; $f_i(z_i)$ is measurable with respect to $\mathcal{H}^k_i$. Therefore we can iterate (19) by applying (18) and (20) to it, and so obtain

$$\begin{aligned} P\left[d_k \le f_k(z) \,|\mathcal{H}^k_{k-1}\right] &= (q^u)^{k-1} \left(P\left[\varepsilon_{k,0} \le f_k(z) - d_0|\mathcal{H}^k_0\right]\right. \\ &+ \left.\sum_{i=1}^{k-1} (1-q^u)(q^u)^{-i} P\left[\varepsilon_{k,i} \le f_k(z) - f_i(z_i)|\mathcal{H}^k_i\right]\right). \end{aligned}$$

Hence $d_k|\mathcal{H}^k_{k-1}$ has continuous distribution and therefore we have

$$P[n_k = l_k] = 1.$$

By induction we have

$$P[n_k = l_k] = 1, \quad \text{for any } k.$$

Hence

$$P[N_T \ne L_T] = \sum_{i=0}^{\infty} P[n_k \ne l_k|N_T = k]\, P[N_T = k]$$

and since $N_T < \infty$ a.s., it follows that

$$P[N_T = L_T] = 1.$$

■

Several important observations follow from this lemma. The first is that

$$\tau_i = \theta_i \quad \text{for } \forall i,$$

and since all the filtrations we consider in this paper are complete we have

$$\mathcal{H}_i = \mathcal{H}^M_i = \tilde{\mathcal{H}}^M_i. \tag{21}$$

Therefore in the rest of the paper we will not distinguish between trade and arrival times and filtrations $\mathcal{H}$, $\mathcal{H}^M$ and $\tilde{\mathcal{H}}^M$.

The second observation (which will be used in deriving the limiting distribution of the price process) is that we can characterize the distribution of z_k conditional on $\sigma(\mathcal{H}_{k-1}, \theta_k)$ as stated in the next remark.

Remark 5 *Denote by*

$$\hat{d}_k = f_k(z_k), \tag{22}$$

where

$$f_k(z) = \frac{\log(\gamma z)}{\gamma} - \left(\mu_\gamma - \frac{r}{\gamma}\right)(T - \theta_k).$$

It can be proved, in an analogous manner to the proof of lemma 1, that

$$\begin{aligned} P\left[\hat{d}_k \leq z | \mathcal{H}_{k-1}, \theta_k\right] = (q^u)^{k-1} & \left(P\left[\varepsilon_{k,0} \leq z - d_0 | \theta_k\right]\right. \\ & \left. + \sum_{i=1}^{k-1} \frac{(1-q^u)}{(q^u)^{i-k+1}} P\left[\hat{d}_i + \varepsilon_{k,i} \leq z | \hat{d}_i, \Delta_{k,i}\right]\right), \end{aligned} \tag{23}$$

where $\varepsilon_{i,j} = \mu\Delta_{i,j} + \sigma\sqrt{\Delta_{i,j}}\nu_{ij}$ *and* ν_{ij} *is a standard normal random variable independent of* $\hat{d}_j, \Delta_{k,j}$.

With these results at hand we can characterize the market maker's learning process.

Theorem 6 *Suppose assumptions* ***A1-A5*** *and conditions* ***C1-C4*** *are satisfied. Then*

$$X_t^M = \sum_{i=0}^{\infty} e^{r(t-\theta_i)} z_i^M 1_{\{N_{t-}=i\}}, \tag{24}$$

with

$$z_i^M = (1 - q^u)\gamma z_i + q^u e^{r(\theta_i - \theta_{i-1})} z_{i-1}^M,$$

and where

$$z_i^M = e^{-r(T-\theta_i)} E\left[e^{\gamma D_T} | \mathcal{H}_i\right].$$

Proof. By Lemma 1 and relationships(21) between filtrations, we have

$$\begin{aligned}
X_t^M &= e^{-r(T-t)} \sum_{i=0}^{\infty} E\left[e^{\gamma D_T} | \mathcal{G}^M_{\tau_i -}, \tau_i \right] \Big|_{\tau_i = t} 1_{\{L_{t-}=i\}} \\
&= e^{-r(T-t)} \sum_{i=0}^{\infty} E\left[e^{\gamma D_T} | \mathcal{G}^M_{\theta_i -}, \theta_i \right] \Big|_{\theta_i = t} 1_{\{N_{t-}=i\}} \\
&= e^{-r(T-t)} \sum_{i=0}^{\infty} E\left[e^{\gamma D_T} | \mathcal{H}_{i-1}, \theta_i \right] \Big|_{\theta_i = t} 1_{\{N_{t-}=i\}} \\
&= e^{-r(T-t)} \sum_{i=0}^{\infty} E\left[e^{\gamma D_T} | \mathcal{H}_{i-1} \right] 1_{\{N_{t-}=i\}} \\
&= e^{r(t-\theta_i)} \sum_{i=0}^{\infty} z_i^M 1_{\{N_{t-}=i\}}
\end{aligned}$$

a.s. where the last but one equality is due to **A2**.

To show that

$$z_i^M = (1-q^u)\,\gamma z_i + q^u e^{r(\theta_i - \theta_{i-1})} z_{i-1}^M,$$

we use Bayes rule and assumptions **A1** and **A3** in a similar manner as in the proof of lemma 1, to get

$$z_i^M = (1-q^u)\,\gamma z_i + e^{-r(T-\theta_i)} q^u E\left[e^{\gamma D_T} | \mathcal{H}_{i-1} \right],$$

a.s.. By **A2**, and the fact that $N_T = L_T \dot{s}.$, we have

$$z_i^M = (1-q^u)\,\gamma z_i + e^{r(\theta_i - \theta_{i-1})} q^u z_i^M,$$

a.s. as claimed. ■

Notice that the resulting process X_t^M is indeed left-continuous with right limits (LCRL for brevity), and so the ask and bid curves satisfy condition **C1**. Hence Theorem 6 completes the proof of existence and uniqueness of the equilibrium of the market.

In the next section we will use these results to derive the price process resulting from the optimal behavior of the market participants.

3 Equilibrium price process

3.1 Equilibrium price process at ultra-high frequency

Suppose the trade at time τ_{i-1} was executed at the ask price i.e. $v_{i-1} > 0$, then by (2) we have

$$\begin{aligned} p_{i-1}^{\gamma} &= A_{\tau_{i-1}}^{\gamma}(v_{i-1}) = e^{-r(T-\tau_{i-1})} E\left[e^{\gamma D_T} | \mathcal{H}_{i-2}^M, v_{i-1}, \tau_{i-1}\right] \\ &= e^{-r(T-\tau_{i-1})} E\left[e^{\gamma D_T} | \mathcal{H}_{i-1}^M\right] = z_{i-1}^M. \end{aligned}$$

Since the same considerations apply if the trade happens at bid, we have

$$p_{i-1}^{\gamma} = z_{i-1}^M, \tag{25}$$

for all i. Therefore it follows from Theorem 4 and equation (24) that the price process satisfies

$$p_i^{\gamma} = e^{r(\tau_i - \tau_{i-1})} p_{i-1}^{\gamma} \left(1 + \xi_i \left(V_{\tau_i} - V_{\tau_{i-1}}\right)^{p\gamma}\right), \tag{26}$$

where V_t is process of cumulative volume of trade and

$$\xi_i = \begin{cases} A & \text{if the } i^{\text{th}} \text{ trade is at the ask price} \\ -B & \text{if the } i^{\text{th}} \text{ trade is at the bid price.} \end{cases}$$

Since the transaction price, P_t, between trades is unchanged we have that

$$\log\left(\frac{P_t}{P_s}\right) = \frac{1}{\gamma} \sum_{i=L_s}^{L_t} \log\left(1 + \xi_i \left(V_{\tau_i} - V_{\tau_{i-1}}\right)^{p\gamma}\right) + \frac{r}{\gamma}\left(\tau_{L_t} - \tau_{L_s}\right), \tag{27}$$

so the model implies a direct relationship between the price change and the volume of trade for the tick-by-tick price movements. However, the relationship is not linear as in Jones, Kaul and Lipson (1994) or Ane and Geman (2000), but nonlinear as in Gallant, Rossi and Touchen (1992).

Equation (27) gives a direct relationship between the volume of trade and the price change. However, the volume of trade is not an exogenous variable since it is a function of z_i. Therefore, to study the limiting distribution of the price process we need an alternative characterization of it in terms of z_i or, equivalently, in terms of $\hat{d}$ given by (22), which is provided by the next remark.

Remark 7 *Using the result of Theorem 6 and equations (25) and (22) we may rewrite (26) as*

$$\begin{aligned} p_i^{\gamma} &= q^u e^{r(\tau_i - \tau_{i-1})} p_{i-1}^{\gamma} + (1 - q^u) \exp\left\{\gamma \hat{d}_i + \left(\gamma \mu_\gamma - r\right)(T - \tau_i)\right\}, \\ p_0 &= \exp\left\{\mu_\gamma T + d_0\right\}. \end{aligned} \tag{28}$$

3.2 Equilibrium price process at low frequency

For simplicity of presentation it is assumed that $r = 0$ in what follows. This is not a restrictive assumption since we can obtain the same results for the discounted price process. Also, all processes are defined on $[0, T]$ unless stated otherwise.

The aim of this section is to establish the limiting behavior of the transaction price process as the intensity of the traders' arrival process increases. This is summarized in the following result:

Theorem 8 *Suppose that process D satisfies A1 and $\check{N}$ is a Poisson process such that $\mathcal{F}^{\check{N}}$ is independent of $\mathcal{F}^W$.*

Then there exists a sequence of Poisson processes N^n satisfying $\mathbb{P}\left[N_t^n = \check{N}_{tn}\right] = 1$ for any $t \in [0, T]$ and a constant $q^* > 0$ such that for any $q^u \in [0, q^*]$ the price process P^n resulting from any sequence of markets $\mathcal{M}^n(q^u, N^n, D, S^n, 1(U^n))$ satisfying A1-A5 weakly converges in the Skorokhod topology ($P^n \to^w P$), and the limit process, P, is geometric Brownian motion independent of $\mathcal{F}^{\check{N}}$ with drift $\mu + \frac{\sigma^2 \gamma}{2}$ and volatility σ.

To set the stage for proving theorem 8 we first define a sequence of traders' arrival process N^n. In the rest of this section we assume that the process D satisfying A1 is given and fixed. For any given Poisson process $\check{N}_t$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with intensity λ and corresponding arrival times

$\check{\tau}_i = \inf\left\{t \geq 0 : \check{N}_t \geq i\right\}$, independent of $\mathcal{F}^W$, consider the following sets

$$
\begin{aligned}
\Omega_1 &= \left\{\omega \in \Omega : \lim_{i\to\infty} \frac{\sum_{j=1}^{\lfloor zi \rfloor} \left[\check{\tau}_j - \check{\tau}_{j-1}\right]^2}{\sum_{j=1}^{i} \left[\check{\tau}_j - \check{\tau}_{j-1}\right]^2} = z \text{ for any } z \in [0,1]\right\} \\
&= \cap_{z\in[0,1]\cap\mathbb{Z}} \left\{\omega \in \Omega : \lim_{i\to\infty} \frac{\sum_{j=1}^{\lfloor zi \rfloor} \left[\check{\tau}_j - \check{\tau}_{j-1}\right]^2}{\sum_{j=1}^{i} \left[\check{\tau}_j - \check{\tau}_{j-1}\right]^2} = z\right\}, \\
\Omega_2 &= \left\{\omega \in \Omega : \max_{i\leq k} \left[\check{\tau}_i - \check{\tau}_{i-1}\right] < \infty \text{ for any } k \in \mathbb{N}_+\right\}, \\
\Omega_3 &= \cup_{k=1}^{\infty} \cap_{i=k}^{\infty} \left\{\omega \in \Omega : \left[\check{\tau}_i - \check{\tau}_{i-1}\right] \leq 2\log(i)\right\} \\
\Omega_4 &= \left\{\omega \in \Omega : \lim_{n\to\infty} \frac{\check{N}_{tn}}{n} = \lambda t \text{ for any } t \in [0,T]\right\} \\
&= \cap_{t\in[0,T]\cap\mathbb{Z}} \left\{\omega \in \Omega : \lim_{n\to\infty} \frac{\check{N}_{tn}}{n} = \lambda t\right\}.
\end{aligned}
$$

Clearly $\Omega_i \subset \Omega$ and $\Omega_i \in \mathcal{F}$ for $i = 1,2,3,4$. Moreover, it directly follows from the Strong Law of Large Numbers and the properties of the Poisson process that $\mathbb{P}(\Omega_i) = 1$, $i = 1,2,4$, and from the Borel-Cantelli lemma that $\mathbb{P}(\Omega_3) = 1$.

Consider a Poisson process $\tilde{N}_t$, a modification of $\check{N}_t$ given by:

$$
\begin{aligned}
\tilde{\tau}_0(\omega) &= 0, \\
\tilde{\tau}_i(\omega) &= \begin{cases} \check{\tau}_i(\omega), \, \omega \in \cap_{j=1}^{4}\Omega_i, \\ \tilde{\tau}_{i-1}(\omega) + \lambda, \, \omega \in \Omega \backslash \left(\cap_{j=1}^{4}\Omega_i\right) \end{cases}, \\
\tilde{N}_t &= \sum_{i=1}^{\infty} 1_{\{\tilde{\tau}_i \leq t\}},
\end{aligned}
$$

and a corresponding sequence of counting processes N_t^n given by

$$
N_t^n = \tilde{N}_{tn}, \tag{29}
$$

$$
\tau_i^n = \frac{\tilde{\tau}_i}{n}, \tag{30}
$$

which defines the sequence of trader's arrival process in the markets $\mathcal{M}^n$ of Theorem 8.

To define the sequence of markets $\mathcal{M}^n$ with the sequence of traders' arrival process given by N^n, we need to define a sequence of processes S^n and $1(U^n)$ such that $\mathcal{M}^n\left(q^u, N^n, D, S^n, 1(U^n)\right)$ satisfy assumptions A1-A5. However, due to assumptions A2, A3 and A5 the law of the price process P^n

conditional on $\mathcal{F}_\infty^{\tilde{N}}$ (notation: $\mathcal{L}(P^n|\mathcal{F}_\infty^{\tilde{N}})$) resulting from any market $\mathcal{M}^n\left(q^u, N^n, D, S^n, 1(U^n)\right)$ satisfying A1-A5 is uniquely defined by (23) and (28). Since we are aiming to prove a weak convergence result, it is enough to define the process $\tilde{P}^n$ such that $\mathcal{L}(\tilde{P}^n|\mathcal{F}_\infty^{\tilde{N}}) = \mathcal{L}(P^n|\mathcal{F}_\infty^{\tilde{N}})$.

Fix any market $\mathcal{M}^n = \mathcal{M}^n\left(q^u, N^n, D, S^n, 1(U^n)\right)$ satisfying assumptions A1-A5 with N^n given by equation (30) and let P^n be the transaction price process resulting from it. Consider processes $\tilde{D}_t^n = \sum_{j=0}^{\infty} \tilde{d}_j^n 1_{\{N_t^n = j\}}$ and $\tilde{P}_t^n = \sum_{j=0}^{\infty} \tilde{p}_j^n 1_{\{N_t^n = j\}}$ with random variables $\tilde{d}_j^n$ and $\tilde{p}_j^n$ given by

$$\begin{aligned}
\tilde{d}_j^n &= \sum_{i=0}^{j-1} \zeta_i^{j-1} \left(\tilde{d}_i^n + \mu\left(\tau_j^n - \tau_i^n\right) + \sigma\sqrt{\tau_j^n - \tau_i^n}\,\nu^{j-1}\right), && (31)\\
\tilde{d}_0^n &= d_0, \\
\left(\tilde{p}_j^n\right)^\gamma &= q^u\left(\tilde{p}_{j-1}^n\right)^\gamma + (1-q^u)\exp\left\{\gamma \tilde{d}_j^n + \gamma\mu_\gamma\left(T - \tau_j^n\right)\right\}, && (32)\\
\tilde{p}_0^n &= \exp\left\{\mu_\gamma T + d_0\right\},
\end{aligned}$$

where ζ and ν are independent, ν^j are independent standard normal random variables, $\zeta_j = \left(\zeta_i^j\right)_{i=1}^n$ are independent random variables with $\sum_{i=1}^j \zeta_i^j = 1$, $\zeta_i^j \in \{0,1\}$,

$$\mathbb{P}(\zeta_i^j = 1) = \begin{cases} (1-q^u)\,(q^u)^{j-i} \text{ for } i > 0 \\ (q^u)^j \text{ for } i = 0 \end{cases}$$

and $\nu \perp \zeta$, $\nu^i \perp \mathcal{F}_{i-1}^{\hat{d}} \vee \mathcal{F}^{N^n}$ and $\zeta_i \perp \mathcal{F}_{i-1}^{\hat{d}} \vee \mathcal{F}^{N^n}$.

Then we will have $\mathcal{L}(\tilde{P}^n|\mathcal{F}_\infty^{\tilde{N}}) = \mathcal{L}(P^n|\mathcal{F}_\infty^{\tilde{N}})$. To demonstrate this, note that due to the one-to-one correspondence between processes p^n (defined by $p_i^n = P_{\tau_i^n}^n$) and $\hat{d}^n$, and between $\tilde{p}^n$ and $\tilde{d}^n$, expressed by (28) and (32) respectively, it is enough to show that $\mathcal{L}(\tilde{D}^n|\mathcal{F}_\infty^{\tilde{N}}) = \mathcal{L}(\hat{D}^n|\mathcal{F}_\infty^{\tilde{N}})$, where $\hat{D}_t^n = \sum_{i=0}^{\infty} \hat{d}_i^n 1_{\{N_t^n = i\}}$ and $\hat{d}_i^n$ is given by (22) applied to the market $\mathcal{M}^n$. From (23), (31) and by Lemma 1, it follows that

$$\mathcal{L}\left(\hat{D}^n|\mathcal{F}_\infty^{\tilde{N}}\right) = \mathcal{L}\left(\hat{D}^n|\mathcal{F}_T^{N^n}\right) = \mathcal{L}\left(\tilde{D}^n|\mathcal{F}_{Tn}^{\tilde{N}}\right) = \mathcal{L}\left(\tilde{D}^n|\mathcal{F}_\infty^{\tilde{N}}\right).$$

Therefore to prove Theorem 8 it is enough to show that $\tilde{P}^n$ defined by equations (32) and (31) converges weakly in the Skorokhod topology to geometric Brownian Motion independent of $\mathcal{F}_\infty^{\tilde{N}}$. Since we consider weak convergence and $\mathcal{L}(\tilde{P}^n|\mathcal{F}_\infty^{\tilde{N}}) = \mathcal{L}(P^n|\mathcal{F}_\infty^{\tilde{N}})$, in what follows we shall not distinguish between the processes $\tilde{P}^n$ and P^n so we drop the "tilde" notation.

Observe that if we define the new (random) probability measure $\tilde{\mathbb{P}}$ by the regular version of the kernel $\mathbb{P}\left(A|\mathcal{F}_\infty^{\tilde{N}}\right)$ on $\tilde{\mathcal{F}} = \sigma\left(\mathcal{F}_\infty^{\tilde{N}} \cup \left(\cup_n \mathcal{F}_T^{\hat{D}^n}\right)\right)$, i.e.

$$\tilde{\mathbb{P}}(A) = \mathbb{P}\left(A|\mathcal{F}_\infty^{\tilde{N}}\right)$$

and define a filtration $\mathcal{F}_t^n = \sigma\left(\mathcal{F}_\infty^{\tilde{N}} \cup \mathcal{F}_t^{P^n}\right)$, then if we show that there exists a process M^P, a geometric Brownian motion such that

$$\tilde{\mathcal{L}}(P^n) \to \tilde{\mathcal{L}}\left(M^P\right), \text{ as } n \to \infty,$$

then we will have that

$$\mathcal{L}(P^n) \to \mathcal{L}\left(M^P\right), \text{ as } n \to \infty,$$

and $\mathcal{F}^{M^P}$ is independent of $\mathcal{F}_\infty^{\tilde{N}}$, so Theorem 8 will be proved.

Based on this observation we derive the convergence result for the process

$$P_t^n = P_t^n(q^u)$$

on $\left(\Omega, \mathcal{F}, \tilde{\mathbb{P}}\right)$ and then the result for the price process on $(\Omega, \mathcal{F}, \mathbb{P})$ follows. First we prove the tightness result presented in the following lemma.

Lemma 2 *Consider the sequence of processes*

$$M_t^n = \sum_{i=0}^{\infty} \left(\hat{d}_i^n + \mu\left(T - \tau_i^n\right)\right) 1_{\{N_t^n = i\}}$$

There exists $q_1^u > 0$ *such that for any* $q^u \in [0, q_1^u]$ *and* (μ, σ) *the sequence of processes* $(P^n, \mathcal{F}^n)$ *and* $(M^n, \mathcal{F}^n)$ *are tight in the Skorokhod topology. Moreover, the limits of* M^n *and* $(P^n)^\gamma$ *are continuous local martingales in their natural filtration.*

To prove this lemma we will need the following result: let $\mathcal{H}_i^n = \mathcal{F}^n_{\tau_i^n}$ and consider

$$\eta_i^n = \eta_i^n(q^u) = (p_i^n(q^u))^\gamma - \left(p_{i-1}^n(q^u)\right)^\gamma, \tag{33}$$

$$\xi_i^n = \xi_i^n(q^u) = \tilde{\mathbb{E}}\left[\hat{D}_T^n(q^u)\,|\mathcal{H}_i^n\right] - \tilde{\mathbb{E}}\left[\hat{D}_T^n(q^u)\,|\mathcal{H}_{i-1}^n\right], \tag{34}$$

and

$$\left(\sigma_{n,i}^\eta\right)^2 = \tilde{\mathbb{E}}\left(\eta_i^n\right)^2, \tag{35}$$

$$\left(\sigma_{n,i}^\xi\right)^2 = \tilde{\mathbb{E}}\left(\xi_i^n\right)^2. \tag{36}$$

Then the following lemma holds:

Lemma 3 *Consider $\sigma_{n,i}^\eta$ and $\sigma_{n,i}^\xi$ given by (35) and (36). If conditions of Theorem 2 are satisfied we have that*

$$\max_{i\le N_t^n} \sigma_{n,i}^\eta \to 0 \text{ as } n\to\infty$$

$$\max_{i\le N_t^n} \sigma_{n,i}^\xi \to 0 \text{ as } n\to\infty$$

a.s. for any $q^u > 0$ and $t \ge 0$.

Moreover, there exists $q_1^u > 0$ such that for any $t \ge 0$, $q^u \in [0, q_1^u]$, the sets

$$U^\eta = \left\{\frac{(\eta_i^n)^2}{\left(\sigma_{n,i}^\eta\right)^2},\ n\in\mathbb{N},\ i\le N_t^n\right\},$$

$$U^\xi = \left\{\frac{(\xi_i^n)^2}{\left(\sigma_{n,i}^\xi\right)^2},\ n\in\mathbb{N},\ i\le N_t^n\right\},$$

are a.s. uniformly integrable and for any $\alpha > 0$

$$\lim_n \tilde{\mathbb{P}}\left(\max_{i\le N_t^n} |\eta_i^n| > \alpha\right) = 0,$$

$$\lim_n \tilde{\mathbb{P}}\left(\max_{i\le N_t^n} |\xi_i^n| > \alpha\right) = 0,$$

a.s.. (Proof: Appendix A)

Proof. With this result at hand we can prove Lemma 2. Consider a sequence of processes $\hat{M}^n_t = \tilde{\mathbb{E}}\left[\hat{D}^n_T(\mu,\sigma,q^u)\,|\mathcal{F}^n_t\right]$. Due to the Lemma 3 and the result of McLeish (1977) we obtain that the processes $(P^n(q^u),\mathcal{F}^n)$ and $\left(\hat{M}^n(q^u),\mathcal{F}^n\right)$ are tight in the Stone's topology for all $q^u \in [0,q^u_1]$. Since Stone's topology is equivalent to the Skorokhod topology on $\mathbb{D}([0,T])$ we have that the processes $(P^n(q^u),\mathcal{F}^n)$ and $\left(\hat{M}^n(q^u),\mathcal{F}^n\right)$ are tight in the Skorokhod topology for all $q^u \in [0,q^u_1]$ as well. Moreover, by Lemma 3, for such a q^u we have

$$\lim_n \tilde{\mathbb{P}}\left(\max_{t\le K}|\Delta P^n_t(q^u)|>\alpha\right) = \lim_n \tilde{\mathbb{P}}\left(\max_{i\le N^n_K}|\eta^n_i|>\alpha\right)=0, \quad (37)$$

$$\lim_n \tilde{\mathbb{P}}\left(\max_{t\le K}\left|\Delta \hat{M}^n_t(q^u)\right|>\alpha\right) = \lim_n \tilde{\mathbb{P}}\left(\max_{i\le N^n_K}|\xi^n_i|>\alpha\right)=0, \quad (38)$$

and the fact that sequences $P^n(q^u)$ and $\hat{M}^n(q^u)$ are tight, we have that they are C-tight, i.e. all limit points of the sequences $\left\{\tilde{\mathcal{L}}(P^n)\right\}$ and $\left\{\tilde{\mathcal{L}}\left(\hat{M}^n\right)\right\}$ are laws of continuous processes (see Jacod, Shiryaev (2003), Proposition VI.3.26). Moreover, consider any convergent subsequence P^{n_k}, $\hat{M}^{n_k}$. Then by (37) and (38) there exists a further subsequence n_{k_i}such that

$$\begin{aligned}
\max_{t\le N^n_T}\left|\Delta P^{n_{k_i}}_t(q^u)\right| &\to 0 \ \tilde{\mathbb{P}} \text{ a.s.},\\
\max_{t\le N^n_T}\left|\Delta \hat{M}^{n_{k_i}}_t(q^u)\right| &\to 0 \ \tilde{\mathbb{P}} \text{ a.s.}.
\end{aligned}$$

Hence there exist N, $b \in \mathcal{F}^{\tilde{N}}_\infty$ such that

$$\begin{aligned}
\left|\Delta P^{n_{k_i}}_t(q^u)\right| &\le b,\\
\left|\Delta \hat{M}^{n_{k_i}}_t(q^u)\right| &\le b,
\end{aligned}$$

a.s. for all $i \ge N$ and therefore the limit processes are local martingales (see Jacod, Shiryaev (2003) Proposition IX.1.17). Since the choice of convergent subsequence was arbitrary we have that the limits of $\hat{M}^n$ and $(P^n)^\gamma$ are continuous local martingales in their natural filtration on $\left(\Omega,\mathcal{F},\tilde{\mathbb{P}}\right)$.

To prove the same result for M^n, observe that

$$M^n_t = \hat{M}^n_t + \frac{q^u}{1-q^u}\xi^n_i,$$

and since ξ_i^n satisfies (38) we have that M_t^n converges to the same limit as $\hat{M}^n$ (see Jacod, Shiryaev (2003) Lemma VI.3.31) $\blacksquare$

Since, as demonstrated above, the processes $(P^n(q^u), \mathcal{F}^n)$ and $(M^n(q^u), \mathcal{F}^n)$ are tight in the Skorokhod topology for all $q^u \in [0, q_1^u]$, identification of the limit of these processes is equivalent to finding their limiting finite dimensional distributions. Since, by **Lemma** 2, M^n and $(P^n)^\gamma$ are continuous local martingales in their natural filtration on $\left(\Omega, \mathcal{F}, \tilde{\mathbb{P}}\right)$, it is possible to show that P^n is geometric Brownian motion in the limit, as demonstrated in the next lemma.

Lemma 4 *For any $q^u \in [0, q_1^u]$, the sequence of price processes $P^n(q^u)$ converges in law to M^P given by*

$$M_t^P = M_0^P \exp\{\sigma W_t - \mu_\gamma t\}$$

where W_t is Brownian motion.

Proof. Consider a subsequence n_i such that P^{n_i} and M^{n_i} converge weakly to M^P and M respectively. Consider a sequence of processes

$$\tilde{P}_t^{n_i} = \sum_{j=0}^{\infty} \exp\left\{\gamma \hat{d}_j^{n_i} + \gamma\mu_\gamma\left(T - \tau_j^{n_i}\right)\right\} 1_{\left\{N_t^{n_i}=j\right\}}.$$

Then, since $\tilde{P}_t^{n_i} = P_t^{n_i} + \frac{q^u}{1-q^u}\sum_{j=0}^{\infty} \xi_j^{n_i} 1_{\left\{N_t^{n_i}=j\right\}}$ and ξ satisfies equation (37), we have as in proof of Lemma 2 that $\tilde{P}^{n_i}$ converges weakly to M^P.

Hence we have that

$$\begin{aligned}
\tilde{\mathbb{P}}\left(M_t^P \le x\right) &= \lim_{i\to\infty} \tilde{\mathbb{P}}\left(\tilde{P}_t^{n_i} \le x\right) \\
&= \lim_{i\to\infty} \tilde{\mathbb{P}}\left(\hat{d}_j^{n_i} \le \frac{\ln x}{\gamma} - \mu_\gamma\left(T - \tau_j^{n_i}\right)\right) 1_{\left\{N_t^{n_i}=j\right\}} \\
&= \tilde{\mathbb{P}}\left(\exp\left\{\gamma M_t + \frac{\sigma^2\gamma^2}{2}(T-t)\right\} \le x\right).
\end{aligned}$$

But by definition of N^n (29), we have that $\sum_{j=0}^{\infty} \tau_j^n 1_{\{N_t^n=j\}} \to t$ a.s. and hence

$$\begin{aligned}
\lim_{i\to\infty}\sum_{j=0}^{\infty} \tilde{\mathbb{P}}\left(\hat{d}_j^{n_i} \le y + (\mu+z)\,\tau_j^{n_i}\right) 1_{\left\{N_t^{n_i}=j\right\}} &= \lim_{i\to\infty}\sum_{j=0}^{\infty} \tilde{\mathbb{P}}\left(N(0,1) \le \frac{y}{\sigma\sqrt{\tau_j^{n_i}}} + \frac{z}{\sigma}\sqrt{\tau_j^{n_i}}\right) 1_{\left\{N_t^{n_i}=j\right\}} \\
&= \tilde{\mathbb{P}}\left(N(0,1) \le \frac{y}{\sigma\sqrt{t}} + \frac{z}{\sigma}\sqrt{t}\right).
\end{aligned}$$

Hence

$$\tilde{\mathbb{P}}\left(M_t^P \leq x\right) = \tilde{\mathbb{P}}\left(\exp\left\{N\left(\mu\gamma T + \frac{\sigma^2\gamma^2}{2}(T-t), \gamma^2\sigma^2 t\right)\right\} \leq x\right)$$

and therefore

$$\tilde{\mathbb{E}}\left[M_T^P\right] = \exp\left\{\mu_\gamma T\right\} = M_0^P$$

. By Lemma 2, M^P is a local martingale, and since it is positive, it follows that it is a true martingale. Therefore $\langle M \rangle_t = \sigma^2 t$ (see Karatzas and Shreve Exercise 3.3.38.ii) and since M_t is a continuous local martingale it follows from the Levy characterization of Brownian motion that $\frac{M}{\sigma}$ is a Brownian motion independent of $\mathcal{F}_\infty^{\tilde{N}}$. Therefore

$$M_t^P = M_0^P \exp\left\{\sigma W_t - \mu_\gamma t\right\}$$

as claimed ■ **Proof.** From Lemmas 2 and 4 and the discussion in the beginning of this section, the proof of Theorem 8 is complete. ■

Therefore, as the number of arrivals tends to infinity the price process converges to a geometric Brownian motion independent of the Poisson arrival process. This demonstrates that the efficient market hypothesis is not a necessary assumption to justify geometric Brownian Motion model of stock prices – it is also consistent with asymmetric information and agents' learning models.

4 Empirical test of the model

This section explores the time series relation between stock returns and volume of trade implied by the model developed in the previous section. Moreover, we assess the relative explanatory power of volume of trade for the price movements compared with the number of trades that, in the previous literature, have been shown to explain a large part of price variability.

Since the degree of information asymmetry is likely to be higher for small stocks, the model at hand implies that volume of trade should matter more for small stocks then large stocks.[10] We therefore focus on two stocks that have been previously employed in the literature for analogous purposes: International Business Machines corp. (IBM) - as the large stock - and Bentley Pharmaceuticals

[10]Suggestive evidence that volume of trade has a larger impact on the price movements of smaller firms than bigger ones can be found in Jones, Kaul and Lipson (1994).

(BNT) - as the small stock.

The data was obtained from the NYSE trade and quote database (TAQ) and consists of one week (for IBM) and one month (for BNT) time stamped observations of bid and ask quotes together with prices and trade sizes. [11]

Since the model developed in the previous section implies that order size affects prices on a trade-by-trade basis, we consider log returns on a tick-by-tick basis rather then on fixed intervals. Furthermore, the models imply that any trade would result in a price change, but it is often observed in the market that there are several trades clustered in time and executed at the same price. Therefore, to be consistent with the theoretical setup, such trades are treated as one,[12] with the order size being the sum of the order sizes of these trades, and the time of execution given by the time of execution of the last trade. To put this formally, define the new trading times recursively as

$$\begin{aligned} \tilde{\tau}_0 &= 0 \\ \tilde{\tau}_i &= \left\{ s \geq \tilde{\tau}_{i-1} : P_s \neq P_{\tilde{\tau}_{i-1}} \right\}. \end{aligned}$$

Then the log return over the period $[\tilde{\tau}_{i-1}, \tilde{\tau}_i]$ is given by

$$Y_i = \log \left(\frac{P_{\tilde{\tau}_i}}{P_{\tilde{\tau}_{i-1}}} \right).$$

Summary statistics of the data are reported in Table 1, where $\left(\tilde{\Delta} N\right)_n = N_{\tilde{\tau}_n} - N_{\tilde{\tau}_{n-1}}$, $\left(\tilde{\Delta} V\right)_n = V_{\tilde{\tau}_n} - V_{\tilde{\tau}_{n-1}}$, $\Delta\tilde{\tau}_n = \tilde{\tau}_n - \tilde{\tau}_{n-1}$ (measured in seconds), N_t is the cumulative number of trades by time t and V_t is the cumulative volume of trade by time t (measured in shares).

[11] A larger sample period is used for BNT since this asset is traded less frequently. The time period is 5/10/04-5/14/04 for IBM (15685 data points) and May 2004 for BNT (877 data points).

[12] The reason to define this sequence of trades at the same price as one trade is that there are many market makers, which post their bid and ask, and when the trade happens they don't withdraw their quotes immediately, which results in several trades being very close to each other and executed at the same price. Also, there is a lag between the execution and the reporting of a trade, hence a market maker doesn't observe the trade of other market makers immediately.

	IBM		BNT	
	mean	standard error	mean	standard error
Y_t	$-4 \cdot 10^{-7}$	$3 \cdot 10^{-4}$	$3 \cdot 10^{-5}$	$2 \cdot 10^{-4}$
$(\tilde{\Delta} V)_n$	1655.7	3888.3	685.2	1270.8
$(\Delta\tilde{\tau})_n$	7.3	8.1	366.2	526
$(\tilde{\Delta} N)$	1.8	1.5	2.1	1.9

Table 1: Summary statistics

4.1 Relative impact of number of trades and volume of trade on price volatility

As a first exercise we test the conjecture of Ane and Geman (2000) and Jones, Kaul and Lipson (1994) that, given the number of trades, the volume of trade does not affect the volatility of stock prices. In contrast to these authors, and consistent with the model presented in Section 3, we introduce volume of trade in a nonlinear way. Like Ane and Geman, we use IBM price data to test this conjecture.

To investigate the relative role of number and volume of trade in explaining price volatility, we follow the procedure developed by Jones, Kaul and Lipson (1994) and employed by Ane and Geman (2000), therefore taking as a proxy for volatility the modulus of the shocks to the series of log returns. The procedure consists of two steps.

First, the log return Y_n is regressed on the 12 lagged returns to remove short term movements in conditional expected returns. Thus, the first regression is

$$Y_n = \sum_{j=1}^{12} a_j Y_{n-j} + \zeta_n \tag{39}$$

Then, to find the relative importance of the number of trades versus the volume of trades, the following three nonlinear regressions[13] are performed

$$\left|\hat{\zeta}_n\right| = const + \nu \left(\left(\tilde{\Delta} V\right)_n\right)^q + \eta \left(\tilde{\Delta} N\right)_n + \mu \Delta\tilde{\tau}_n + \sum_{j=1}^{12} b_j \left|\zeta_{n-j}\right| + \theta_n, \tag{a}$$

[13] The nonlinear regressions are estimated by nonlinear least squares.

$$\left|\hat{\zeta}_n\right| = const + \eta\left(\tilde{\Delta}N\right)_n + \mu\Delta\tilde{\tau}_n + \sum_{j=1}^{12} b_j \left|\zeta_{n-j}\right| + \theta_n, \tag{b}$$

$$\left|\hat{\zeta}_n\right| = const + \nu\left(\left(\tilde{\Delta}V\right)_n\right)^q + \mu\Delta\tilde{\tau}_n + \sum_{j=1}^{12} b_j \left|\zeta_{n-j}\right| + \theta_n, \tag{c}$$

where $\hat{\zeta}_n$ are the estimated residuals for equation (4.1), $\left(\tilde{\Delta}N\right)_n = N_{\tilde{\tau}_n} - N_{\tilde{\tau}_{n-1}}$, $\left(\tilde{\Delta}V\right)_n = V_{\tilde{\tau}_n} - V_{\tilde{\tau}_{n-1}}$ and $\Delta\tilde{\tau}_n = \tilde{\tau}_n - \tilde{\tau}_{n-1}$ is added to control for the fact that the observations are not equally spaced in time, and the lagged volatility proxies are included to control for the persistence in volatility process.

Remark 9 *Ane and Geman perform different regressions, of the form*

$$\left|\zeta_n\right| = const + \nu\left(V_{t+\Delta} - V_t\right) + \eta\left(N_{t+\Delta} - N_t\right) + \sum_{j=1}^{12} b_j \left|\zeta_{n-j}\right| + \theta_n$$

$$\left|\zeta_n\right| = const + \eta\left(N_{t+\Delta} - N_t\right) + \sum_{j=1}^{12} b_j \left|\zeta_{n-j}\right| + \theta_n$$

$$\left|\zeta_n\right| = const + \nu\left(V_{t+\Delta} - V_t\right) + \sum_{j=1}^{12} b_j \left|\zeta_{n-j}\right| + \theta_n.$$

However, since our model implies a nonlinear relation between the price movement and the order size, we have replaced the $(V_{t+\Delta} - V_t)$ *term with* $\nu\left(V_{\tilde{\tau}_n} - V_{\tilde{\tau}_{n-1}}\right)^q \approx \log(1 + \nu\left(V_{\tilde{\tau}_n} - V_{\tilde{\tau}_{n-1}}\right)^q)$ *(if* ν *is small). Also, since the price observations are not equally spaced, the term* $(\tilde{\tau}_n - \tilde{\tau}_{n-1})$ *appears.*

Regression results are reported in the Table 2. Column a) reports the estimation output of the regression in equation (a), and shows that the regressors considered are able to explain 10 percent of price volatility. Moreover, both q and η are significant. Even though ν individually is not statistically different from zero, the joint test of ν and q being zero is rejected at any standard level of marginal significance[14] consistent with the model proposed in the previous sections. Moreover, the sign of the estimated coefficient for q agrees with the theoretical one ($q = \gamma\frac{q^u}{1-q^u} > 0$) as well as the estimated sign of ν.

[14]A Wald type test of joint significance of these two coefficients produces a $\chi^2(2)$ statistic that is in the order of the thousands, delivering a p-value that is less than10^{-6}.

	IBM		
	Regression (a)	Regression (b)	Regression (c)
R^2	0.10	0.05	0.09
$\bar{R}^2$	0.10	0.05	0.09
ν	$3{\cdot}10^{-6}$		$1{\cdot}10^{-6}$
	(1.4)		(1.2)
q	0.47		0.5
	(6.3)		(6.1)
η	$-2{\cdot}10^{-5}$	$-6{\cdot}10^{-6}$	
	(14.1)	(5.5)	
μ	$-2{\cdot}10^{-6}$	$4{\cdot}10^{-6}$	$-3{\cdot}10^{-5}$
	(7.8)	(12.9)	(3.6)
$const$	$-3{\cdot}10^{-5}$	$8{\cdot}10^{-6}$	$1{\cdot}10^{-6}$
	(2.3)	(3.5)	(4.3)

Table 2: Relative impact of the number of trades and the volume of trades on volatility (All regressions use Newey and West (1987) correction of standard errors for generalized serial correlation and heteroscedasticity of the residuals. The t -statistics is reported in parenthesis)

The second column of Table 2 reports the results of regression (b), which uses only number of trades as main explanator of price volatility. All the estimated coefficients are significant. However, having dropped volume of trade as an explanatory variable, the measure of fit is reduced by one half.

The last column considers volume of trade as the main regressor. The first thing to notice, comparing these regression results with the ones in the first column, is that removing the number of trades $\left(\tilde{\Delta}N\right)_n$ causes a reduction of R^2 of less then 1%. This suggests that, on a tick-by-tick basis, once the volume of trade is introduced as explanatory variable, the number of trades do not carry much additional information, consistent with the model presented in the previous sections. Moreover, ν and q are jointly highly statistically significant.[15]

[15]The result of the joint significance test for ν and q is basically the same as in regression (a): the p-value of this restriction is less than 10^{-6}.

What is the relative economic importance of this regressor for price volatility? The estimated coefficients in the first column of Table 2 and statistics of the data given in Table 1 imply that a one standard deviation change in volume of trade ($\left(\tilde{\Delta}V\right)_n$) causes a 0.13 basis point change in the volatility proxy, while a one standard deviation change in the number of trades ($\left(\tilde{\Delta}N\right)_n$) causes only a 0.026 change in price volatility. Therefore, the economic significance of volume of trade, on a tick-by-tick basis, is about one order of magnitude larger than the one of number of trades.

Overall, the results in Table 2 suggest that, in accordance with the model presented, volume of trade is the main factor affecting volatility at very high frequency, and that the number of trades adds little additional information.

4.2 Relative impact of number of trades and volume of trade on price process

As a second check, we test the empirical implications of the baseline model, by directly estimating equation (27) through considering the approximation[16]

$$\log\left(1+\xi_{N_s+k}\left(\tilde{V}_{N_s+k}-\tilde{V}_{N_s+k-1}\right)^{p\gamma}\right)^{\frac{1}{\gamma}} \simeq \frac{\xi_{N_s+k}}{\gamma}\left(\tilde{V}_{N_s+k}-\tilde{V}_{N_s+k-1}\right)^{p\gamma},$$

where

$$\xi_n = \begin{cases} A & \text{if the } n^{\text{th}} \text{ trade is at the ask ,} \\ -B & \text{if the } n^{\text{th}} \text{ trade is at the bid.} \end{cases}$$

We therefore estimate the following regressions:

$$\begin{aligned} Y_n &= \text{const} + \left(\left(\nu_1-\nu_2\right)\varphi_n+\nu_2\right)\left(\left(\tilde{\Delta}V\right)_n\right)^q \qquad \text{(a)} \\ &\quad + \left(\left(\eta_1-\eta_2\right)\varphi_n+\eta_2\right)\left(\tilde{\Delta}N\right)_n + \mu\Delta\tilde{\tau}_n + \theta_n, \\ Y_n &= \text{const} + \left(\left(\eta_1-\eta_2\right)\varphi_n+\eta_2\right)\left(\tilde{\Delta}N\right)_n + \mu\Delta\tilde{\tau}_n + \theta_n, \qquad \text{(b)} \\ Y_n &= \text{const} + \left(\left(\nu_1-\nu_2\right)\varphi_n+\nu_2\right)\left(\left(\tilde{\Delta}V\right)_n\right)^q + \mu\left(\Delta\tilde{\tau}_n\right) + \theta_n, \qquad \text{(c)} \end{aligned}$$

[16] We consider this approximation since ξ_{N_s+k}, as shown below, is of an order of magnitude smaller than 10^{-4}., thus nonlinear regression will have too many parameters otherwise

where φ_n is a dummy variable that takes value 1 if the trade is at the ask, and 0 otherwise. Note also that, given the model at hand, equation (c) should be able to explain the variability of log returns as well as equation (a)

	IBM			BNT		
	a	b	c	a	b	c
R^2	0.34	0.20	0.33	0.30	0.13	0.28
$\bar{R}^2$	0.34	0.20	0.33	0.29	0.13	0.27
ν_1	$3 \cdot 10^{-5}$		$3 \cdot 10^{-5}$	$2 \cdot 10^{-4}$		$2 \cdot 10^{-4}$
	(10.6)		(8.9)	(3.0)		(2.1)
ν_2	$3 \cdot 10^{-5}$		$3 \cdot 10^{-5}$	$3 \cdot 10^{-4}$		$2 \cdot 10^{-4}$
	(11.2)		(9.5)	(2.7)		(2.3)
q	0.26		0.26	0.3		0.28
	(22.1)		(18.5)	(6.6)		(4.6)
η_1	$-1 \cdot 10^{-5}$	$5 \cdot 10^{-5}$		$-5 \cdot 10^{-5}$	$3 \cdot 10^{-4}$	
	(5.6)	(18.2)		(0.7)	(5.3)	
η_2	$-2 \cdot 10^{-5}$	$6 \cdot 10^{-5}$		$-3 \cdot 10^{-4}$	$3 \cdot 10^{-4}$	
	(8.4)	(21.5)		(4.1)	(3.3)	
μ	$-4 \cdot 10^{-7}$	$-6 \cdot 10^{-7}$	$-4 \cdot 10^{-7}$	$-2 \cdot 10^{-7}$	$-3 \cdot 10^{-7}$	$-2 \cdot 10^{-7}$
	(1.4)	(1.6)	(1.4)	(1.8)	(2.5)	(1.7)
$const$	$-7 \cdot 10^{-7}$	$7 \cdot 10^{-6}$	$1 \cdot 10^{-6}$	$-2 \cdot 10^{-4}$	$-5 \cdot 10^{-5}$	$-3 \cdot 10^{-4}$
	(0.1)	(1.8)	(0.2)	(0.8)	(0.5)	(1.0)

Table 3: Empirical test of the model(All regressions use Newey and West (1987) correction of standard errors for generalized serial correlation and heteroscedasticity of the residuals. The t -statistics is reported in parenthesis)

The results of these regressions are summarized by the Table 3, where the left panel focuses on IBM data and the right panel focuses on BNT data. Columns a) of the two panels show that volume of trade and number of trades are able to explain, jointly, between 29 to 34 percent of the variability in log returns. Moreover, the regression coefficients associated with $\left(\tilde{\Delta}N\right)_n$ and $\left(\tilde{\Delta}V\right)_n$ are all significant, and the signs of ν_1, ν_2 and q agree with the theory proposed. On the other hand, the coefficients associated with number of trades (η_1 and η_2) have counterintuitive signs.

Restricting the coefficients associated with order size to be equal to zero (column b)), causes a significant reduction of the explanatory power of the regression for both stocks (a reduction in $\bar{R}^2$ of 14 points for IBM and of 16 points for BNT). Moreover, the coefficients associated with the number of trades change sign moving from column a) to column b).

On the other hand, if we remove number of trades as an explanatory variable (column c), the volume of trade is able to explain 33 and 27 percent of the variability in log returns for, respectively, IBM and BNT, with a reduction of $\bar{R}^2$ of less then 1 percent for both stocks.

Looking at the economic significance of the regressors, once again (for both stocks) the volume of trade has an impact on log returns that is of an order of magnitude bigger then that of number of trades. For the IBM stocks, a one standard deviation increase in $\left(\tilde{\Delta} V\right)_n$ above its mean causes an increase in log returns of about one third of its standard deviation. On the other side, a one standard deviation change in $\left(\tilde{\Delta} N\right)_n$ causes a change of only 6% of a standard deviation in logreturns. For the BNT stock analogous shocks in the regressors generates, respectively, a 26% and 3% of a standard deviation change in log returns.

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A Proof of Lemma 3

Lemma 5 *Consider the sequences $\sigma^{\eta}_{n,i}$, $\sigma^{\eta}_{n,i}$ defined by (35) and (36) then there exist constants $c^{\eta}_{\min}$, $c^{\eta}_{\max}$, c^{ξ}, $b(q^u) \in \mathcal{F}_0$ independent of n and i with $b(q^u) < 1$ and $c^{\eta}_{\min} > 0$ such that*

$$\frac{\left(\sigma^{\eta}_{n,i}\right)^2}{L^n_i} \in [c^{\eta}_{\min}, c^{\eta}_{\max}] \text{ for all } i \leq N^n_T \tag{40}$$

$$\frac{\left(\sigma^{\xi}_{n,i}\right)^2}{L^n_i} = c^{\xi} \text{ for all } i \leq N^n_T \tag{41}$$

where $L^n_i = \sum_{j=1}^{i} b^{i-j}\left[\tau^n_j - \tau^n_{j-1}\right]$

Proof. Let $\mathcal{H}^n_i = \mathcal{F}^n_{\tau^n_i}$. Notice that due to (31) we have

$$\tilde{\mathbb{E}}\left[\hat{D}^n_T|\mathcal{H}^n_i\right] = (1-q^u)\left(\hat{d}^n_i + \mu\left(T-\tau^n_i\right)\right) + q^u\tilde{\mathbb{E}}\left[\hat{D}^n_T|\mathcal{H}^n_{i-1}\right]$$

therefore

$$\frac{\left(\sigma^{\xi}_{n,i}\right)^2}{(1-q^u)^2} = \sigma^2\tau^n_i - \sum_{j=1}^{i-1}\left(\sigma^{\xi}_{n,j}\right)^2$$

and due to (32) we have

$$\begin{aligned}\frac{\left(\sigma^{\eta}_{n,i}\right)^2}{(1-q^u)^2} &= \tilde{\mathbb{E}}\left(e^{\gamma\hat{d}^n_i + \gamma\mu_\gamma(T-\tau^n_i)} - \left(p^n_{i-1}\right)^{\gamma}\right)^2 \\ &= \left(P^n_0\right)^{2\gamma}\left[e^{\sigma^2\gamma^2\tau^n_i} - 1\right] - \sum_{j=1}^{i-1}\left(\sigma^{\eta}_{n,j}\right)^2\end{aligned}$$

where $P^n_0 = e^{D_0+\mu_\gamma T}$ i.e.

$$\begin{aligned}\frac{\left(\sigma^{\eta}_{n,i}\right)^2}{(1-q^u)^2} &= \left(P^n_0\right)^{2\gamma}\sum_{j=1}^{i}\left(q^u\left(2-q^u\right)\right)^{i-j}\left[\exp\left\{\sigma^2\gamma^2\tau^n_j\right\} - \exp\left\{\sigma^2\gamma^2\tau^n_{j-1}\right\}\right] \\ \frac{\left(\sigma^{\xi}_{n,i}\right)^2}{(1-q^u)^2} &= \sigma^2\sum_{j=1}^{i}\left(q^u\left(2-q^u\right)\right)^{i-j}\left[\tau^n_j - \tau^n_{j-1}\right]\end{aligned}$$

therefore by Taylor expansion we have

$$
\begin{aligned}
1 &\le \frac{\left(\sigma^{\eta}_{n,i}\right)^2}{\left(1-q^u\right)^2\left(P^n_0\right)^{2\gamma}\sigma^2\gamma^2\sum_{j=1}^{i}\left(q^u\left(2-q^u\right)\right)^{i-j}\left[\tau^n_j-\tau^n_{j-1}\right]} \le e^{\sigma^2\gamma^2 T}\\
\frac{\left(\sigma^{\xi}_{n,i}\right)^2}{\left(1-q^u\right)^2} &= \sigma^2\sum_{j=1}^{i}\left(q^u\left(2-q^u\right)\right)^{i-j}\left[\tau^n_j-\tau^n_{j-1}\right]
\end{aligned}
$$

for all $i \le N^n_T$. Denoting by

$$
\begin{aligned}
c^{\eta}_{\min} &= e^{-\sigma^2\gamma^2 T}\\
c^{\eta}_{\max} &= \left(1-q^u\right)^2\left(P^n_0\right)^{2\gamma}\sigma^2\gamma^2\\
c^{\xi} &= \left(1-q^u\right)^2\sigma^2
\end{aligned}
$$

and $b\left(q^u\right) = q^u\left(2-q^u\right) < 1$ we get the statement of the lemma. ■

Lemma 6 *Consider the sequences η^n_i, ξ^n_i defined by (33) and (34) and*

$$
\begin{aligned}
\left(\kappa^{\eta}_{n,i}\right)^4 &= \tilde{\mathbb{E}}\left(\eta^n_i\right)^4\\
\left(\kappa^{\xi}_{n,i}\right)^4 &= \tilde{\mathbb{E}}\left(\xi^n_i\right)^4
\end{aligned}
$$

then there exist constants C^{η}, C^{ξ}, $a\left(q^u\right) \in \mathcal{F}_0$ independent of n and i such that

$$
\left(\kappa^{\eta}_{n,i}\right)^4 \le C^{\eta}\left[\sum_{j=1}^{i}a^{i-j}\left[\tau^n_j-\tau^n_{j-1}\right]^2\right] \text{ for all } i \le N^n_T \tag{42}
$$

$$
\left(\kappa^{\xi}_{n,i}\right)^4 \le C^{\xi}\left[\sum_{j=1}^{i}a^{i-j}\left[\tau^n_j-\tau^n_{j-1}\right]^2\right] \text{ for all } i \le N^n_T \tag{43}
$$

Moreover, there exists $q^u_1 > 0$ such that for any $q^u \in [0, q^u_1]$ we will have $a\left(q^u\right) < 1$.

Proof. *D*ue to (31) and (32) and since

$$
\tilde{\mathbb{E}}\left[\hat{D}^n_T|\mathcal{H}^n_i\right] = \left(1-q^u\right)\left(\hat{d}^n_i + \mu\left(T-\tau^n_i\right)\right) + q^u\tilde{\mathbb{E}}\left[\hat{D}^n_T|\mathcal{H}^n_{i-1}\right]
$$

we have

$$\left(\kappa_{n,i}^{\eta}\right)^4 \leq 4\left(1-q^u\right)^4 \tilde{\mathbb{E}}\left[Y_i^{\eta}\right]^4 + 4(q^u)^4\left(\kappa_{n,i-1}^{\eta}\right)^4 \tag{44}$$

$$\left(\kappa_{n,i}^{\xi}\right)^4 \leq 4\left(1-q^u\right)^4 \tilde{\mathbb{E}}\left[Y_i^{\xi}\right]^4 + 4(q^u)^4\left(\kappa_{n,i-1}^{\xi}\right)^4 \tag{45}$$

where

$$\begin{aligned} Y_i^{\eta} &= e^{\gamma\hat{d}_i^n+\gamma\mu_\gamma(T-\tau_i^n)} - e^{\gamma\hat{d}_{i-1}^n+\gamma\mu_\gamma\left(T-\tau_{i-1}^n\right)} \\ Y_i^{\xi} &= \hat{d}_i^n - \hat{d}_{i-1}^n + \mu\left(\tau_{i-1}^n-\tau_i^n\right) \end{aligned}$$

Since by (31) and (32)

$$\begin{aligned} \tilde{\mathbb{E}}\left[Y_i^{\eta}\right]^4 &= \tilde{\mathbb{E}}\Bigg[\Bigg(\left(1-q^u\right)\sum_{j=1}^{i-1}(q^u)^{j-1}\left(e^{\gamma\left(\hat{d}_{i-j}^n+\mu\Delta_{i,i-j}^n+\sigma\sqrt{\Delta_{i,i-j}^n}\nu^{i-1}-\mu_\gamma\Delta_i^n\right)} - e^{\gamma\hat{d}_{i-1}^n}\right)^4 \\ &\quad +(q^u)^{i-1}\left(e^{\gamma\left(\hat{d}_0^n+\mu\Delta_{i,0}^n+\sigma\sqrt{\Delta_{i,0}^n}\nu^{i-1}-\mu_\gamma\Delta_i^n\right)} - e^{\gamma\hat{d}_{i-1}^n}\right)^4\Bigg)e^{4\gamma\mu_\gamma\left(T-\tau_{i-1}^n\right)}\Bigg] \\ \tilde{\mathbb{E}}\left[Y_i^{\xi}\right]^4 &= \tilde{\mathbb{E}}\Bigg[\Bigg(\left(1-q^u\right)\sum_{j=1}^{i-1}(q^u)^{j-1}\left(\hat{d}_{i-j}^n-\hat{d}_{i-1}^n+\mu\Delta_{i,i-j}^n+\sigma\sqrt{\Delta_{i,i-j}^n}\nu^{i-1}-\mu\Delta_i^n\right)^4 \\ &\quad +(q^u)^{i-1}\left(\hat{d}_0^n-\hat{d}_{i-1}^n+\mu\Delta_{i,0}^n+\sigma\sqrt{\Delta_{i,0}^n}\nu^{i-1}-\mu\Delta_i^n\right)^4\Bigg)\Bigg] \end{aligned}$$

where ν^i are as in (31) and $\Delta_{i,i-j}^n = \tau_i^n - \tau_{i-j}^n$, $\Delta_i^n = \tau_i^n - \tau_{i-1}^n$. Then

$$\begin{aligned} \tilde{\mathbb{E}}\left[Y_i^{\eta}\right]^4 &= \tilde{\mathbb{E}}\Bigg[e^{4\gamma\mu_\gamma\left(T-\tau_{i-1}^n\right)}\Bigg(\left(1-q^u\right)\left(e^{\gamma\left(\hat{d}_{i-1}^n+\sigma\sqrt{\Delta_i^n}\nu^{i-1}-\frac{\sigma^2\gamma}{2}\Delta_i^n\right)} - e^{\gamma\hat{d}_{i-1}^n}\right)^4 \\ &\quad +q^u\left(e^{\gamma\left(\left(\hat{d}_{i-1}^n\right)'+\sigma\sqrt{\Delta_i^n}\nu^{i-1}-\frac{\sigma^2\gamma}{2}\Delta_i^n\right)} - e^{\gamma\hat{d}_{i-1}^n}\right)^4\Bigg)\Bigg] \\ \tilde{\mathbb{E}}\left[Y_i^{\xi}\right]^4 &= \tilde{\mathbb{E}}\left[\left(1-q^u\right)\left(\sigma\sqrt{\Delta_i^n}\nu^{i-1}\right)^4 + q^u\left(\left(\hat{d}_{i-1}^n\right)'-\hat{d}_{i-1}^n+\sigma\sqrt{\Delta_i^n}\nu^{i-1}\right)^4\right] \end{aligned}$$

where $\left(\hat{d}_{i-1}^n\right)'|\sigma\left(\mathcal{H}_{i-2}^n\cup\sigma(\tau_{i-1}^n)\right) \sim^d \hat{d}_{i-1}^n|\sigma\left(\mathcal{H}_{i-2}^n\cup\sigma(\tau_{i-1}^n)\right)$ and $\left(\hat{d}_{i-1}^n\right)'$ is independent of $\hat{d}_{i-1}^n$ conditionally on $\sigma\left(\mathcal{H}_{i-2}^n\cup\sigma(\tau_{i-1}^n)\right)$. Therefore conditioning on $\left(\hat{d}_{i-1}^n\right)'$, Taylor expansion and elementary inequality $ab \leq \frac{a^2+b^2}{2}$ give

$$\begin{aligned}\tilde{\mathbb{E}}\left[Y_i^\eta\right]^4 &\leq K^\eta \left(\Delta_i^n\right)^2 + 4q^u \tilde{\mathbb{E}}\left[e^{\gamma\left(\hat{d}_{i-1}^n\right)'} - e^{\gamma \hat{d}_{i-1}^n}\right]^4 \\ \tilde{\mathbb{E}}\left[Y_i^\xi\right]^4 &\leq K^\xi \left(\Delta_i^n\right)^2 + 4q^u \tilde{\mathbb{E}}\left[\left(\hat{d}_{i-1}^n\right)' - \hat{d}_{i-1}^n\right]^4\end{aligned}$$

where $K^\eta = \sigma^4\gamma^4\left(21+3q^u\right) e^{4\gamma\left(D_0+\mu_\gamma T\right)+6\sigma^2\gamma^2 T}$ and $K^\xi = 3\left(1+q^u\right)\sigma^4$.

Moreover, since by (31) and (32) we have

$$\begin{aligned}\eta_{i-1}^n &= \left(1-q^u\right)\left[e^{\gamma\left(\hat{d}_{i-1}^n+\mu_\gamma\left(T-\tau_{i-1}^n\right)\right)} - \left(p_{i-2}^n\right)^\gamma\right] \\ \xi_{i-1}^n &= \left(1-q^u\right)\left[\left(\hat{d}_{i-1}^n+\mu\left(T-\tau_{i-1}^n\right)\right) - \tilde{\mathbb{E}}\left[\hat{D}_T^n|\mathcal{H}_{i-1}^n\right]\right]\end{aligned}$$

and since $\left(\hat{d}_{i-1}^n\right)' |\sigma\left(\mathcal{H}_{i-2}^n \cup \sigma(\tau_{i-1}^n)\right) \sim^d \hat{d}_{i-1}^n|\sigma\left(\mathcal{H}_{i-2}^n \cup \sigma(\tau_{i-1}^n)\right)$ and $\left(\hat{d}_{i-1}^n\right)'$ is independent of $\hat{d}_{i-1}^n$ conditionally on $\sigma\left(\mathcal{H}_{i-2}^n \cup \sigma(\tau_{i-1}^n)\right)$ we have

$$\begin{aligned}\tilde{\mathbb{E}}\left[Y_i^\eta\right]^4 &\leq K^\eta \left(\Delta_i^n\right)^2 + \frac{32q^u}{\left(1-q^u\right)^4}\left(\kappa_{n,i-1}^\eta\right)^4 \\ \tilde{\mathbb{E}}\left[Y_i^\xi\right]^4 &\leq K^\xi \left(\Delta_i^n\right)^2 + \frac{32q^u}{\left(1-q^u\right)^4}\left(\kappa_{n,i-1}^\xi\right)^4\end{aligned}$$

and therefore by (44) and (45) we have

$$\begin{aligned}\left(\kappa_{n,i}^\eta\right)^4 &\leq 4\left(1-q^u\right)^4 K^\eta \left(\Delta_i^n\right)^2 + 4\left(\left(q^u\right)^4+32q^u\right)\left(\kappa_{n,i-1}^\eta\right)^4 \\ \left(\kappa_{n,i}^\xi\right)^4 &\leq 4\left(1-q^u\right)^4 K^\xi \left(\Delta_i^n\right)^2 + 4\left(\left(q^u\right)^4+32q^u\right)\left(\kappa_{n,i-1}^\xi\right)^4\end{aligned}$$

If we take $C^\eta = 4\left(1-q^u\right)^4 K^\eta$, $C^\xi = 4\left(1-q^u\right)^4 K^\xi$ and $a = 4\left(\left(q^u\right)^4+32q^u\right)$ then the conclusion of the lemma follows (clearly, for q^u small enough $a<1$). ■

Lemma 7 *Consider $\sigma_{n,i}^\eta$ and $\sigma_{n,i}^\xi$ given by (35) and (36) then*

$$\begin{aligned}\max_{i\leq N_t^n} \sigma_{n,i}^\eta &\rightarrow 0 \text{ as } n\rightarrow\infty \\ \max_{i\leq N_t^n} \sigma_{n,i}^\xi &\rightarrow 0 \text{ as } n\rightarrow\infty\end{aligned}$$

a.s. for any $q^u > 0$ and $t \in [0, T]$.

Proof. Notice that by definition of $\tilde{N}$ we have that for any $\omega \in \Omega$ there exist $k(\omega)$ ($k \in \mathcal{F}_{\infty}^{\tilde{N}}$) such that

$$n\left[\tau_i^n(\omega) - \tau_{i-1}^n(\omega)\right] < 2\log(i)$$

for any $i > k(\omega)$.

Moreover, by definition of $\tilde{N}$ for any $\omega \in \Omega$ we have

$$\lim_{n\to\infty} \frac{N_{Tn}(\omega)}{n} = \lambda T$$

Fix $\omega \in \Omega$. Since by Lemma 5 we have that there exist constants $c_{\max}^{\eta}, c^{\xi}, b(q^u) \in \mathcal{F}_0$ independent of n and i with $b(q^u) < 1$ such that

$$\begin{aligned} \left(\sigma_{n,i}^{\eta}\right)^2 &\leq c_{\max}^{\eta} \sum_{j=1}^{i} b^{i-j}\left[\tau_j^n - \tau_{j-1}^n\right] \text{ for all } i \leq N_T^n \\ \left(\sigma_{n,i}^{\xi}\right)^2 &\leq c^{\xi} \sum_{j=1}^{i} b^{i-j}\left[\tau_j^n - \tau_{j-1}^n\right] \text{ for all } i \leq N_T^n \end{aligned}$$

from above it follows that

$$\begin{aligned} \max_{i\leq N_T^n(\omega)} \left(\sigma_{n,i}^{\eta}\right)^2(\omega) &\leq c_{\max}^{\eta}\left[\sum_{j=1}^{k(\omega)} \Delta_j^n(\omega) + \frac{2}{n}\log\left(N_T^n(\omega)\right)\frac{1}{1-b}\right] \\ \max_{i\leq N_T^n(\omega)} \left(\sigma_{n,i}^{\xi}\right)^2(\omega) &\leq c^{\xi}\left[\sum_{j=1}^{k(\omega)} \Delta_j^n(\omega) + \frac{2}{n}\log\left(N_T^n(\omega)\right)\frac{1}{1-b}\right] \end{aligned}$$

Since $\Delta_j^n(\omega) = \frac{\Delta_j(\omega)}{n}$ and $\lim_{x\to\infty} \frac{\log(x)}{x} = 0$ we have conclusion of the lemma. ■

Lemma 8 *Consider $\sigma_{n,i}^{\eta}$ and $\sigma_{n,i}^{\xi}$ given by (35) and (36) then there exists $q_1^u > 0$ such that for any $t \geq 0$, $q^u \in [0, q_1^u]$ the sets*

$$\begin{aligned} U^{\eta} &= \left\{\frac{\left(\eta_i^n\right)^2}{\left(\sigma_{n,i}^{\eta}\right)^2}, n \in \mathbb{N}, i \leq N_t^n\right\} \\ U^{\xi} &= \left\{\frac{\left(\xi_i^n\right)^2}{\left(\sigma_{n,i}^{\xi}\right)^2}, n \in \mathbb{N}, i \leq N_t^n\right\} \end{aligned}$$

are uniformly integrable with respect to $\tilde{\mathbb{P}}$.

Proof. ¿From lemmas 5 and 6 it follows that there exists $q_1^u > 0$ such that for any $q^u \in [0, q_1^u]$ there exist constants $c_{\min}^{\eta}$, c^{ξ}, $b(q^u)$, C^{η}, C^{ξ}, $a \in \mathcal{F}_0$ independent of n and i with $b(q^u) < 1$, $a(q^u) < 1$ and $c_{\min}^{\eta} > 0$ such that

$$\begin{aligned}
\frac{\tilde{\mathbb{E}}\left[(\eta_i^n)^4\right]}{\left(\sigma_{n,i}^{\eta}\right)^4} &\leq \frac{C^{\eta}\left[\sum_{j=1}^{i} a^{i-j}\left[\tau_j^n - \tau_{j-1}^n\right]^2\right]}{\left(c_{\min}^{\eta}\right)^2 \sum_{j=1}^{i} b^{2(i-j)}\left[\tau_j^n - \tau_{j-1}^n\right]^2} \\
\frac{\tilde{\mathbb{E}}\left[(\xi_i^n)^4\right]}{\left(\sigma_{n,i}^{\xi}\right)^4} &\leq \frac{C^{\xi}\left[\sum_{j=1}^{i} a^{i-j}\left[\tau_j^n - \tau_{j-1}^n\right]^2\right]}{\left(c^{\xi}\right)^2 \sum_{j=1}^{i} b^{2(i-j)}\left[\tau_j^n - \tau_{j-1}^n\right]^2}
\end{aligned}$$

therefore to prove uniform integrability of the sets U^{η} and U^{ξ} it is enough to prove a.s. uniform boundedness of

$$\frac{\sum_{j=1}^{i} a^{i-j}\left[\tau_j^n - \tau_{j-1}^n\right]^2}{\sum_{j=1}^{i} \tilde{b}^{i-j}\left[\tau_j^n - \tau_{j-1}^n\right]^2} = \frac{\sum_{j=1}^{i} a^{i-j}\left[\tilde{\tau}_j - \tilde{\tau}_{j-1}\right]^2}{\sum_{j=1}^{i} \tilde{b}^{i-j}\left[\tilde{\tau}_j - \tilde{\tau}_{j-1}\right]^2}$$

where $\tilde{b} = b^2 < 1$ and $a < 1$.

To do so, consider a random variable Y_i on $\mathcal{F}_{\infty}^{\tilde{N}}$ with the (random) distribution given by

$$\check{\mathbb{P}}\left(Y_i = \frac{i-j}{i}\right) = \frac{\left[\tilde{\tau}_j - \tilde{\tau}_{j-1}\right]^2}{\sum_{j=1}^{i}\left[\tilde{\tau}_j - \tilde{\tau}_{j-1}\right]^2}$$

Then for any $s \in [0,1]$ we will have that

$$F_i(s) = \check{\mathbb{P}}(Y_i \leq s) = \frac{\sum_{j=1}^{\lfloor si \rfloor}\left[\tilde{\tau}_j - \tilde{\tau}_{j-1}\right]^2}{\sum_{j=1}^{i}\left[\tilde{\tau}_j - \tilde{\tau}_{j-1}\right]^2}$$

and by definition of $\tilde{N}$ we have that

$$\lim_{i \to \infty} F_i(s) = s$$

for any $s \in [0,1]$. Therefore, (see Shiryaev (1996), Theorem III.1.2) we have that $Y_i \to_{i\to\infty}^{w} Y$ where Y has uniform distribution on $[0,1]$ and in particular we get $\lim_{i\to\infty} \check{\mathbb{E}}\left[e^{-cY_i}\right] = \frac{1-e^{-c}}{c}$ for any $c > 0$. Notice that

$$\frac{\sum_{j=1}^{i} a^{i-j}\left[\tilde{\tau}_j - \tilde{\tau}_{j-1}\right]^2}{\sum_{j=1}^{i} \tilde{b}^{i-j}\left[\tilde{\tau}_j - \tilde{\tau}_{j-1}\right]^2} = \frac{\check{\mathbb{E}}\left[e^{\log(a) i Y_i}\right]}{\check{\mathbb{E}}\left[e^{\log(\tilde{b}) i Y_i}\right]}$$

where $\log(a)$, $\log\left(\tilde{b}\right) < 0$ and therefore to prove the lemma we need uniform (in c) convergence of $\check{\mathbb{E}}\left[e^{-cY_i}\right]$.

To do so, consider $\mathcal{G} = \left\{g : \mathbb{R}_+ \to \mathbb{R}_+ : g(x) = \frac{e^{-cx}}{c},\ c \in [1,\infty)\right\}$ – a class of equicontinuous, uniformly bounded functions (easy to demonstrate). Then (see Shiryaev (1996), Theorem III.8.3) we have that

$$\lim_{i\to\infty} \sup_{c\in[1,\infty)} \left|\check{\mathbb{E}}\left[\frac{e^{-cY_i}}{c}\right] - \left(1-e^{-c}\right)\right| = 0$$

and therefore for any $\epsilon \in (0,1)$ there exists $k \in \mathcal{F}_\infty^{\tilde{N}}$ such that for any $i \geq k$ we will have

$$\frac{\sum_{j=1}^{i} a^{i-j}\left[\tilde{\tau}_j - \tilde{\tau}_{j-1}\right]^2}{\sum_{j=1}^{i} \tilde{b}^{i-j}\left[\tilde{\tau}_j - \tilde{\tau}_{j-1}\right]^2} \leq \frac{\log(a)}{\log\left(\tilde{b}\right)}(1+\epsilon)$$

Fix

$$C = \max\left(\max_{i\leq k}\left(\frac{\sum_{j=1}^{i} a^{i-j}\left[\tilde{\tau}_j - \tilde{\tau}_{j-1}\right]^2}{\sum_{j=1}^{i} \tilde{b}^{i-j}\left[\tilde{\tau}_j - \tilde{\tau}_{j-1}\right]^2}\right), \frac{\log(a)}{\log\left(\tilde{b}\right)}(1+\epsilon)\right) \in \mathcal{F}_\infty^{\tilde{N}}$$

Then by above we will have that

$$\begin{aligned}
\frac{\tilde{\mathbb{E}}\left[\left(\eta_i^{n,q^u}\right)^4\right]}{\left(\sigma_{n,i}^{\eta}\right)^4} &\leq C\frac{C^{\eta}}{\left(c_{\min}^{\eta}\right)^2} < \infty \\
\frac{\tilde{\mathbb{E}}\left[\left(\xi_i^{n,q^u}\right)^4\right]}{\left(\sigma_{n,i}^{\xi}\right)^4} &\leq C\frac{C^{\xi}}{\left(c^{\xi}\right)^2} < \infty
\end{aligned}$$

and therefore the lemma is proved. ■

Proof. of Lemma 3

We proved that

$$\begin{aligned}
\max_{i\leq N_t^n} \sigma_{n,i}^{\eta} &\to 0 \text{ as } n\to\infty \\
\max_{i\leq N_t^n} \sigma_{n,i}^{\xi} &\to 0 \text{ as } n\to\infty
\end{aligned}$$

and that there exists $q_1^u > 0$ such that for any $q^u \in [0, q_1^u]$ the sets $\left\{\frac{\left(\eta_i^{n,q^u}\right)^2}{\left(\sigma_{n,i}^{\eta}\right)^2},\ n\in\mathbb{N},\ i\leq N_t^n\right\}$, $\left\{\frac{\left(\xi_i^{n,q^u}\right)^2}{\left(\sigma_{n,i}^{\xi}\right)^2},\ n\in\mathbb{N},\ i\leq N_t^n\right\}$ are uniformly integrable in lemmas 7 and 8.

To prove that there exists $q_1^u > 0$ such that for any $q^u \in [0, q_1^u]$ the jumps of the processes $P^n(q^u)$ and $\hat{D}^n(q^u)$ uniformly converge in probability to zero consider stopping times

$$\begin{aligned}
\tau_\alpha^{n,\eta} &= \inf\{i \geq 0 : |\eta_i^n| \geq \alpha\} \\
\tau_\alpha^{n,\xi} &= \inf\{i \geq 0 : |\xi_i^n| \geq \alpha\}
\end{aligned}$$

in the filtration $\mathcal{F}_t^n$ then by lemma 6 and Chebychev inequality we have that there exists $q_1^u > 0$ such that for any $q^u \in [0, q_1^u]$

$$\begin{aligned}
\tilde{\mathbb{P}}\left(\max_{i \leq N_t^n} |\eta_i^n| > \alpha\right) &\leq \frac{\sum_{i \leq N_t^n}\left(\tilde{\mathbb{E}}\left[\eta_i^n\right]^4 \tilde{\mathbb{P}}\left(\tau_\alpha^{n,\eta} = i\right)\right)^{\frac{1}{2}}}{\alpha^2} \\
&\leq \frac{\left[\frac{N_t^n}{n^2} C^\eta \left[\sum_{j=1}^{N_t^n} a^{i-j}\left[\tilde{\tau}_j - \tilde{\tau}_{j-1}\right]^2\right]\right]^{\frac{1}{2}}}{\alpha^2} \\
\tilde{\mathbb{P}}\left(\max_{i \leq N_t^n} |\xi_i^n| > \alpha\right) &\leq \frac{\left[\frac{N_t^n}{n^2} C^\xi \left[\sum_{j=1}^{N_t^n} a^{i-j}\left[\tilde{\tau}_j - \tilde{\tau}_{j-1}\right]^2\right]\right]^{\frac{1}{2}}}{\alpha^2}
\end{aligned}$$

with $a < 1$. Notice that by definition of $\tilde{N}$ we have that for any $\omega \in \Omega$ there exists $k_1(\omega)$ ($k \in \mathcal{F}_\infty^{\tilde{N}}$) such that

$$\left[\tilde{\tau}_i(\omega) - \tilde{\tau}_{i-1}(\omega)\right] < 2\log(i)$$

for any $i > k_1(\omega)$.

Moreover, by definition of $\tilde{N}$ for any $\omega \in \Omega$ we have

$$\lim_{n\to\infty} \frac{N_t^n(\omega)}{n} = \lim_{n\to\infty} \frac{\tilde{N}_{tn}(\omega)}{n} = \lambda t$$

i.e. for any $\omega \in \Omega$ and any $\epsilon > 0$ there exists $k_2(\omega)$ ($k \in \mathcal{F}_\infty^{\tilde{N}}$) such that $\frac{N_t^n(\omega)}{n} \leq \lambda t + \epsilon$ for any $n \geq k_2$. Therefore we have ($k \geq k_1$, $n \geq k_2$)

$$\begin{aligned}
\tilde{\mathbb{P}}\left(\max_{i \leq N_t^n} |\eta_i^n| > \alpha\right) &\leq \left[\frac{(\lambda t + \epsilon)}{a} C^\eta \left[\frac{\sum_{j=1}^{k}\left[\tilde{\tau}_j - \tilde{\tau}_{j-1}\right]^2}{N_t^n} + 2\frac{\log(N_t^n)}{N_t^n}\frac{1}{1-a}\right]\right]^{\frac{1}{2}} \\
\tilde{\mathbb{P}}\left(\max_{i \leq N_t^n} |\xi_i^n| > \alpha\right) &\leq \left[\frac{(\lambda t + \epsilon)}{a} C^\xi \left[\frac{\sum_{j=1}^{k}\left[\tilde{\tau}_j - \tilde{\tau}_{j-1}\right]^2}{N_t^n} + 2\frac{\log(N_t^n)}{N_t^n}\frac{1}{1-a}\right]\right]^{\frac{1}{2}}
\end{aligned}$$

Since $\max_{i \le k} \left[\tilde{\tau}_j - \tilde{\tau}_{j-1}\right]^2 < \infty$, $\lim_{n\to\infty} N_t^n = \infty$ and $\lim_{x\to\infty} \frac{\log(x)}{x} = 0$ we have

$$\begin{aligned} \lim_n \tilde{\mathbb{P}}\left(\max_{i \le N_t^n} |\eta_i^n| > \alpha\right) &= 0 \\ \lim_n \tilde{\mathbb{P}}\left(\max_{i \le N_t^n} |\xi_i^n| > \alpha\right) &= 0 \end{aligned}$$

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