SIAM J. CONTROL OPTIM.
Vol. 0, No. 0, pp. 000–000

OPTIMAL STOCHASTIC CONTROL AND CARBON PRICE FORMATION*

RENÉ CARMONA†, MAX FEHR‡, AND JURI HINZ§

Abstract. To meet the targets of the Kyoto Protocol, the European Union established the European Emission Trading Scheme, a mandatory market for carbon emission allowances. This regulatory framework has introduced a market for emission allowances and created a variety of emission-related financial instruments. In this work, we show that the economic mechanism of carbon allowance price formation can be formulated in the framework of competitive stochastic equilibrium models, and we show that its solution reduces to an optimal stochastic control problem. Using this mathematical setup, we identify the main allowance price drivers and show how stochastic control can be used to treat quantitative problems in carbon price risk management.

Key words. stochastic control, commodity options, environmental risk, carbon trading

AMS subject classifications. 93E20, 91B60, 91B62, 91B70, 91B76

DOI. 10.1137/080736910

1. Introduction.

The world's changing climate and the pressing need for measures to curtail the man-made effects on ecosystems continue to challenge policy makers. One of the issues is to establish regulations which would provide a certain amount of flexibility, enabling agents to apply commitments best suited to their circumstances. Presently, there are several international, national, and corporate policy frameworks aimed at pollution reduction in a flexible, cost-effective manner by the introduction of marketable emission credits. In this paper, the focus is on the European Union Emission Trading Scheme (EU ETS), designed by the European Union as an instrument to meet targets under the Kyoto Protocol.

In 1997 governments adopted the Kyoto Protocol, which broke new ground with its mandatory requirements to reduce emissions of greenhouse gases (GHG). Each Annex B member, who has ratified the Kyoto Protocol, is assigned a number of tradable credits AAU (assigned amount units), each of which represents an allowance to emit one metric ton of carbon dioxide equivalent. Moreover, there are a number of regional and national emission reduction projects, where agents trade diverse emission certificates.

The EU ETS, launched by the Directive 2003/87/EC of the European Parliament and the Council of October 13, 2003, is a remarkable example of this project. This scheme is intended to ensure the reduction of carbon dioxide emissions from large industrial sources within the European Union in order to contribute to the EU's targets under the Kyoto Protocol.

*Received by the editors October 1, 2008; accepted for publication (in revised form) March 27, 2009; published electronically DATE.
http://www.siam.org/journals/sicon/x-x/73691.html

†Department of Operations Research and Financial Engineering, Princeton University, Princeton, NJ 08544 (rcarmona@princeton.edu), and Bendheim Center for Finance and the Applied and Computational Mathematics Program.

‡Institute for Operations Research, ETH Zurich, CH-8092 Zurich, Switzerland (maxfehr@ifor.math.ethz.ch).

§Department of Mathematics, National University of Singapore, 2 Science Drive, 117543 Singapore (mathj@nus.edu.sg). The work of this author was partially supported by WBS R-703-000-020-720 / C703000 of the Risk Management Institute at the National University of Singapore.

The EU ETS imposes mandatory participation of nearly 10,000 [1] installations. These are power plants and industrial users responsible for approximately 44% of the entire EU carbon emission in 2003 [1]. For these installations, carbon emission allowances (EUAs) are allocated yearly by the responsible governments, according the National Allocation Plans (NAPs). Installations must cover their emissions by allowances every single year. The precise regulation is as follows: There are two compliance periods 2005–2007 and 2008–2012. Within each period, allowances are valid regardless of the year in which they are allocated. Compliance is to be met yearly. Each operator that does not surrender sufficient allowances by the 30th of April of each year to cover emissions for the preceding year is liable for the payment of a penalty for each ton of excess emissions. The size of the potential payment is considerable: Within the first period, the penalty was 40 Euros, whereas for the second period, agents must pay 100 Euros per excess ton of carbon dioxide. Moreover, the penalty payment does not release offenders from the obligation of surrendering the missing allowances in the following calendar year.

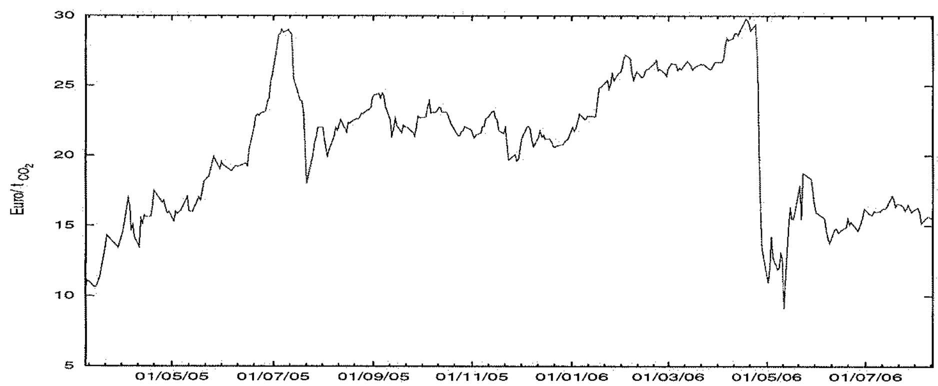

FIG. 1. *EUA spot price, listed at the European Energy Exchange EEX. The price drop occurred right after carbon emission data became public, showing that the overall market position was* long.

Several exchanges in Europe are now committed to the trading of EUAs. The products listed there are spot (see Figure 1) and forward contracts with physical delivery. The stochastic nature of this price process is obvious. Consequently, conventional deterministic models are hardly applicable to model allowance price formation in the EU ETS. In line with this insight [1] observe that the carbon market is to a large extent trading on changes in stochastic fundamentals, such as relative coal/gas prices and temperature/precipitation.

The present work proposes a dynamic stochastic equilibrium model that aims to explain allowance price formation with respect to these fundamentals. The fundamental contribution of the paper is in developing a model for equilibrium price formation of emission allowances when the abatement cost is stochastic. We show that the equilibrium price process is related to the solution of an appropriate optimal control problem.

We now review the recent literature on *equilibrium modeling* of the emissions markets, and we introduce the main mathematical assumptions of our approach.

To the best of our knowledge, the first equilibrium model for environmental trading schemes was proposed by [6]. In his seminal paper, Montgomery considers a static perfect market of several agents obliged to cover their entire emissions through allowances. To this end, the agents are allocated a certain initial amount of permits, reflecting the overall emission target. Agents face different reduction costs and, based on allowance prices, optimize their emissions and trade allowances. It turns out that optimal emission rates equate marginal control costs and allowance prices. Montgomery [6] proves the existence of a market equilibrium, where the emission target is reached at minimal costs.

Numerous subsequent studies have addressed environmental trading schemes within a static framework of nonbankable permits. This line of research is not considered in the present work as our focus is on the dynamic aspects of emission trading. Publications The works [14], [3], and [10] provide a foundation for dynamic permit trading in a deterministic environment. They derive market equilibria, similar to [6], but consider different multiperiod settings where banking and borrowing of allowances are allowed. The terminology *banking* stands for a mechanism where credits remain valid for compliance in all future periods, while *borrowing* means that credits from future periods can be used for compliance. The work [14] considers one dynamic compliance period with discrete time steps, where the abatement costs curve is static and reflects only short-term abatement. All allowances are allocated at the beginning of the compliance period, while the total emissions have to be covered by allowances only at the end of the period. It is concluded that before the permits are used up, their price must rise at the rate of interest, while also in this dynamic framework, emission levels are chosen such that marginal control costs and allowance price coincide. In [3] a permit system with banking is examined, where permits are allocated to firms in each of T compliance periods. Further, [10] extends the work of [14] and [3] by providing a more general treatment of permit trading in continuous time through the use of optimal-control theory. Considering both borrowing and banking, with restrictions on borrowing as a special case, this formulation allows the extension of the results from [14] and [3]. Further generalization is given in [13], where also explicit constraints on permit trading are examined.

In the area of stochastic modeling, to the best of our knowledge, only [5] takes into account the uncertainty aspects in an equilibrium with banking. In this two-step stochastic equilibrium model, it is assumed that both emissions and abatement costs are stochastic, although obeying a specific structure. The study [5] discusses how uncertainty, technological progress, and different types of market participants affect allowance prices. Beyond equilibrium modeling, the issues of uncertainty are addressed in several papers. We single out [11], which discusses the optimal decisions of the representative agent under uncertainty. The approach of [11] is based on [10] and presents a model where both allowance demand and abatement costs are stochastic although in a very restrictive way. However, equilibrium formation is not considered. Finally, we mention the recent works [12] and [16], which deal with the allowance price formation within the EU ETS. Here, rather than modeling equilibrium allowance prices, the EUA's evolution is introduced as the marginal abatement costs when the market follows an overall minimal-cost abatement policy. In [2] valuation of options on emission allowances is addressed, and [7] treats econometric aspects of emission allowance prices.

Our contribution to equilibrium modeling of emission markets is twofold. On the one hand, we address the need for a model which takes into account the particularities of the EU ETS. Indeed, due to the existence of a penalty payment, the fundamental connection with electricity production, the stochastic nature of demand for power and fuel prices, the abatement costs, etc., there is no obvious way to extend the existing models reviewed above to fit the EU ETS. On the other hand, the theoretical value of our model will go beyond EU ETS since it encompasses most of the stylized facts in a generic cap-and-trade system, and yields a straightforward extension of [14] (see [15, Chapter 2]) to a stochastic framework. In particular, we expect that our work will be relevant for the cap-and-trade schemes to be implemented in Japan, Canada, Australia, and the U.S.

2. Emissions markets modeling.

2.1. Allowance prices and abatement costs. In a cap-and-trade system, the allowance price is determined by the existing abatement strategies, their flexibility, and costs. Moreover, it is important to distinguish abatement measures according to the time they require to return a profit. In this regard, one can conceive a continuum of measures ranging from short-term measures (no initial investment, savings returned within days) to long-term measures (high and irreversible investments, savings returned over decades). Examples of long-term measures are optimization/substitution of highly polluting production units, installation of scrubbers, investment in Clean Development Mechanism (CDM), and Joint Implementation (JI) projects. On the other end of the time scale, typical short-term abatement measures yield emission savings by replacing fuels or skipping/rescheduling the production.

Because of their differences in time scales, we assume in this work that decisions of middle- and long-term investments are made in a different manner than those of short-term abatements. Long-term investments are on a much larger time scale than a single compliance period. For instance, according to [8] the time scale for new electricity generating capacity is 20–30 years. Thus, the influence of such projects can be modeled as an endogenous stochastic process which stands for the residual demand on short-term reduction, when the uncertainty about the amount of realized long- and middle-term measures and their allowance supply is taken into account. Our prime focus is on short-term measures, referred to as abatement measures in the following. The study of Dresdner Kleinwort Wasserman Research (see [4, p. 57]) finds that in the case of the EU ETS, the main short-term abatement potential stems from the electricity sector, where the production can be switched from hard coal to gas. Clearly marginal abatement costs need to be modeled as stochastic, but before turning to the modeling of this stochasticity, we discuss marginal abatement cost curves for fixed fuel prices.

In the existing literature, most of the research is done under the assumption that marginal abatement costs increase with abatement volumes, while for mathematical reasons they restrict marginal abatement curves to be continuous. The recent study [4] reveals, however, that for fixed fuel prices, marginal abatement curves are stepwise constant and increasing. Each step is associated with a certain abatement measure, its reduction capacity, and marginal costs. Notice that the marginal abatement cost curves in [4] could look continuous, as in most plots the steps are linearly interpolated. In this work, we follow the same strategy in the sense that we suppose that a finite number of abatement measures (here abatement measures are on the level of single plants) are available and assume that there is a limit on the maximally possible emission reduction of each measure per time unit. Furthermore, at each time step, marginal reduction costs of each abatement measure are supposed to be constant, though dependent on instantaneous fuel prices. Consequently, the marginal abatement curves in our model are nondecreasing and piecewise constant, with a finite number of steps. Note, however, that since in our model marginal abatement costs are stochastic, abatement curves and merit order change randomly over time.

To provide the reader with further insight, we illustrate the price of fuel switching from coal to gas in electricity generation. This illustration is of great practical importance since it represents one of the major short-term abatement measures within the EU ETS.

Consider an agent i switching from a hard coal plant to a cleaner CCGT plant. CCGT is an acronym for combined cycle gas turbine, a relatively novel technology where the waste heat from the gas turbine is used to run a steam turbine in order to enhance the efficiency of electricity generation. Using CCGT, the agent's technology possesses specific emissions for gas,

$$e_g^i = 0.202 \frac{\mathrm{t_{CO_2}}}{\mathrm{MWh_{therm}}} \cdot \frac{1}{0.52} \frac{\mathrm{MWh_{therm}}}{\mathrm{MWh_{el}}} = 0.388 \frac{\mathrm{t_{CO_2}}}{\mathrm{MWh_{el}}}, \tag{1}$$

and coal,

$$e_c^i = 0.341 \frac{\mathrm{t_{CO_2}}}{\mathrm{MWh_{therm}}} \cdot \frac{1}{0.38} \frac{\mathrm{MWh_{therm}}}{\mathrm{MWh_{el}}} = 0.897 \frac{\mathrm{t_{CO_2}}}{\mathrm{MWh_{el}}}, \tag{2}$$

measured in tons of emitted carbon for the generation of one MWh of electricity. Here, $\mathrm{t_{CO_2}}$ and $\mathrm{MWh_{therm}}$, $\mathrm{MWh_{el}}$ denote a ton of carbon dioxide and a megawatt of thermic and electrical power, respectively. The carbon dioxide emission factors are default values provided by the Intergovernmental Panel on Climate Change (IPCC). The switch of production technology at time t yields per MWh of electricity a reduction of $e_c^i - e_g^i = 0.509$ tons of carbon dioxide. At the same time, this fuel switch causes costs of $h_g^i G_t^i - h_c^i C_t^i$ Euro per MWh, where G_t^i, C_t^i are gas and coal spot prices for the agent i at time t (expressed in Euros per $\mathrm{MWh_{therm}}$ and in Euros per ton, respectively). The coefficients

$$h_g^i = \frac{1}{0.52} \frac{\mathrm{MWh_{therm}}}{\mathrm{MWh_{el}}} = 1.92 \frac{\mathrm{MWh_{therm}}}{\mathrm{MWh_{el}}},$$
$$h_c^i = \frac{1}{6.961} \frac{\mathrm{t_{coal}}}{\mathrm{MWh_{therm}}} \frac{1}{0.38} \frac{\mathrm{MWh_{therm}}}{\mathrm{MWh_{el}}} = 0.378 \frac{\mathrm{t_{coal}}}{\mathrm{MWh_{el}}}$$

are specific heat rates expressing how much fuel is consumed for the generation of one MWh of electricity. Here we have assumed that the amount of coal is measured in tons, whereas the amount of gas is expressed in MWh hours of thermal power since gas prices we use are given in Euro/$\mathrm{MWh_{therm}}$. The calculation of h_g^i is based on the reference value of 6000kcal/kg reported in McCloskey's NWE Steam Coal Marker. With these quantities, we have the fuel switching price

$$E_t^i = \frac{h_g^i G_t^i - h_c^i C_t^i}{e_g^i - e_c^i} \qquad \text{for all } t = 0, \ldots, T-1 \tag{3}$$

measured in Euros per ton of carbon dioxide. Based on a given time series for coal and gas spot prices, formula (3) yields the corresponding fuel switching price process appearing in Figure 1.

Remark. In the case of Figure 1, gas prices are too high to trigger a notable number of fuel switches from CCGT technology. For the actual abatement from other measures, we refer the interested reader to the comprehensive study [4].

Remark. In the commodity business, companies exposed to risks from fluctuations in the prices of commodities used as production inputs, hedge themselves with an appropriate portfolio of futures contracts. Hence correlations between the prices of the diverse commodities become essential. In particular, the European energy business is concerned about the correlation between EUA and fuel prices. However, as Figure 2 shows, their interdependence is not obvious. A study based on our model could shed light on this important problem.

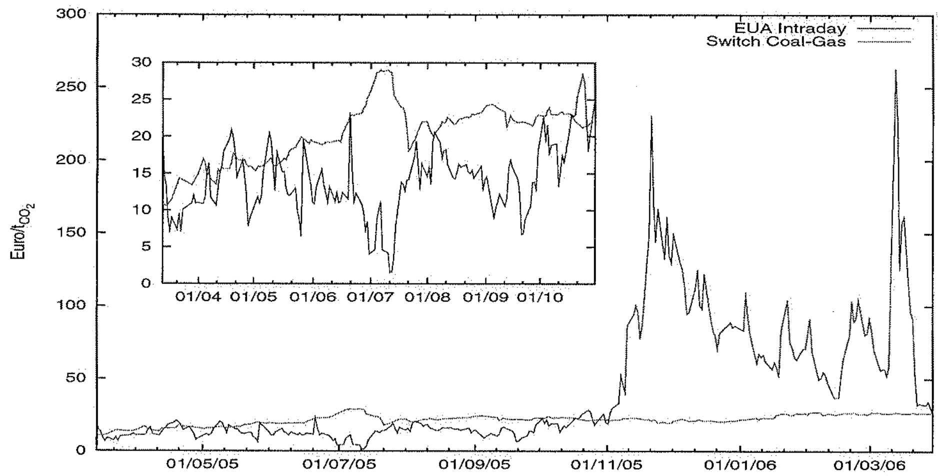

FIG. 2. *The price for EUA versus fuel switching price calculated from gas/coal spot prices.*

2.2. Mathematical model. In this paper, we consider a stylized cap-and-trade scheme where market participants trade allowances and apply abatement policies to comply with the emissions targets. We suppose that each agent acts rationally: to increase her own wealth, she applies abatement and trading based on past and current information. Our goal is to identify and study an equilibrium elucidating the EUA price formation mechanism. However, in order to deal with a tractable model we have to make simplifications which we articulate below.

Since trading and production decisions occur at discrete times, we choose to work in a discrete-time setting. However, notice that the quantitative study in section 4 is based on parameter estimation for discretely sampled diffusion processes. A further simplification is to suppose that the penalty consists only of a financial penalty which is paid once at the end of the compliance period. Current regulation stipulates annual compliance, and allowances are valid during the entire period, even though companies are issued allowances on February 28, two months prior to the date when they are required to cover their emissions from the previous year. Consequently for compliance in one given year, companies may surrender allowances which are issued for the following year, a form of borrowing. Considering that the yearly allocations are sufficiently high, the penalty must only be paid at the end of the compliance period, if ever.

Once a frictionless transfer of allowances over time is agreed upon, we can assume that EUA is a purely financial asset, and the difference between spot and forward prices is captured by interest rate effects only. That is, it suffices to consider either the EUA spot or the EUA forward price. To avoid discounting, we decided to model the forward price for EUA with maturity at the final compliance date. Remember that a forward is an agreement to exchange an asset at a future date, at the price which is fixed when the parties enter the contract. Because of this, all incentives are described by the maximization of the individual terminal wealths, expected from the perspective of the historical *objective measure*. This is a rather strong assumption, and the advantages of the ensuing simplification are far-reaching: due to social optimality (which applies only to the nonrisk-averse case), the model boils down to a low-dimensional optimal control problem and is amenable to numerical treatment.

In some sense, we obtain a similarity to auction theory, where the celebrated revenue equivalence theorem plays an essential role and enjoys applications, although it applies only to nonrisk-averse agents.

We start from the realization that carbon price development reflects the private economic interests of installations, concerned with emission regulations in a cap-and-trade framework. Thus, the main aspect in our modeling is to face the individual strategy optimization of single market participants exposed to carbon price risk. We consider personal incentives of stylized agents who possess the flexibility of short-term emission reduction, which is exercised whenever emission allowance prices indicate that this is reasonable. Here, the decisions may range from a simple replacement of the output by a supplementary material to a partial shutdown of production. For instance, in the cement industry, customers are offered cinders remaining from the iron production, whereas in the energy business, electricity producers occasionally cancel their long-term supply contracts.

As already mentioned, the study [4] indicates that the *major carbon emission reduction resource* is *fuel switch* (in the simplest case, from coal to gas) in heating and electricity generation. On this account, commodity price models (in particular, fuel price models) form an intrinsic part of carbon price description. Consequently, we attempt to find how the emission allowances price evolution relates to the fuel price development.

We consider $N \in \mathbb{N}$ market participants producing electricity from fossil fuel and trading carbon allowances at discrete times $t \in \{0, 1, \ldots, T\} \subset \mathbb{N}$. The entire time horizon corresponds to one compliance period; that is, at maturity T, all agents have to cover their carbon emissions by allowances or pay penalties. We agree to describe all prices and strategies by adapted stochastic processes on a filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t=0}^T)$. So the decisions of all market participants are based on the information flow given by the filtration $(\mathcal{F}_t)_{t=0}^T$. In particular, each agent observes the allowance price and the short-term abatement prices of all market participants. From a practical perspective, it is not obvious whether modeling of private information is crucial for the EU ETS, since the collection and distribution of news, reports, and analyses of the carbon market is becoming a business line for many consultancies. Another model simplification is that each agent $i = 1, \ldots, N$ observes her *own short-term abatement* price $(E_t^i)_{t=0}^{T-1}$ which is supposed to follow an adapted stochastic process (note that the fuel switch price is not necessarily positive).

We write $A = (A_t)_{t=0}^T$ for the forward price with delivery date T of one carbon allowance certificate, and we assume that it is a positive-valued stochastic process. Note that due to the definition of a forward price, A_T equals the spot price for carbon allowances at the final time T. A forward trading strategy for agent i is a process $\theta^i = (\theta_t^i)_{t=0}^{T-1}$, where θ_t^i for $t = 0, \ldots, T-1$ stands for the number of forward contracts held by agent $i = 1, \ldots, N$ at time t. Note that holding position θ_t^i from t to $t+1$ yields

$$\theta_t^i(A_T - A_t) - \theta_t^i(A_T - A_{t+1}) = \theta_t^i(A_{t+1} - A_t)$$

at T. That is, at compliance date T, the wealth of strategy $(\theta_t^i)_{t=0}^{T-1}$ is given by

$$\sum_{t=0}^{T-1} \theta_t^i(A_{t+1} - A_t). \tag{4}$$

Moreover, each agent/producer i must face the cost associated with the difference between emitted carbon and allowances allocated at the beginning of the period.

We model this quantity by an $\mathcal{F}_T$-measurable random variable Γ^i. We allow for both negative and positive realizations of Γ^i, which occur if the credit exceeds or drops below the realized emission. In order to comply at the end of the period, each agent adjusts the number of credits depending on the actually realized allowance demand. This action is described by the number of allowances θ^i_T which agent i purchases at T giving the final emission balance $\Gamma^i - \theta^i_T$. In the case $\Gamma^i - \theta^i_T \geq 0$ emissions top allowances, a penalty of $\pi \in (0, \infty)$ Euros is to be paid for each ton which is not covered. Thus, the final profit from trading allowances equals

$$\sum_{t=0}^{T-1} \theta^i_t (A_{t+1} - A_t) - \theta^i_T A_T - \pi(\Gamma^i - \theta^i_T)^+. \tag{5}$$

According to our previous discussion, we should include the effect of long-term abatement measures by adjusting the emission expectation with the estimated reduction from the proposed long-term measures, but we shall refrain from doing that in the present contribution.

We suppose that each producer i possesses a technology which at any time $t = 0, \dots, T-1$ allows a reduction ξ^i_t of at most $\lambda^i \in [0, \infty)$ tons of carbon emitted within the period $[t, t+1]$ by fuel switching. It is straightforward to extend this work to several abatement measures per agent. Then each agent would face a stepwise constant increasing marginal abatement cost curve in each state and at each time point. At time t, the decision is based on the public information $\mathcal{F}_t$. The fuel switching policy $\xi^i = (\xi^i_t)_{t=0}^{T-1}$ yields expenses which are modeled by a cash payment of the amount

$$\sum_{t=0}^{T-1} \xi^i_t \frac{E^i_t}{p_t(T)} = \sum_{t=0}^{T-1} \xi^i_t \mathcal{E}^i_t \tag{6}$$

at maturity T. Discounting is taken care of by relating the fuel switch spot price E_t to the price $p_t(T)$ of the zero bond maturing at T by

$$\mathcal{E}^i_t := E^i_t / p_t(T) \qquad t = 0, \dots, T.$$

The fuel switch helps to meet compliance, since instead of the allowance demand Γ^i merely $\Gamma^i - \sum_{t=0}^{T-1} \xi^i_t$ tons of carbon dioxide are to be covered at the end of the period. Thus, we correct Γ^i in (6) by $\Gamma^i - \sum_{t=0}^{T-1} \xi^i_t$, which, combined with (5), expresses the profit/loss of the producer i by

$$I^{A,i}(\theta^i, \xi^i) = \sum_{t=0}^{T-1} \theta^i_t (A_{t+1} - A_t) - \theta^i_T A_T - \pi \left(\Gamma^i - \sum_{t=0}^{T-1} \xi^i_t - \theta^i_T \right)^+ - \sum_{t=0}^{T-1} \xi^i_t \mathcal{E}^i_t. \tag{7}$$

3. Mathematical analysis of the model. In order to guarantee the existence of expected values of (7) and later on (13), we suppose that

$$\Gamma^i, \mathcal{E}^i_t \text{ are integrable for } i = 1, \dots, N,\ t = 0, \dots, T-1. \tag{8}$$

We shall use the Banach spaces $L_1 = L_1(\mathcal{F}_T)$ and $L_\infty = L_\infty(\mathcal{F}_T)$ of $\mathbb{P}$-equivalent classes of integrable and essentially bounded $\mathcal{F}_T$-measurable random variables, respectively. Further, we introduce the following spaces of adapted processes:

$$\begin{aligned}
\mathcal{L}_1 &:= \{(\Xi_t)_{t=0}^{T-1} : \Xi_t \in L_1(\mathcal{F}_t), t = 0, \dots, T-1\}, \\
\mathcal{L}_\infty &:= \{(\xi_t)_{t=0}^{T-1} : \xi_t \in L_\infty(\mathcal{F}_t), t = 0, \dots, T-1\}, \\
\mathcal{U}^i &:= \{(\xi^i_t)_{t=0}^{T-1} : \text{adapted } [0, \lambda^i]\text{-valued process}\}, \\
\mathcal{U} &:= \times_{i=1}^N \mathcal{U}^i.
\end{aligned}$$

Given that each market participant maximizes her own profit by trading allowances and applying abatement measures, given an allowance price process $A = (A_t)_{t=0}^{T}$, we formulate the individual optimization problem for agent i as

$$\sup_{(\theta^i,\xi^i)\in(\mathcal{L}_1\times L_1)\times\mathcal{U}^i} \mathbb{E}\{I^{A,i}(\theta^i,\xi^i)\}. \tag{9}$$

With these notations, we define the equilibrium as follows.

DEFINITION 1. *Given a fuel switch price process* $(\mathcal{E}_t^i)_{t=0}^{T-1} \in \mathcal{L}_1^N$ *for each agent* $i = 1, \dots, N$*, an adapted process* $A^* = (A_t^*)_{t=0}^{T}$ *is said to be an equilibrium carbon price process if there exists for each agent* $i = 1, \dots, N$ *a trading and an abatement strategy* $(\theta^{i*}, \xi^{i*}) \in (\mathcal{L}_1 \times L_1) \times \mathcal{U}^i$ *for which financial positions are in zero net supply,*

$$\sum_{i=1}^{N} \theta_t^{*i} = 0 \quad \text{at any time } t = 0, \dots, T, \tag{10}$$

and individual agents cannot do better in the sense that

$$\mathbb{E}\{I^{A^*,i}(\theta^{i*},\xi^{i*})\} \ge \mathbb{E}\{I^{A^*,i}(\theta^{i},\xi^{i})\} \quad \text{for all } (\theta^i,\xi^i) \in (\mathcal{L}_1 \times L_1) \times \mathcal{U}^i \tag{11}$$

for $i = 1, \dots, N$.

Remark. It should be emphasized that zero net supply (10) is stated at $t = 0, \dots, T-1$ for different reasons than at $t = T$. Indeed, for $t = 0, \dots, T-1$, $(\theta_t^{*i})_{i=1}^{N}$ are forward positions, whereas at maturity, $(\theta_T^{*i})_{i=1}^{N}$ stand for the *change* in the initial physical allocation of the agents $i = 1, \dots, N$. Let us elaborate on the economic meaning of this remark. In the above definition, we use a standard equilibrium notion, based on the intuition that at low prices most agents will have a tendency to purchase allowances, while at high prices they will most likely produce and sell the goods. Hence, the equilibrium price is characterized by vanishing excess demand. Note that according to this concept, (θ^{i*}, ξ^{i*}) is indeed a maximizer of $(\theta^i, \xi^i) \mapsto \mathbb{E}\{I^{A^*,i}(\theta^{i*},\xi^{i*})\}$ on the entire set $(\mathcal{L}_1 \times L_1) \times \mathcal{U}^i$ without any restrictions from zero net supply.

The most important property of the above notion of equilibrium is that it enjoys the property of *social optimality*. As we are about to show, an equilibrium in the above sense automatically results in the solution of a certain global optimization problem, where the total pollution is reduced at minimal overall cost. Beyond the economic interpretations of social optimality, the importance of the global optimization is that its solutions help to show the existence of an equilibrium, and to calculate the corresponding carbon prices.

Suppose we are given the fuel switch prices $(\mathcal{E}_t^i)_{t=0}^{T-1} \in \mathcal{L}_1$ for $i = 1, \dots, N$. For any given switching policy $\xi = (\xi_t^1, \dots, \xi_t^N)_{t=0}^{T-1} \in \mathcal{U}$ of the agents $i = 1, \dots, N$, we denote the final overall switching costs by

$$F(\xi) = \sum_{i=1}^{N} \sum_{t=0}^{T-1} \xi_t^i \mathcal{E}_t^i.$$

Further, we write

$$\Pi(\xi) = \sum_{i=1}^{N} \sum_{t=0}^{T-1} \xi_t^i \tag{12}$$

for the total savings in emissions when the abatement policy ξ is used, and we denote by

$$\Gamma = \sum_{i=1}^{N} \Gamma^i$$

the overall allowance demand. Finally, we define the total costs from fuel switching and penalty payments as

$$G(\xi) = -F(\xi) - \pi(\Gamma - \Pi(\xi))^+, \quad \xi \in \mathcal{U}, \tag{13}$$

and we introduce the global optimization problem

$$\xi^* = \arg \sup_{\xi \in \mathcal{U}} \mathbb{E}\{G(\xi)\}, \tag{14}$$

where a switching policy $\xi^* \in \mathcal{U}$ for all agents is to be determined, which minimizes the social costs of noncompliance. We now prove that the existence of such an optimal policy ξ^* is ensured by standard functional analytic arguments.

PROPOSITION 1. *With the above notation and under the above assumptions, there exists a solution $\xi^* \in \mathcal{U}$ to the global optimal control problem* (14).

Remark. In economic terms, this proposition states that the financial trading of allowances decouples from physical abatement. Indeed, the proof actually shows that there is no contribution from financial trading to individual wealth because the equilibrium allowance price process is a martingale. In other words, the individual wealth is effectively increased only by physical abatement and by the adjustment of the final physical position.

Proof. First, note that $\mathcal{L}_1^N$ equipped with the norm

$$\|\Xi\|_1 = \sum_{t=0}^{T-1} \sum_{i=1}^{N} \mathbb{E}\{|\Xi_t^i|\}$$

is a Banach space with dual $\mathcal{L}_\infty^N$, the canonical bilinear pairing being

$$\langle \Xi, \xi \rangle := \sum_{t=0}^{T-1} \sum_{i=1}^{N} \mathbb{E}\{\Xi_t^i \xi_t^i\}, \quad \Xi \in \mathcal{L}_1^N, \xi \in \mathcal{L}_\infty^N.$$

For the weak topology $\sigma(\mathcal{L}_\infty^N, \mathcal{L}_1^N)$ on $\mathcal{L}_\infty^N$ (see [9]), a basis of neighborhoods of a point $\xi \in \mathcal{L}_\infty^N$ is given by the finite intersections of sets

$$B_\xi(\Xi, \delta) := \{\xi' \in \mathcal{L}_\infty^N : |\langle \Xi, \xi' - \xi \rangle| < \delta\}, \qquad \Xi \in \mathcal{L}_1^N,\ \delta > 0. \tag{15}$$

In other words, $\sigma(\mathcal{L}_\infty^N, \mathcal{L}_1^N)$ is the weakest topology for which all the linear functionals

$$\mathcal{L}_\infty^N \to \mathbb{R}, \quad \xi \mapsto \langle \Xi, \xi \rangle, \quad \Xi \in \mathcal{L}_1^N \tag{16}$$

are continuous. A function $f : \mathcal{L}_\infty^N \to \mathbb{R}$ is lower semicontinuous at ξ if for each $\varepsilon > 0$ there exists a neighborhood B_ξ of ξ such that $f(\xi') > f(\xi) - \varepsilon$ for all $\xi' \in B_\xi$. Such a function is called lower semicontinuous if it is lower semicontinuous at each point. We prove existence of a minimizer ξ^* of $\xi \mapsto \mathbb{E}\{-G(\xi)\}$ on $\mathcal{U}$ by proving that $\mathcal{U}$ is compact and that $\xi \mapsto \mathbb{E}\{-G(\xi)\}$ is lower semicontinuous with respect to $\sigma(\mathcal{L}_\infty^N, \mathcal{L}_1^N)$, and the fact that lower semicontinuous functions attain their minima on compact sets.

Given $\xi \in \mathcal{L}_\infty^N$,

$$\mathbb{E}\{-G(\xi)\} = \sum_{t=0}^{T-1} \sum_{i=1}^{N} \mathbb{E}\{\mathcal{E}_t^i \xi_t^i\} + \pi \mathbb{E}\{(\Gamma - \Pi(\xi))^+\},$$

and since the first term is a continuous linear functional of type (16) (evaluated at ξ), it suffices to prove lower semicontinuity of

$$\xi \mapsto \mathbb{E}\{(\Gamma - \Pi(\xi))^+\}.$$

In order to do so, we fix the point ξ. For any $\xi' \in \mathcal{L}_\infty^N$ we have

$$\begin{aligned}(\Gamma - \Pi(\xi'))^+ &\geq (\Gamma - \Pi(\xi'))1_{\{\Gamma - \Pi(\xi) \geq 0\}} \\ &\geq (\Gamma - \Pi(\xi))1_{\{\Gamma - \Pi(\xi) \geq 0\}} - (\Pi(\xi') - \Pi(\xi))1_{\{\Gamma - \Pi(\xi) \geq 0\}} \\ &\geq (\Gamma - \Pi(\xi))^+ - (\Pi(\xi') - \Pi(\xi))1_{\{\Gamma - \Pi(\xi) \geq 0\}},\end{aligned}$$

and thus

$$\begin{aligned}\mathbb{E}\{(\Gamma - \Pi(\xi'))^+\} &\geq \mathbb{E}\{(\Gamma - \Pi(\xi))^+\} - \mathbb{E}\{(\Pi(\xi') - \Pi(\xi))1_{\{\Gamma - \Pi(\xi) \geq 0\}}\} \\ &\geq \mathbb{E}\{(\Gamma - \Pi(\xi))^+\} - \langle \Xi, \xi' - \xi \rangle, \end{aligned} \tag{17}$$

where $\Xi \in \mathcal{L}_1^N$ is given by $\Xi_t^i = \mathbb{E}\{1_{\{\Gamma - \Pi(\xi) \geq 0\}} | \mathcal{F}_t\}$ for all $i = 1, \ldots, N$ and $t = 0, \ldots, T-1$. Hence, given ε, define the neighborhood $B_\xi(\Xi, \varepsilon)$ of ξ as in (15), which ensures that $|\langle \Xi, \xi' - \xi \rangle| < \varepsilon$ for all $\xi \in B_\xi(\Xi, \varepsilon)$ and finally with (17) yields the lower semicontinuity

$$\mathbb{E}\{(\Gamma - \Pi(\xi'))^+\} \geq \mathbb{E}\{(\Gamma - \Pi(\xi))^+\} - \varepsilon \quad \text{for all } \xi' \in B_\xi(\Xi, \varepsilon).$$

Now let $(\xi(n))_{n \in \mathbb{N}} \subset \mathcal{U}$ be a sequence approaching the infimum

$$\lim_{n \to \infty} \mathbb{E}\{-G_T(\xi(n))\} = \inf_{\xi \in \mathcal{U}} \mathbb{E}\{-G_T(\xi)\}.$$

By the Banach–Alaoglu theorem, it contains a subsequence $(\xi(n_k))_{k \in \mathbb{N}}$ which converges to ξ^* in the weak topology. Since $\mathcal{U}$ is convex and norm-closed in $\mathcal{L}_1^N$, Lemma 1 shows that

$$\mathcal{U} \text{ is } \sigma(\mathcal{L}_\infty^N, \mathcal{L}_1^N)\text{-closed,} \tag{18}$$

and therefore $\xi^* \in \mathcal{U}$. Finally, the semicontinuity ensures that $\mathbb{E}\{-G_T(\xi^*)\} = \inf_{\xi \in \mathcal{U}} \mathbb{E}\{-G_T(\xi)\}$. ☐

Since $\mathcal{L}_1^N$ is not the dual space of $\mathcal{L}_\infty^N$ we cannot conclude that the $\|\cdot\|_\infty$-closed subset $\mathcal{U}$ of $\mathcal{L}_\infty^N$ is $\sigma(\mathcal{L}_\infty^N, \mathcal{L}_1^N)$-closed. However, since $\mathcal{U}$ is $\|\cdot\|_1$-closed in $\mathcal{L}_\infty^N$, we obtain the assertion.

LEMMA 1. *For the convex subset $\mathcal{U}$ of $\mathcal{L}^\infty$ it holds that*

(i) *$\mathcal{U}$ is norm-closed in $\mathcal{L}^1$,*

(ii) *$\mathcal{U}$ is $\sigma(\mathcal{L}_\infty^N, \mathcal{L}_1^N)$-closed in $\mathcal{L}^\infty$.*

Proof. (i) If $(\xi_n)_{n \in \mathbb{N}}$ is a sequence in $\mathcal{U}$ converging in $\mathcal{L}^1$ to some random variable ξ, then, extracting a subsequence if necessary, one concludes that it converges almost surely, showing that the constraints defining $\mathcal{U}$ are satisfied in the limit, which implies that $\xi \in \mathcal{U}$.

(ii) Since $\mathcal{U}$ is a convex and a norm-closed subset of $\mathcal{L}^1$ it follows from the Hahn–Banach theorem that $\mathcal{U}$ is the intersection of half spaces $H_{\xi,c} = \{X \in \mathcal{L}^1 | \langle X, \xi \rangle \leq c\}$

with $\xi \in \mathcal{L}^\infty$ and $c \in \mathbb{R}$. Since $\mathcal{L}^\infty \subseteq \mathcal{L}^1$ it holds for each of these half spaces $H_{\xi,c}$ that $\xi \in \mathcal{L}^1$. Thus we conclude that $H_{\xi,c} \cap \mathcal{L}^\infty = \{X \in \mathcal{L}^\infty | \langle X, \xi \rangle \leq c\}$ is closed in $(\mathcal{L}^\infty, \sigma(\mathcal{L}^\infty, \mathcal{L}^1))$. Since by definition it holds that $\mathcal{U} \subseteq \mathcal{L}^\infty$ it follows that $\mathcal{U}$ is given by the intersection of the sets $H_{\xi,c} \cap \mathcal{L}^\infty$ which are $\sigma(\mathcal{L}^N_\infty, \mathcal{L}^N_1)$-closed, so (18) follows. $\square$

The following result will be crucial in our analysis of the equilibrium.

THEOREM 1. *Suppose that*

$$\text{the } \mathcal{F}_{T-1}\text{-conditional distribution of } \Gamma \text{ possesses almost surely no point mass.} \tag{19}$$

Then the equilibrium carbon price process is a martingale given in terms of the global-optimal policy $\xi^* \in \mathcal{U}$ from Proposition 1 by

$$A^*_t = \pi \mathbb{E}\{1_{\{\Gamma - \Pi(\xi^*) \geq 0\}} \mid \mathcal{F}_t\} \quad \text{for } t = 0, \ldots, T. \tag{20}$$

Remark. Let us highlight the connection of this theorem to results given in the literature. Due to (20), the equilibrium allowance price is economically interpreted as the marginal contribution of an additional allowance to lower the potential penalty payment, when the global-optimal policy $\xi^* \in \mathcal{U}$ is followed. Indeed, expressing the indicator function appearing in (20) as a derivative, we get

$$A^*_t = -\frac{\partial}{\partial x} \mathbb{E}\{\pi(\Gamma - \Pi(\xi^*) - x)^+ \mid \mathcal{F}_t\}|_{x=0}.$$

This justifies rigorously the *folk principle* often found in the literature on emission markets and considered as crucial in the economic analysis of these markets:

(21) *the equilibrium allowance price equals the marginal abatement costs.*

In the proof given below, the equilibrium allowance price plays the role of an exercise boundary which drives the abatement measures to a global optimum. To understand this, we point out that in the presence of allowance trading, the optimal decision of each market participant i is to apply abatement at full intensity if the value of the allowance price indicates that this is reasonable $\{A_t > \mathcal{E}^i_t\} \subseteq \{\xi^i_t = \lambda\}$. Indeed, all saved credits can be immediately sold on the market. Otherwise, no-abatement action is optimal, $\{A_t < \mathcal{E}^i_t\} \subseteq \{\xi^i_t = 0\}$, since instead of saving emissions in the producers own business, the agent is better off purchasing emissions allowances on the market. In the proof (see assertions (32) and (33)) we show that the equilibrium allowance price A^*_t actually triggers the overall optimal abatements ξ^{i*}_t $(i = 1, \ldots, N)$ in this way. Now we explain why this issue is actually nothing but a version of the classical result (21). Being the optimal-exercise boundary (in the above sense), A^*_t is greater than or equal to the costs of the most expansive among all active abatement measures. Moreover, there is no inactive abatement measure whose cost is below A^*_t. Therefore, by an appropriate interpretation of the notion of *marginality*, we see that (21) holds in our model.

Proof. The equilibrium property of $(A^*_t)^T_{t=0}$ can be shown by an explicit construction of $\theta^{i*} \in \mathcal{L}_1 \times L_1$ such that the individual strategies

$$(\theta^{i*}, \xi^{i*}) \in (\mathcal{L}_1 \times L_1) \times \mathcal{U}^i, \quad i = 1, \ldots, N,$$

fulfill (11) and (10). To proceed, let $\xi^* \in \mathcal{U}$ be as given in Proposition 1, and define $(\theta^{i*})_{i=1}^N \in (\mathcal{L}_1 \times L_1)^N$ by

$$
\begin{aligned}
\theta_t^{*i} &= 0 \quad \text{for all } i = 1, \dots, N,\ t = 0, \dots, T-1, \\
\theta_T^{i*} &= \Gamma^i - \sum_{t=0}^{T-1} \xi_t^{i*} - (\Gamma - \Pi(\xi^*))/N.
\end{aligned}
\tag{22}
$$

Note that due to the martingale property of $(A_t^*)_{t=0}^T$, any other choice of (22) which respects the zero net supply condition is also admissible. Since (10) is obviously fulfilled, we focus on the proof of (11).

For $\theta \in (\mathcal{L}_1 \times L_1)^N$, $\xi \in \mathcal{U}$, and carbon price processes (20), we can express the expectation of (7) as

$$
\mathbb{E}\{I^{A^*,i}(\theta^i, \xi^i)\} = \mathbb{E}\left\{ -\theta_T^i A_T^* - \pi \left(\Gamma^i - \sum_{t=0}^{T-1} \xi_t^i - \theta_T^i \right)^+ \right\} - \mathbb{E}\left\{ \sum_{t=0}^{T-1} \xi_t^i \mathcal{E}_t^i \right\}
$$

since the process $A^* \in \mathcal{L}_\infty$ is a martingale by definition (20) and $(\theta_t^i)_{t=0}^{T-1}$ is an element of $\mathcal{L}_1$. Thus, in order to show (11), it suffices to prove that for each $\xi^i \in \mathcal{U}^i$, the supremum

$$
m(\xi^i) := \sup_{\theta_T^i \in L^1} \mathbb{E}\left\{ -\theta_T^i A_T^* - \pi \left(\Gamma^i - \sum_{t=0}^{T-1} \xi_t^i - \theta_T^i \right)^+ \right\}
\tag{23}
$$

is attained on L_1 at

$$
\theta_T^i(\xi^i) = \Gamma^i - \sum_{t=0}^{T-1} \xi_t^i - (\Gamma - \Pi(\xi^*))/N
\tag{24}
$$

and

$$
\xi^i \mapsto m(\xi) + \mathbb{E}\left\{ \sum_{t=0}^{T-1} \xi_t^i \mathcal{E}_t^i \right\} \qquad \text{is maximized on } \mathcal{U}^i \text{ at } \xi^{i*}.
\tag{25}
$$

First, we turn to (23) and (24), showing that the maximum is attained pointwise. In view of (20), $\omega \in \{\Gamma - \Pi(\xi^*) < 0\}$ implies that

$$
A_T^*(\omega) = 0 \ \text{ and } \ \theta_T^{i*}(\xi^i)(\omega) > \Gamma^i(\omega) - \sum_{t=0}^{T-1} \xi_t^i(\omega).
$$

Moreover, the maximum of

$$
z \mapsto -z A_T^*(\omega) - \pi \left(\Gamma^i(\omega) - \sum_{t=0}^{T-1} \xi_t^i(\omega) - z \right)^+
\tag{26}
$$

is attained at each point of the interval $[\Gamma^i(\omega) - \sum_{t=0}^{T-1} \xi_t^i(\omega), \infty)$, thus $\theta_T^i(\xi^i)(\omega)$ is a maximizer. In the other case, namely, when $\omega \in \{\Gamma - \Pi(\xi^*) \geq 0\}$, we have

$$
A_T^*(\omega) = \pi \ \text{ and } \ \theta_T^i(\xi^i)(\omega) \leq \Gamma^i(\omega) - \sum_{t=0}^{T-1} \xi_t^i(\omega).
$$

Here the maximum of (26) is attained at each point of the interval $[0, \Gamma^i(\omega) - \sum_{t=0}^{T-1} \xi_t^i(\omega)]$, thus $\theta_T^i(\xi^i)(\omega)$ is again a maximizer. In both cases, the maximum of (23) is

$$m(\xi^i) = \mathbb{E}\left\{ -\left(\Gamma^i - \sum_{t=0}^{T-1} \xi_t^i\right) A_T^* \right\}.$$

If we plug the above expression for $m(\xi^i)$ into (25) and use the martingale property of $(A_t^*)_{t=0}^{T-1}$, we conclude that

$$\begin{aligned} m(\xi^i) - \mathbb{E}\left\{ \sum_{t=0}^{T-1} \xi_t^i \mathcal{E}_t^i \right\} &= -\mathbb{E}\{\Gamma^i A_T^*\} + \mathbb{E}\{\xi_t^i (A_T^* - \mathcal{E}_t^i)\} \\ &= -\mathbb{E}\{\Gamma^i A_T^*\} + \mathbb{E}\{\xi_t^i (A_t^* - \mathcal{E}_t^i)\}. \end{aligned}$$

To show (25), it suffices to check that for each $i = 1, \ldots, N$ and $t \in \{0, \ldots, T-1\}$ the following inclusions hold almost surely:

$$\{A_t^* - \mathcal{E}_t^i > 0\} \subseteq \{\xi_t^{*i} = \lambda^i\} \quad \text{and} \quad \{A_t^* - \mathcal{E}_t^i < 0\} \subseteq \{\xi_t^{*i} = 0\}. \tag{27}$$

First, we remark that

$$\begin{array}{c} \text{for any } \xi \in \mathcal{U} \text{ with } \xi_s = \xi_s^* \text{ for } s = 0, \ldots, t-1, \\ \mathbb{E}\{G(\xi)|\mathcal{F}_t\} \leq \mathbb{E}\{G(\xi^*)|\mathcal{F}_t\} \quad \text{holds almost surely.} \end{array} \tag{28}$$

This assertion is proved as follows. If untrue, the $\mathcal{F}_t$-measurable set

$$M := \{\mathbb{E}\{G(\xi)|\mathcal{F}_t\} > \mathbb{E}\{G(\xi^*)|\mathcal{F}_t\}\}$$

would be of positive measure, $\mathbb{P}\{M\} > 0$, and could be used to construct an abatement strategy ξ' which would outperform ξ^*. Indeed, setting

$$\xi_s' = 1_M \xi_s + 1_{\Omega \setminus M} \xi_s^* \tag{29}$$

for all $s = 0, \ldots, T-1$ we see that since ξ^* and ξ' coincide at times $0, \ldots, t-1$, this definition indeed yields an adapted process $\xi' \in \mathcal{U}$. The decomposition

$$G(\xi') = 1_M G(\xi) + 1_{\Omega \setminus M} G(\xi^*)$$

leads to a contradiction to the optimality of ξ^*. Indeed,

$$\begin{aligned} \mathbb{E}\{G(\xi')\} &= \mathbb{E}\{\mathbb{E}\{1_M G(\xi) + 1_{\Omega \setminus M} G(\xi^*)|\mathcal{F}_t\}\} \\ &= \mathbb{E}\{1_M \mathbb{E}\{G(\xi)|\mathcal{F}_t\} + 1_{\Omega \setminus M} \mathbb{E}\{G(\xi^*)|\mathcal{F}_t\}\} \\ &> \mathbb{E}\{1_M \mathbb{E}\{G(\xi^*)|\mathcal{F}_t\} + 1_{\Omega \setminus M} \mathbb{E}\{G(\xi^*)|\mathcal{F}_t\}\} = \mathbb{E}\{G(\xi^*)\}. \end{aligned}$$

To prove (27) we consider for each λ in the countable set $Q := [0, \lambda^i] \cap \mathbb{Q}$, where $\mathbb{Q}$ denotes the set of rational numbers, the strategy $\xi(\lambda, i) \in \mathcal{U}$ defined by

$$\xi_s^k(\lambda, i) = \begin{cases} \lambda & \text{if } s = t \text{ and } k = i, \\ \xi_s^{*k} & \text{otherwise.} \end{cases}$$

In other words, $\xi(\lambda, i)$ coincides with ξ^* with the exception of time t, where only for agent i is the fuel switch intensity changed from ξ_t^{*i} to a deterministic value $\lambda \in Q$. This abatement policy $\xi(\lambda, i)$ satisfies

$$\begin{array}{l} \Pi(\xi(\lambda, i)) = \Pi(\xi^*) - (\xi_t^{*i} - \lambda) \\ F(\xi(\lambda, i)) = F(\xi^*) - (\xi_t^{i*} - \lambda)\mathcal{E}_t^i \end{array} \quad \text{for all } \lambda \in Q. \tag{30}$$

Since the set Q is countable due to (28), there exists a set $\tilde{\Omega}$ with $\mathbb{P}\{\tilde{\Omega}\} = 1$ such that

$$\frac{\mathbb{E}\{G(\xi^*|\mathcal{F}_t)\}(\omega) - \mathbb{E}\{G(\xi(\lambda,i)|\mathcal{F}_t)\}(\omega)}{|\xi_t^{*i}(\omega) - \lambda|} \geq 0 \ \text{ for all } \omega \in \tilde{\Omega} \text{ with } \lambda \neq \xi_t^{*i}(\omega).$$

Using (30) and (13), we conclude from this inequality that

$$\begin{aligned} 0 \leq & -\frac{\xi_t^{*i}(\omega) - \lambda}{|\xi_t^{*i}(\omega) - \lambda|}\mathcal{E}_t^i(\omega) \\ & -\mathbb{E}\left\{\pi \frac{(\Gamma_T - \Pi(\xi^*))^+ - (\Gamma_T - \Pi(\xi^*) + (\xi_t^{*i} - \lambda))^+}{|\xi_t^{*i} - \lambda|} \,|\, \mathcal{F}_t\right\}(\omega) \end{aligned} \tag{31}$$

holds for all $\omega \in \tilde{\Omega}$ with $\lambda \neq \xi_t^{*i}(\omega)$. Let us denote the term in (31) by $D(\xi^*, \lambda)(\omega)$. Approaching $\xi_t^{*i}(\omega)$ by $\lambda \in Q\backslash\{\xi_t^{*i}(\omega)\}$, we apply the dominated convergence theorem to obtain

$$\lim_{\lambda \uparrow \xi_t^{i*}(\omega)} D(\xi^*, \lambda)(\omega) = -\mathbb{E}\left\{\pi 1_{\{\Gamma - \Pi(\xi^*) \geq 0\}} \,|\, \mathcal{F}_t\right\}(\omega) \ \text{ for } \xi_t^{*i}(\omega) \in \,]0, \lambda^i],$$

$$\lim_{\lambda \downarrow \xi_t^{i*}(\omega)} D(\xi^*, \lambda)(\omega) = \mathbb{E}\left\{\pi 1_{\{\Gamma - \Pi(\xi^*) > 0\}} \,|\, \mathcal{F}_t\right\}(\omega) \ \text{ for } \xi_t^{*i}(\omega) \in [0, \lambda^i[.$$

Now (19) gives

$$\mathbb{E}\left\{\pi 1_{\{\Gamma - \Pi(\xi^*) \geq 0\}} \,|\, \mathcal{F}_t\right\} = \mathbb{E}\left\{\pi 1_{\{\Gamma - \Pi(\xi^*) > 0\}} \,|\, \mathcal{F}_t\right\} = A_t^*,$$

which with (31) implies that the following inclusions hold almost surely: Calculating the left limit $\lambda \uparrow \xi_t^i(\omega)$, we have

$$\{\xi_t^{*i} \in]0, \lambda^i]\} \subseteq \{A_t^* - \mathcal{E}_t^i \geq 0\} \ \Leftrightarrow \ \{A_t^* - \mathcal{E}_t^i < 0\} \subseteq \{\xi_t^{*i} = 0\}. \tag{32}$$

For the right limit $\lambda \downarrow \xi_t^i(\omega)$, we obtain

$$\{\xi_t^{*i} \in [0, \lambda^i[\} \subseteq \{A_t^* - \mathcal{E}_t^i \leq 0\} \ \Leftrightarrow \ \{A_t^* - \mathcal{E}_t^i > 0\} \subseteq \{\xi_t^{*i} = \lambda^i\}. \tag{33}$$

The assertions (32) and (33) give (27). □

4. Quantitative analysis. This section is devoted to the numerical analysis of some of the quantitative aspects of the carbon market equilibrium model introduced in this paper. Recall that it deals with only one compliance period. First, we emphasize the main differences between our generic model and the actual EU ETS implementation.

Streamlined the EU ETS. As explained earlier, we took care of the discounting effects by working with forward prices. In this section, we go even further and suppose that the interest rate is equal to zero.

(1) Working with one compliance period in isolation does not fully reflect the situation of the EU ETS in the period 2005–2007, since a certain amount of allowances could be banked into the next period 2008–2012 and, more importantly, the penalty structure was different from what we modeled. Indeed, at the end of the first EU ETS period, for each ton of carbon dioxide equivalent uncovered by an EUA, one EUA from the second period was charged for noncompliance in the first period, *in addition* to the fine of 40 Euros.

Another simplification is that we do not consider the impact of allowances gained from CDM and JI projects. As mentioned earlier, in our model, the impact of long-term projects is captured by the anticipated short-term reduction demand $(\mathbb{E}\{\Gamma|\mathcal{F}_t\})_{t=0}^T$. Hence, the role of the stochastic process $(\mathbb{E}\{\Gamma|\mathcal{F}_t\})_{t=0}^T$ is to account for the aggregate impact of all uncertainty sources, including ambiguity from emitting factors (weather, climate, business activity), the success of long- and middle-term reduction projects, imperfections in information flow, etc. Because of this complexity, one of the most difficult problems in the present framework is to find an appropriate quantitative description for $(\mathbb{E}\{\Gamma|\mathcal{F}_t\})_{t=0}^T$. In the present study, we choose for this martingale a discrete version of a Brownian motion independent from the fuel switch processes. This choice is motivated by the connection of $(\mathbb{E}\{\Gamma|\mathcal{F}_t\})_{t=0}^T$ with the *Emission-to-Cap Indicator* listed by PointCarbon; see, e.g., [1]. This index is designed to monitor the estimated emission savings required to meet the compliance. Note, however, that our $(\mathbb{E}\{\Gamma|\mathcal{F}_t\})_{t=0}^T$ is not the Emission-to-Cap Indicator since the latter does not consider potential demand reduction from CDM projects and long-term investments.

Finding out if $(\mathbb{E}\{\Gamma|\mathcal{F}_t\})_{t=0}^T$ and $(\mathcal{E}_t^i)_{t=0}^T$ can be modeled by independent processes will require an empirical correlation analysis between historical values of the Emission-to-Cap Indicator and representative fuel switch price processes. Unfortunately, historical data for the Emission-to-Cap Indicator is not available yet.

Remark. The problem of the correct choice for $(\mathbb{E}\{\Gamma|\mathcal{F}_t\})_{t=0}^T$ is part of the model calibration, which amounts to choosing a specific parametric family of martingales, and tuning the parameters in such a way that the prices produced within the model match the listed carbon prices. No-arbitrage valuation of a European derivative written on EUA could then be obtained by computation of its expected payoff within the calibrated model.

(2) A further simplification is to consider a single fuel switch price process. The argument is that if the cheapest technology is applied first, and if the switch capacity is small enough, other technologies are rarely used. So, the decision to consider only one fuel switch price process does not mean that we omit other short-term abatement measures. We merely suppose that their impact on carbon price can be neglected, due to high capacity of cheaper abatement measures from CCGT. Even though there are numerous plants with different efficiencies, we aggregate all fuel switch possibilities from CCGT to coal plants into one representative capacity. This approximation seems acceptable, since according to [4], fuel switch prices vary within a 4 Euro range, which is small compared to the changes of the fuel switch prices caused by gas and coal price fluctuations.

Based on [4] we assume that the European yearly switch capacity is 52 megatons of carbon dioxide. For the fuel switch, we suppose the efficiencies of 38% for coal and 52% for CCGT plants. Next, we discuss the time resolution of our calculations. Since the model is set up in terms of

$$(\mathcal{E}_t)_{t=0}^{T-1} \quad \text{and} \quad (\Gamma_t = \mathbb{E}\{\Gamma|\mathcal{F}_t\})_{t=0}^T, \tag{34}$$

which represent the evolutions of the fuel switch cost and of expected demand, respectively, these processes are obtained as time discretizations of continuous-time processes

$$(\mathcal{E}(t))_{t\in[0,\mathcal{T}]} \quad \text{and} \quad (\Gamma(t))_{t\in[0,\mathcal{T}]} \tag{35}$$

for fuel switch price and expected demand evolutions. Note that we write the time parameter in parentheses instead of using subscript, to indicate continuous-time processes. Moreover, the horizon for continuous time is $[0,\mathcal{T}]$, where we suppose that the

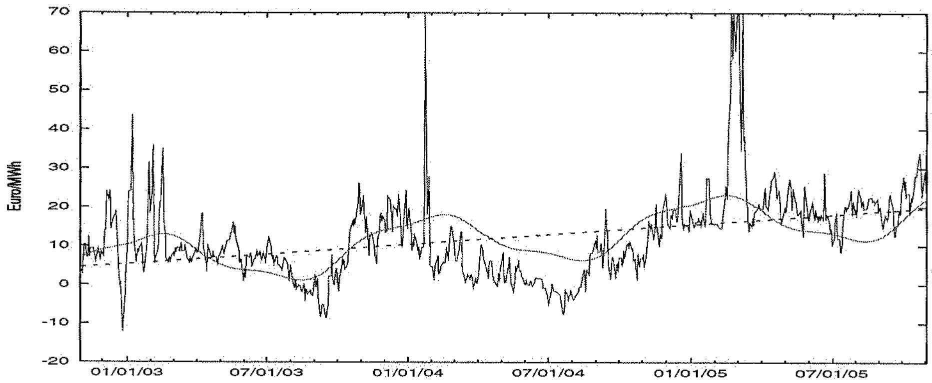

FIG. 3. *Historical fuel switch prices for CCGT technology calculated with* (3) *and based on historical data from McCloskey Index and NBP natural gas spot prices.* .

time unit is one year. Given the fact that a reasonable time step should not fall below the time required to reschedule a CCGT turbine, we sample the continuous-time evolutions daily.

Fuel switch process. Because during the pre-Kyoto period 2005–2007 at least half of the entire EU fuel switch capacity was located within the United Kingdom, we decided to base fuel switch prices on the McCloskey North-West Europe Steam Coal Index and on natural gas prices from NBP (National Balancing Point, which specifies delivery location within the UK). The continuous-time fuel switch price process is modeled by

$$\mathcal{E}(t) = P(t) + X(t), \qquad t \in [0, \mathcal{T}], \tag{36}$$

where the deterministic part

$$P(t) = a + bt + \sum_{j=0}^{2} c_j \cos(2\pi\varphi_j t + l_j), \qquad t \in [0, \mathcal{T}], \tag{37}$$

accounts for a linear price increase superimposed onto seasonal price fluctuations. The stochastic part $(X(t))_{t\in[0,\mathcal{T}]}$ is modeled by an Ornstein–Uhlenbeck process, whose evolution follows the stochastic differential equation

$$dX(t) = \gamma(\alpha - X(t))dt + \sigma dW(t) \tag{38}$$

with parameters $\gamma, \alpha, \sigma \in \mathbb{R}$. Here, $(W(t))_{t\in[0,\mathcal{T}]}$ is a Brownian motion process. After performing estimation (see the appendix) based on the historical data shown in Figure 3, the process (36) is identified with the parameters shown in Tables 1 and 2.

TABLE 1
Stochastic part $(X(t))_{t\in[0,\mathcal{T}]}$

γ	α	σ
31.82	−0.12	68.24

TABLE 2
Deterministic part $(P(t))_{t\in[0,\mathcal{T}]}$

a	b	c_0	φ_0	l_0	c_1	φ_1	l_1	c_2	φ_2	l_2
21.42	6.19	7.62	1	5.95	0.55	2	1.14	1.11	3	3.24

Expected allowance demand. The continuous-time counterpart of the expected allowance demand is described by

$$\Gamma(t) := m + vW'(t), \qquad t \in [0, \mathcal{T}], \tag{39}$$

where $(W'(t))_{t\in[0,\mathcal{T}]}$ is a Brownian motion independent of $(W(t))_{t\in[0,\mathcal{T}]}$. In this context, the parameters m and v are interpreted as the mean and the standard deviation of the final allowance demand. In accordance with [4] we set $m = 30$m for the total required abatement, which equals to 60 to 65m tons of nominal emissions minus 20 to 25m tons of savings which are automatically effected (being at very low costs). The parameter $v = 20$m is chosen to reflect the possible deviations of the nominal emissions caused by dry/wet years, since the required abatement may change by 18 to 20m tons, due to changes in hydroelectric capacity.

Numerical implementation. As stated earlier, $(\mathcal{E}(t))_{t\in[0,\mathcal{T}]}$ and $(\Gamma(t))_{t\in[0,\mathcal{T}]}$ are both interpreted as continuous-time counterparts of fuel switch prices and expected allowance demand, respectively. For numerical purposes we used a standard trinomial tree discretization of each component of the two-dimensional diffusion process $(\mathcal{E}(t), \Gamma(t))_{t\in[0,\mathcal{T}]}$ to solve the corresponding dynamic optimization problem (14) through a backward induction method. Figure 4 graphically illustrates this technique.

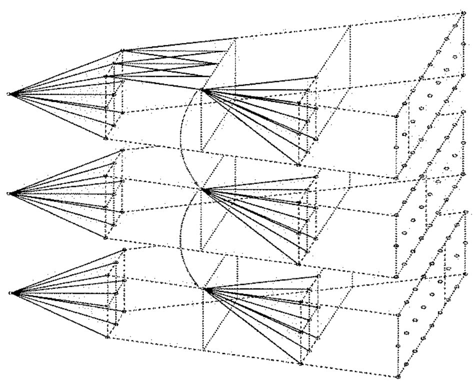

FIG. 4. *Backward induction optimal control schematic, as applied to the trinomial tree discretization to each component of the processes* $(\mathcal{E}(t), \Gamma(t))_{t\in[0,\mathcal{T}]}$.

At each node, we see a splitting into three vertical and three horizontal directions, giving nine branches in all. The vertical direction describes the movement of the fuel switch price, whereas the horizontal branches model the expected demand dynamics. At maturity, paths finish either at positive realizations of the allowance demand or at nonpositive. The optimally controlled fuel switch process is calculated by backward induction: At each node the maximum principle is applied to decide whether to apply the fuel switch or not. If the fuel switch is performed, then the state is changed due to the effectively reduced allowance demand, indicated by a move to the next lower tree in the forest diagram.

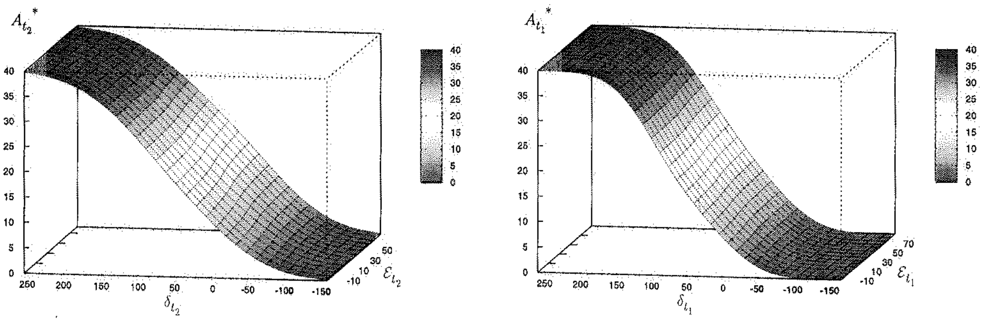

FIG. 5. *The dependence of allowance price on the present expected demand δ_t and $\mathcal{E}_t$ for different times (right: $t = t_1$ beginning of March; left: $t = t_2$ beginning of September).*

We now discuss the impact of parameters on carbon price. The following numerical illustration is based on the discrete-time model $(\mathcal{E}_t)_{t=0}^{T-1}, (\Gamma_t)_{t=0}^{T}$ which corresponds to the parameters in Tables 1 and 2 estimated in the appendix. Here, the time horizon $T = 253$ stands for the number of working days in 2005. The fuel switch process is based on the deterministic component fitted to that year's data, and the starting point $\mathcal{E}_0$ is set at the value of the deterministic component at the beginning of January 2005.

Present values. For commensurability reasons, we decided to show the dependence of the allowance price A_t^* on allowance demand Γ_t in terms of the *relative demand*

$$\delta_t = \frac{\Gamma_t - \sum_{s=0}^{t} \xi_s}{\lambda(T-t)}, \quad t = 0, \ldots, T-1,$$

which stands for the percentage of time steps at which the fuel switch at full intensity is needed in order to meet the compliance. (Note that we have to take into account carbon $\sum_{s=0}^{t} \xi_s$ saved by the previous fuel switches $(\xi_s)_{s=0}^{t}$.) The dependence illustrated in Figure 5 is obvious. The price A_t^* is increasing in δ_t and $\mathcal{E}_t$. Moreover, for $\delta_t \to +\infty$, the allowance price approaches the boundary π of 40 Euros, whereas for $\delta_t \to -\infty$ it tends to 0. Further, A_t^* changes significantly with moderate deviations in δ_t, which is in line with the high correlation of allowance prices and values of the Emission-to-Cap Indicator observed by [1]. On the contrary, the impact of the present fuel switch price $\mathcal{E}_t$ is weak due to the distinct mean reversion. This poor correlation between instantaneous fuel switch price and allowance price is accurate also in reality, as can be observed in Figure 2. Despite the low degree of dependence of A_t^* on $\mathcal{E}_t$, we suppose though that fuel switch price is a significant factor, whose impact is effected through the expected long-term fuel switch prices (to be deduced from fuel futures, whose price dynamics is not modeled here).

Dependence on model parameters. In accordance, the left picture in Figure 6 shows high sensitivity of allowance price on α, which settles the level of expected long-term fuel prices. For this reason, we decided to visualize the effect of α by a plot of A_0^* against $\mathbb{E}\{\sum_{t=0}^{T-1} \mathcal{E}_t\}/T$. Moreover, this figure shows a weak dependence of the allowance price on σ, which we illustrate by a plot of A_0^* against the stationary fuel switch price variance $\sigma^2/(2\gamma)$. The right picture in Figure 6 shows that the dependence of A_0^* with respect to changes in $\mathbb{E}\{\Gamma\}/(\lambda T)$ is higher than in v. In other words, the

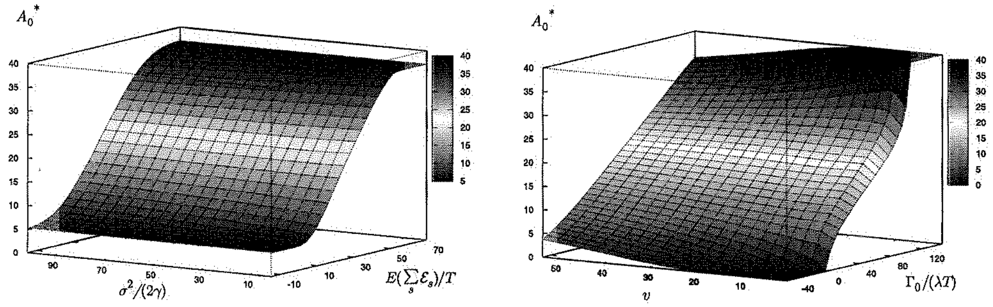

FIG. 6. *On the left: the impact of α and σ on allowance price, expressed for $\gamma = 31.82$ through long-term fuel switch price mean $\mathbb{E}(\sum_{t=0}^{T-1} \mathcal{E}_t)/T$ and variance $\sigma^2/(2\gamma)$, respectively. On the right: the impact of v and $\mathbb{E}(\Gamma)/(\lambda T)$.*

dependence of allowance price on the need for emission reductions is high, whereas the uncertainty about necessitative emission reduction is of secondary importance.

Remark. Note that $\Gamma(t) = m + vW(t)$ stands for the market expectation on allowance demand occurring without short-term abatement. Thus, v is more related to market uncertainty about future emissions than to the actual emissions dynamics.

Remark. We deliberately set the parameter intervals in Figure 6 larger than their physical range in order to show boundaries where allowances price approaches its limits of 0 and 40 Euros. For instance, we estimate that a realistic value for the expected percentage switching $\mathbb{E}(\Gamma)/(\lambda T)$ is between 0.4 and 0.9. Note that for this range, the price dependence on v is very weak (right picture in Figure 6). This observation indicates that the ambiguity about v is not crucial and that m in (39) could be an appropriate parameter for the implicit calibration. Such a calibration seems easily possible, since the allowance price is monotone in m.

Let us summarize our findings. The allowance price should be significantly correlated to the expected long-term fuel prices and to the expected need for emission reductions. These are the main price drivers, since the remaining factors (recent fuel switch prices, their volatility, uncertainty on the required emission reduction amount) have minor effects on the carbon price formation.

Regulatory controls. Designing a legally binding scheme, one of the main concerns of regulatory authorities is, on the one hand, to fulfill environmental targets (at least with a certain probability) and, on the other hand, to achieve this emission reduction at the lowest possible costs for the final consumer. Thus, we have studied the dependence of compliance probability and allowance price on the penalty level and on the initially expected allowance demand (note that this value is controlled by the total amount of allocated allowances). The diagrams in Figure 7 show the corresponding calculations. Again we show this influence in terms of the relative demand $\delta_0 = \mathbb{E}\{\Gamma\}/(\lambda T)$, which stands for the percentage of time steps at which the fuel switch at full intensity is needed in order to meet the initially expected allowance demand. One concludes that up to the relative demand of 50% the penalty can be increased without a notable increase of the allowance price, giving, however, a strong *increase of the compliance* probability. If the relative demand is far above 50%, then the situation changes. The moderate increase of compliance probability is reached only at the expense of a high allowance price. Note that the initial allowance price is directly related to the consumers costs since EUAs are added to electricity prices as

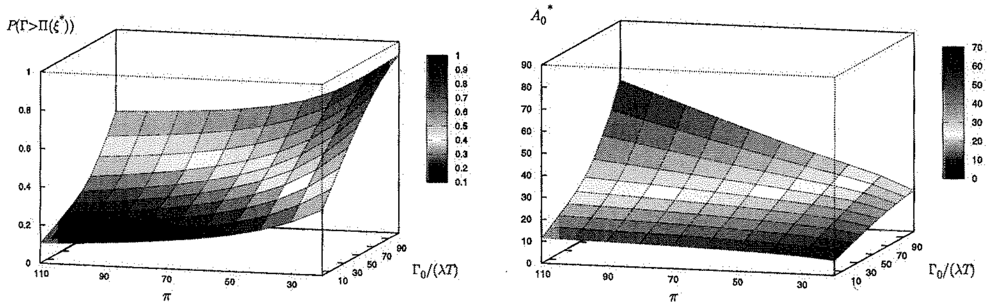

FIG. 7. *The probability of noncompliance and the initial allowance price depending on penalty size and fuel switch demand.*

an extra consumed commodity.

5. Conclusion. In this work, emission allowance price formation is investigated by equilibrium methodology. The mathematical model reflects a generic cap and trade system, naturally generalized to a fully stochastic framework, where both short-term abatement prices and the total emission volumes evolve randomly. Under mild assumptions, the existence of market equilibrium is proved. We show that the calculation of allowance prices reduces to the solution of an optimal control problem. By implementing a stylized model of the first phase of the EU ETS, we demonstrate that important calculations become tractable. We elaborate on the sensitivity with respect to model parameters and initial data to identify main price drivers for the EUA. Since problems of this type frequently arise in risk management, we hope that our findings are useful. Furthermore, we believe that quantitative modeling in the spirit of our approach can help regulatory authorities to optimally design market rules, when a new emission trading mechanism is being established. At this stage, we have to address to future research the question of fair pricing and efficient hedging of allowance derivatives. One pathway is through our structural modeling, which has to be accomplished by an implicit model calibration. The other possibility is by means of a reduced-form approach, targeted on direct modeling for the martingale process of allowance prices. Both directions entail interesting challenges for future research which we hope to encourage by this contribution.

Appendix. Parameters of the fuel switch price process. Our estimation is based on a series of $n = 758$ daily observations,

$$(\mathcal{E}(t\Delta)(\omega))_{t=1}^{n}$$

(where $\Delta = 1/253$ corresponds to one day), which are shown in Figure 3. The deterministic harmonics (37) in the fuel switch price process are identified with the parameters shown in Table 2 obtained from peaks in the Fourier transform. After removing the deterministic part $(P(t\cdot\Delta)(\omega))_{t=1}^{n}$ (smooth line in this figure) the residual component

$$X(t\Delta)(\omega) = \mathcal{E}(t\Delta)(\omega) - P(t\Delta)(\omega), \quad t = 1, \dots, n, \tag{40}$$

is modeled as a realization of the Ornstein–Uhlenbeck process (38) whose parameters γ, α, σ are estimated from the data (40) by a standard linear regression method applied

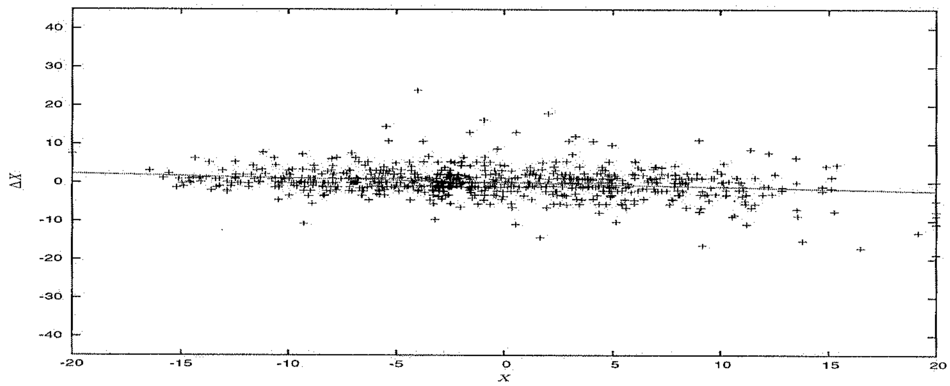

FIG. 8. *Scatter plot of* $(Y(t\Delta)(\omega), X(t\Delta)(\omega))_{t=1}^{n-1}$ *calculated by* (43) *and based on historical fuel switch prices from the Figure* 3. *The straight line depicts the estimated linear regression.*

as follows: From the formulas for conditional mean and variance

$$\mathbb{E}\{X(t)|\mathcal{F}_s\} = X(s)e^{-\gamma(t-s)} + \alpha(1 - e^{-\gamma(t-s)}), \quad s \le t, \tag{41}$$

$$\mathrm{Var}\{X(t)|\mathcal{F}_s\} = \frac{\sigma^2}{2\gamma}(1 - e^{-2\gamma(t-s)}), \quad s \le t, \tag{42}$$

we obtain the regression

$$Y(t\Delta) := X((t+1)\Delta) - X(t\Delta) = \beta_0 + \beta_1 X(t\Delta) + \beta_2 \epsilon_t, \qquad t = 1, \dots, n-1, \tag{43}$$

where $(\epsilon_t)_{t=1}^{n-1}$ are independent, standard Gaussian random variables and $\beta_0, \beta_1, \beta_2$ are connected to α, γ, σ by

$$\begin{aligned} \alpha &= -\frac{\beta_0}{\beta_1}, \\ \gamma &= -\frac{1}{\Delta}\ln(1+\beta_1), \\ \sigma &= \sqrt{\frac{2\gamma\beta_2}{1 - e^{-2\gamma\Delta}}}. \end{aligned}$$

Figure 8 shows a scatter plot of $(Y(t\Delta)(\omega), X(t\Delta)(\omega))_{t=1}^{n-1}$. Maximum likelihood parameter estimation gave $\beta_0 = -0.0147$, $\beta_1 = -0.1182$, $\beta_2 = 16.2708$ from which we computed the original parameters α, β, σ displayed in Table 1.

REFERENCES

[1] *Carbon* 2006, *Towards a Truly Global Market*, Point Carbon, 2006.

[2] M. CHESNEY AND P. TASCHINI, *The endogenous price dynamics of the emission allowances and an application to co2 option pricing*, Preprint, Institut für Schweizerisches Bankwesen, Zürich, 2008.

[3] M. CRONSHAW AND J. B. KRUSE, *Regulated firms in pollution permit markets with banking*, J. Regulatory Econom., 9 (1996), pp. 179–89.

[4] DRESDNER KLEINWORT WASSERSTEIN RESEARCH DEPARTMENT, *Emission Trading, Carbon Derby Part 2: And They're Off*, Dresdner Kleinwort Wasserstein, Frankfurt, Germany, 2003; available online from http://www.unepfi.org/fileadmin/documents/materiality1/emissions_trading_dresdner_2004.pdf.

[5] A. MAEDA, *Impact of banking and forward contracts on tradable permit markets*, Environ. Econom. Policy Stud., 6 (2004), pp. 81–102.

[6] W. D. MONTGOMERY, *Markets in licenses and efficient pollution control programs*, J. Econom. Theory, 5 (1972), pp. 395–418.

[7] T. PAOLELLA AND L. TASCHINI, *An econometric analysis of emission-allowance prices*, J. Banking Finance, 32 (2008), pp. 2022–2032.

[8] J. REINAUD, *Emissions Trading and Its Possible Impacts on Investment Decisions in the Power Sector*, IEA Information Paper, 2003.

[9] A. ROBERTSON AND W. ROBERTSON, *Topological Vector Spaces*, Cambridge University Press, Cambridge, UK, 1973.

[10] J. RUBIN, *A model of intertemporal emission trading, banking and borrowing*, J. Environ. Econom. Management, 31 (1996), pp. 269–286.

[11] S. M. SCHENNACH, *The economics of pollution permit banking in the context of Title* IV *of the* 1990 *Clean Air Act amendments*, J. Environ. Econom. Management, 40 (2000), pp. 189–21.

[12] J. SEIFERT, M. UHRIG-HOMBURG, AND M. WAGNER, *Dynamic Behavior of Carbon Spot Prices, Theory and Empirical Evidence*, preprint, 2006.

[13] B. STEVENS AND A. ROSE, *A dynamic analysis of the marketable permits approach to global warming policy: A comparison of spatial and temporal flexibility*, J. Environ. Econom. Management, 44 (2002), pp. 45–69.

[14] T. TIETENBERG, *Emission trading: An exercise in reforming pollution policy*, Resources for the Future, Washington D.C., 1985.

[15] T. TIETENBERG, *Environmental Natural and Resource Economics*, Addison–Wesley, Boston, 2003.

[16] M. WAGNER, *CO2-Emissionszertifikate, Preismodellierung und Derivatebewertung*, Ph.D. thesis, Universität Karlsruhe, Karlsruhe, Germany, 2006.