

# AN INFINITE DIMENSIONAL STOCHASTIC ANALYSIS APPROACH TO LOCAL VOLATILITY DYNAMIC MODELS

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ABSTRACT. The difficult problem of the characterization of arbitrage free dynamic stochastic models for the equity markets was recently given a new life by the introduction of market models based on the dynamics of the local volatility. Typically, market models are based on Itô stochastic differential equations modeling the dynamics of a set of basic instruments including, but not limited to, the option underliers. These market models are usually recast in the framework of the HJM philosophy originally articulated for Treasury bond markets. In this paper we streamline some of the recent results on the local volatility dynamics by employing an infinite dimensional stochastic analysis approach as advocated by the pioneering work of L. Gross and his students.

## 1. Introduction and Notation

The difficult problem of the characterization of arbitrage free dynamic stochastic models for the equity markets was recently given a new life in [3] by the introduction of market models based on the dynamics of the local volatility surface. Market models are typically based on the dynamics of a set of basic instruments including, but not limited to, the option underliers. These dynamics are usually given by a continuum of Itô's stochastic differential equations, and the first order of business is to check that such a large set of degrees of freedom in the model specification does not introduce arbitrage opportunities which would render the model practically unacceptable.

Market models originated in the groundbreaking original work of Heath, Jarrow and Morton [11] in the case of Treasury bond markets. These authors modeled the dynamics of the instantaneous forward interest rates and derived a no-arbitrage condition in the form of a drift condition. This approach was extended to other fixed income markets and more recently to credit markets. The reader interested in the HJM approach to market models is referred to the recent review article [2]. However, despite the fact that they were the object of the first success of the mathematical theory of option pricing, the equity markets have offered the strongest resistance to the characterization of no-arbitrage in dynamic models. This state of affair is due to the desire to accommodate the common practice of using the Black-Scholes implied volatility to code the information contained in the prices of derivative instruments. Indeed while defining stochastic dynamics

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for the implied volatility surface is rather natural, *see also* for example [5, 6, 9], deriving no-arbitrage conditions is highly technical and could be only done in specific particular cases [8, 14, 15, 16].

In the present paper, we streamline some of the recent results on local volatility dynamics by employing an infinite dimensional stochastic analysis approach as advocated by the pioneering work of L. Gross and his students.

One of the main technical results of [3] is the semi-martingale property of call option prices corresponding to a local volatility surface which evolves over time according to a set of Itô's stochastic differential equations. We denote by  $C_t(T, K)$  the price at time  $t$  of a European call option with maturity  $T \geq t$  and strike  $K > 0$ . For each fixed  $t > 0$  we have

$$\begin{cases} \partial_T C_t(T, K) = \frac{1}{2} a_t^2(T, K) K^2 \partial_K^2 C_t(T, K), & t < T \\ C_t(t, K) = (S_t - K)^+ \end{cases} \quad (1.1)$$

To be more specific, if for each maturity  $T > 0$  and strike  $K > 0$ , we have

$$da_t(T, K) = \alpha_t(T, K)dt + \beta_t(T, K) \cdot dW_t, \quad (1.2)$$

the result we revisit here says that the solution of the Dupire PDE (1.1) is a semi-martingale whenever the second order term coefficient  $a_t^2(T, K)$  has for each  $T > 0$  and  $K > 0$ , a stochastic Itô's differential of the form (1.2).

The goal of this paper is to simplify the proof of this result, while at the same time extending it to the case of infinitely many driving Wiener processes  $W_t$ . Our new proof uses the general framework of infinite dimensional analysis. It streamlines the main argument and gets rid of a good number of technical lemmas proved in [3].

The theoretical results from functional analysis and infinite dimensional stochastic analysis which are needed in this paper can be found in Kuo's original Lecture Notes in Mathematics [13], and in the more recent book by Carmona and Tehranchi [4]. Already, this book was dedicated to Leonard Gross for his groundbreaking work on abstract Wiener spaces and the depth of his contribution to infinite dimensional stochastic analysis. Contributing the present paper to a volume in the honor of his 70th birthday is a modest way to show our deep gratitude.

## 2. Solutions of the Pricing Equations

As explained in the introduction, we denote by  $C_t(T, K)$  the price at time  $t$  of a European Call option with trike  $K$  and maturity  $T$ . It is a random variable measurable with respect to the  $\sigma$ -field  $\mathcal{F}_t$  of the natural filtration of a Wiener process  $\mathbf{W} = \{W_t\}_t$ . Throughout the paper, we use the notation  $\tau = T - t$  for the time to maturity, and we find it convenient to use the notation  $x$  for the log-moneyness  $x = \log(K/S)$ .

**2.1. Pricing PDEs.** We will find convenient to use the notation

$$\tilde{C}_t(\tau, x) := \frac{1}{S_t} C_t(t + \tau, S_t e^x), \quad \tau > 0, x \in \mathbb{R}.$$

for call prices and

$$D_x := \frac{1}{2} (\partial_{x^2}^2 - \partial_x), \quad D_x^* := \frac{1}{2} (\partial_{x^2}^2 + \partial_x)$$

for partial differential operators which we use throughout the paper. Then, if we consider the local volatility  $a_t^2(T, K)$  as given, and introduce the notation

$$\tilde{a}_t^2(\tau, x) = a_t^2(t + \tau, S_t e^x),$$

then we can conclude that the call price  $\tilde{C}_t(\cdot, \cdot)$  satisfies the following initial-value problem

$$\begin{cases} \partial_\tau w = \tilde{a}_t^2(\tau, x) D_x w(\tau, x) \\ w(0, x) = (1 - e^x)^+ \end{cases} \quad (2.1)$$

We will introduce more notation later in the paper, but for the time being we denote by  $p(\tilde{a}^2; \tau, x; u, y)$ , with  $\tau > u$ , the fundamental solution of the *forward* partial differential equation (PDE for short) in (2.1) with coefficient  $\tilde{a}^2$ . Similarly, we introduce  $q(\tilde{a}^2; u, y; \tau, x)$ , with  $u < \tau$ , the fundamental solution of the *backward* equation

$$\partial_u w = -\tilde{a}^2(u, y) D_y w(u, y), \quad (2.2)$$

which is, in a sense, *dual* to (2.1). We will sometimes drop the argument  $\tilde{a}^2$  of the fundamental solutions  $p$  and  $q$ , when the coefficient  $\tilde{a}^2$  is assumed to stay the same. Notice that, if  $w$  is the solution of (2.1), we have

$$D_x w(\tau, x) = \frac{1}{2} e^x q(0, 0; \tau, x). \quad (2.3)$$

This equality will be used later in the paper.

**2.2. Fréchet Differentiability.** For each fixed  $\bar{\tau} > 0$  and integers  $k, m \geq 1$ , and for any smooth function  $(\tau, x) \mapsto f(\tau, x)$  defined in the strip  $\mathcal{S} = [0, \bar{\tau}] \times \mathbb{R}$ , we define the norm

$$\|f\|_{C^{k,m}(\mathcal{S})} = \sup_{(\tau, x) \in \mathcal{S}} \left( \sum_{i=0}^k |\partial_{\tau^i} f(\tau, x)| + \sum_{j=1}^m |\partial_{x^j} f(\tau, x)| \right).$$

Next we denote by  $\tilde{\mathcal{B}}$  the space of functions  $f$  on  $\mathcal{S}$  which are continuously differentiable in the first argument and five times continuously differentiable in the second argument, and for which the norm  $\|\cdot\|_{C^{1,5}(\mathcal{S})}$  is finite. We subsequently denote  $\|\cdot\|_{\tilde{\mathcal{B}}} := \|\cdot\|_{C^{1,5}(\mathcal{S})}$ .

Now we fix  $\bar{\varepsilon} > 0$ , and we define the strip  $\mathcal{S}_{\bar{\varepsilon}}$  by  $\mathcal{S}_{\bar{\varepsilon}} = [\bar{\varepsilon}, \bar{\tau}] \times \mathbb{R}$ . We then define  $\tilde{\mathcal{W}}_{\bar{\varepsilon}} = C^{1,2}(\mathcal{S}_{\bar{\varepsilon}})$  and the mapping

$$\mathbf{F}_{\bar{\varepsilon}} : \tilde{\mathcal{B}} \mapsto \tilde{\mathcal{W}}_{\bar{\varepsilon}},$$

where, for any  $h \in \tilde{\mathcal{B}}$ , the image  $\mathbf{F}_{\bar{\varepsilon}}(h)$  is the restriction to  $\mathcal{S}_{\bar{\varepsilon}}$  of the solution of (2.1) with  $e^h$  in lieu of the coefficient  $\tilde{a}^2$ . Notice that  $e^h \in \tilde{\mathcal{B}}$ , and that it is bounded away from zero, implying that  $\mathbf{F}_{\bar{\varepsilon}}(h)$  is well defined.

We are ready to state and prove the main functional analytic result of the paper. This result is technical in nature, but it should be viewed as the work horse for the paper.

**Proposition 2.1.** *The mapping  $\mathbf{F}_\varepsilon : \tilde{\mathcal{B}} \hookrightarrow \tilde{\mathcal{W}}_\varepsilon$  defined above is twice continuously Fréchet differentiable and for any  $h, h', h'' \in \tilde{\mathcal{B}}$ , we have*

$$\mathbf{F}_\varepsilon'(h)[h'](\tau, x) = \frac{1}{2} \int_0^\tau \int_{\mathbb{R}} h'(u, y) e^{h(u, y) + y} p(e^h; \tau, x; u, y) q(e^h; 0, 0; u, y) dy du,$$

and

$$\begin{aligned} \mathbf{F}_\varepsilon''(h)[h', h''](\tau, x) &= \frac{1}{2} \int_0^\tau \int_{\mathbb{R}} h'(u, y) e^{h(u, y) + y} \cdot \\ &\left[ \left( \int_u^\tau \int_{\mathbb{R}} p(e^h; \tau, x; v, z) e^{h(v, z)} h''(v, z) D_z p(e^h; v, z; u, y) dz dv \right) q(e^h; 0, 0; u, y) \right. \\ &\quad \left. - p(e^h; \tau, x; u, y) \cdot \left( \int_0^u \int_{\mathbb{R}} q(e^h; 0, 0; v, z) e^{h(v, z)} h''(v, z) D_z q(e^h; v, z; u, y) dz dv \right) \right] dy du \end{aligned}$$

*Proof.* Our proof is based on a systematic use of uniform estimates on the fundamental solutions of the parabolic equations (2.1) and (2.2), and their derivatives. These estimates are known as Gaussian estimates. Typically, they hold when the second order coefficients are uniformly bounded together with a certain number of its derivatives. As a preamble to the technical details of the proof, we first state the Gaussian estimates on the fundamental solutions that we will use in this paper.

If  $\Gamma$  denotes the fundamental solution of (2.1) or (2.2), then the following estimates hold

$$\left| \partial_{x^m}^m \partial_{y^k}^k \Gamma(\tau, x; u, y) \right| \leq \frac{C}{|\tau - u|^{(1+m+k)/2}} \exp\left(-c \frac{(x - y)^2}{|\tau - u|}\right), \quad (2.4)$$

and consequently

$$\left| \partial_{x^i}^i \partial_{y^k}^k \Gamma(\tau, x + y; u, y) \right| \leq \frac{C}{|\tau - u|^{(1+i)/2}} \exp\left(-c \frac{x^2}{|\tau - u|}\right), \quad (2.5)$$

for  $0 \leq k + m \leq 4$ ,  $i = 0, 1$ ,  $\tau \neq u \in [0, \bar{\tau}]$  and  $x, y \in \mathbb{R}$ . Here, the constants  $c$  and  $C$  depend only upon the lower bound of  $\tilde{a}^2(\tau, x)$  and the norm  $\|\tilde{a}^2\|_{C^{1,5}(\mathcal{S})}$ , where  $\tilde{a}^2$  is the coefficient in the PDEs (2.1) and (2.2).

Inequality (2.4) is derived on pp. 251-261 of [10]. The comments on the dependence of constants  $c$  and  $C$  on  $\tilde{a}^2$  are given in [12].

Let us fix  $h \in \tilde{\mathcal{B}}$ . Estimate (2.4) holds for  $p(e^{h+h'}; \tau, x; u, y)$  and  $q(e^{h+h'}; \tau, x; u, y)$ , uniformly over  $h'$  varying in a neighborhood of zero, say  $U(0) \subset \tilde{\mathcal{B}}$ . In the following we consider only  $h' \in U(0)$ .

We now extend the properties of the fundamental solutions to a larger class of functions. For each integer  $s \geq 0$  we introduce the space  $\tilde{\mathcal{G}}^s$ :

**Definition 2.2.** We say that a family of functions  $\Gamma = \{\Gamma(\lambda; \cdot, \cdot, \cdot, \cdot)\}_{\lambda \in \Lambda}$  belongs to  $\tilde{\mathcal{G}}^s(\Lambda)$  if, for each  $\lambda \in \Lambda$ , the function  $\Gamma(\lambda; \tau, x; u, y)$  is defined for all  $0 \leq u < \tau \leq \bar{\tau}$ ,  $x, y \in \mathbb{R}$ , and:

- (1)  $\Gamma$  is  $s$  times differentiable in  $(x, y)$ , and its derivatives are jointly continuous in  $(\tau, x; u, y)$ , moreover,  $\Gamma$  satisfies estimates (2.4), for  $0 \leq k + m \leq s$ , uniformly over  $\lambda \in \Lambda$ ;
- (2) for any  $g \in C_0^1(\mathbb{R})$  and all  $\lambda \in \Lambda$ ,

$$\lim_{u \rightarrow \tau} \int_{\mathbb{R}} \Gamma(\lambda; \tau, x; u, y) g(y) dy = c_i g(x),$$

where the  $c_i$ 's are real constants which depend only on  $\Gamma$ .

We will need another class of functions:

**Definition 2.3.** The family of function  $\Gamma$  is said to belong to class  $\mathcal{G}^s(\Lambda)$ , for some integer  $s \geq 0$ , if it belongs to  $\tilde{\mathcal{G}}^s(\Lambda)$ , and, in addition, satisfies the following: if  $s \geq 2$ , then  $\Gamma$  is continuously differentiable in  $\tau$ , and, for all  $\lambda \in \Lambda$ ,

$$\partial_\tau \Gamma(\lambda; \tau, x; u, y) = \sum_{i=0}^2 f_i(\lambda; \tau, x) \partial_{x^i} \tilde{\Gamma}_i(\lambda; \tau, x; u, y),$$

where each  $\tilde{\Gamma}_i \in \tilde{\mathcal{G}}^s$ , and each  $\|f_i(\lambda; \cdot, \cdot)\|_{C^{1, s-2}(\mathcal{S})}$  is bounded over  $\lambda \in \Lambda$ .

For the most part of this proof we assume that the functions are parameterized by the set  $\Lambda = U(0) \subset \tilde{\mathcal{B}}$ , and therefore drop the argument  $\Lambda$  of the class  $\mathcal{G}^s$ . Notice that the families of fundamental solutions

$$\left\{ p(e^{h+h'}; \cdot, \cdot, \cdot, \cdot) \right\}_{h' \in U(0)}, \quad \text{and} \quad \left\{ q(e^{h+h'}; \cdot, \cdot, \cdot, \cdot) \right\}_{h' \in U(0)}$$

belong to  $\mathcal{G}^4$ .

We now derive some important properties of the classes of functions introduced above. Let us consider  $\Gamma_1, \Gamma_2 \in \mathcal{G}^s$  with  $s \geq 2$ , let us fix integers  $i, k, j, m$  satisfying  $0 \leq i + k + j + m \leq s + 1$  and  $(i + k) \vee (j + m) \leq s$ , and let  $f \in C^{1, (i+k-1)^+ \vee (j+m-1)^+}(\mathcal{S})$ . Then, for all  $\lambda_1, \lambda_2 \in \Lambda$ ,  $x_1, x_2 \in \mathbb{R}$ , and  $0 \leq$

$\tau_1 < \tau_2 \leq \bar{\tau}$ , we have:

$$\begin{aligned}
& \int_{\tau_1}^{\tau_2} \left| \int_{\mathbb{R}} \frac{\partial^{i+k}}{\partial x_2^i \partial y^k} \Gamma_2(\lambda_2; \tau_2, x_2; u, y) f(u, y) \frac{\partial^{j+m}}{\partial x_1^j \partial y^m} \Gamma_1(\lambda_1; u, y; \tau_1, x_1) dy \right| du \\
&= \int_{\tau_1}^{\frac{\tau_1+\tau_2}{2}} \left| \int_{\mathbb{R}} \frac{\partial^{m \wedge 1}}{\partial y^{m \wedge 1}} \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial y} - \frac{\partial}{\partial y} \right)^j \Gamma_1(\lambda_1; u, y; \tau_1, x_1) \cdot \right. \\
&\quad \left. \frac{\partial^{m-m \wedge 1}}{\partial y^{m-m \wedge 1}} \left[ f(u, y) \frac{\partial^{i+k}}{\partial x_2^i \partial y^k} \Gamma_2(\lambda_2; \tau_2, x_2; u, y) \right] dy \right| du \\
&+ \int_{\frac{\tau_1+\tau_2}{2}}^{\tau_2} \left| \int_{\mathbb{R}} \frac{\partial^{k \wedge 1}}{\partial y^{k \wedge 1}} \left( \frac{\partial}{\partial x_2} + \frac{\partial}{\partial y} - \frac{\partial}{\partial y} \right)^i \Gamma_2(\lambda_2; \tau_2, x_2; u, y) \cdot \right. \\
&\quad \left. \frac{\partial^{k-k \wedge 1}}{\partial y^{k-k \wedge 1}} \left[ f(u, y) \frac{\partial^{j+m}}{\partial x_1^j \partial y^m} \Gamma_1(\lambda_1; u, y; \tau_1, x_1) \right] dy \right| du \\
&\leq c_1 \|f\| \int_{\tau_1}^{\frac{\tau_1+\tau_2}{2}} \int_{\mathbb{R}} \frac{1}{(u - \tau_1)^{\frac{1+(j+m) \wedge 1}{2}}} \exp\left(-c_2 \frac{y^2}{u - \tau_1}\right) \cdot \\
&\quad \frac{1}{(\tau_2 - \tau_1)^{\frac{1+i+k+(j+m-1)^+}{2}}} \exp\left(-c_2 \frac{(x_2 - x_1 - y)^2}{\tau_2 - u}\right) dy du \\
&+ c_3 \|f\| \int_{\frac{\tau_1+\tau_2}{2}}^{\tau_2} \int_{\mathbb{R}} \frac{1}{(\tau_2 - u)^{\frac{1+(i+k) \wedge 1}{2}}} \exp\left(-c_2 \frac{y^2}{\tau_2 - u}\right) \cdot \\
&\quad \frac{1}{(\tau_2 - \tau_1)^{\frac{1+j+m+(i+k-1)^+}{2}}} \exp\left(-c_2 \frac{(x_1 - x_2 - y)^2}{u - \tau_1}\right) dy du \\
&= c_4 \|f\| (\tau_2 - \tau_1)^{\frac{1-i-j-k-m}{2}} \exp\left(-c_2 \frac{(x_2 - x_1)^2}{\tau_2 - \tau_1}\right). \tag{2.6}
\end{aligned}$$

To derive the above inequality, we integrated by parts in  $y$  and applied the Gaussian estimates (2.4), (??), then used Lemma 3 on p.15 of [10] to compute integrals of the form

$$\int_{\tau_1}^{\tau_2} \int_{\mathbb{R}} \frac{1}{(\tau_2 - u)^\alpha (u - \tau_1)^\beta} \exp\left(-c \frac{(x_2 - y)^2}{\tau_2 - u}\right) \exp\left(-c \frac{(y - x_1)^2}{u - \tau_1}\right) dy du.$$

Now, fix some  $s \geq 2$ , choose some  $\Gamma_1, \Gamma_2 \in \tilde{\mathcal{G}}^s$  and any family of functions  $\{f(\lambda; \cdot, \cdot) \in C^{1, s-1}(\mathcal{S})\}_{\lambda \in \Lambda}$ , and define

$$\begin{aligned}
& I[\Gamma_2, f, \Gamma_1](\lambda; \tau_2, x_2; \tau_1, x_1) \\
&:= \int_{\tau_1}^{\tau_2} \int_{\mathbb{R}} \Gamma_2(\lambda; \tau_2, x_2; u, y) f(\lambda, u, y) D_y \Gamma_1(\lambda; u, y; \tau_1, x_1) dy du.
\end{aligned}$$

We are going to show that

$$I[\Gamma_2, f, \Gamma_1](\lambda \tau_2, x_2; \tau_1, x_1) = \|f(\lambda)\| \Gamma_3(\lambda; \tau_2, x_2; \tau_1, x_1),$$

for some  $\Gamma_3 \in \tilde{\mathcal{G}}^{s-1}$ .

If, for some  $\lambda \in \Lambda$ ,  $f(\lambda) \equiv 0$ , the statement of the claim is obvious. Therefore we will assume that  $\|f(\lambda)\| > 0$ . The smoothness of  $\Gamma_3$  in  $(x_1, x_2)$ , and estimate

(2.4) follow from (2.5), after we integrate by parts in the definition of  $I$ . To obtain inequality (??), we only need to make a shift of the integration variable and proceed as in (2.5).

We now verify the second condition of Definition 2.2. Pick some  $g \in C_0^1(\mathbb{R})$ , and, assuming that  $s \geq 2$ , proceed as follows

$$\begin{aligned}
& \left| \int_{\mathbb{R}} g(x_1) \Gamma_3(\lambda; \tau_2, x_2; \tau_1, x_1) dx_1 \right| \\
&= \left| \int_{\mathbb{R}} g(x_1) \int_{\tau_1}^{\tau_2} \int_{\mathbb{R}} \Gamma_2(\lambda; \tau_2, x_2; u, y) \frac{f(\lambda; u, y)}{\|f(\lambda)\|} \right. \\
&\quad \left. (\partial_y + \partial_{x_1} - \partial_{x_1}) (\partial_y + \partial_{x_1} - \partial_{x_1} - 1) \Gamma_1(\lambda; u, y; \tau_1, x_1) dy du dx_1 \right| \\
&\leq c_6 \int_{\mathbb{R}} (|g(x_1)| + |g'(x_1)|) \int_{\tau_1}^{\tau_2} \int_{\mathbb{R}} \Gamma_2(\lambda; \tau_2, x_2; u, y) \frac{|f(\lambda; u, y)|}{\|f(\lambda)\|} \cdot \quad (2.7) \\
&\quad \sum_{i=0}^2 \sum_{j=0}^1 \left| (\partial_y + \partial_{x_1})^i \partial_{x_1}^j \Gamma_1(\lambda; u, y; \tau_1, x_1) \right| dy du dx_1 \\
&\leq c_7 \sqrt{\tau_2 - \tau_1},
\end{aligned}$$

which goes to zero as  $\tau_1 \rightarrow \tau_2$ . We integrated by parts in  $x_1$ , and applied estimates (2.4), (??) to obtain the above inequality. The interchangeability of integration and differentiation is justified by (2.5) (just notice that, as it is clear from the first line of (2.5), the integrals are, sometimes, understood as iterated rather than double integrals). The above estimate proves that  $\Gamma_3$  satisfies the second condition in Definition 2.2.

Now, assume that, in addition,  $\Gamma_1$  and  $\Gamma_2$  belong to  $\mathcal{G}^s$ . We claim that, in this case,  $\Gamma_3$  is in  $\mathcal{G}^{s-1}$ . We only need to verify the additional property in the Definition 2.3. Assume  $s - 1 \geq 2$ , then, using the expression for the  $\tau_2$  - derivatives of  $\Gamma_2$ , and the fact that  $\Gamma_3 \in \tilde{\mathcal{G}}^{s-1}$ , we obtain the following

$$\begin{aligned}
& \frac{\partial}{\partial \tau_2} \left[ \int_{\tau_1}^{\tau_2} \int_{\mathbb{R}} \Gamma_2(\lambda; \tau_2, x_2; u, y) \frac{f(\lambda; u, y)}{\|f(\lambda)\|} D_y \Gamma_1(\lambda; u, y; \tau_1, x_1) dy du \right] \\
&= c_6 \frac{f(\lambda; \tau_2, x_2)}{\|f(\lambda)\|} D_{x_2} \Gamma_1(\lambda; \tau_2, x_2; \tau_1, x_1) + \sum_{i=0}^2 f_i(\lambda; \tau_2, x_2) \\
&\quad \partial_{x_2}^i \int_{\tau_1}^{\tau_2} \int_{\mathbb{R}} \tilde{\Gamma}_i(\lambda; \tau_2, x_2; u, y) \frac{f(\lambda; u, y)}{\|f(\lambda)\|} D_y \Gamma_1(\lambda; u, y; \tau_1, x_1) dy du
\end{aligned}$$

where each  $f_i(\lambda; \cdot, \cdot)$  is in  $C^{1, s-2}(\mathcal{S})$ , and the  $\tilde{\Gamma}_i$ 's belong to  $\tilde{\mathcal{G}}^s$ . The above decomposition completes the proof of the claim:  $\Gamma_3 \in \mathcal{G}^{s-1}$ .

It is easy to see, integrating by parts, that the operator  $J$  defined by

$$\begin{aligned}
& J[\Gamma_2, f, \Gamma_1](\lambda; \tau_2, x_2; \tau_1, x_1) \\
&:= \int_{\tau_1}^{\tau_2} \int_{\mathbb{R}} D_y \Gamma_2(\lambda; \tau_2, x_2; u, y) f(\lambda, u, y) \Gamma_1(\lambda; u, y; \tau_1, x_1) dy du
\end{aligned}$$

has the same properties as  $I$ .

Similarly, for any  $\{f(\lambda; \cdot, \cdot) \in C^{1,2}(\mathcal{S})\}_{\lambda \in \Lambda}$ , and  $\Gamma_1, \Gamma_2 \in \mathcal{G}^2$ , we define the function  $K[\Gamma_2, f, \Gamma_1]$  by:

$$\begin{aligned} K[\Gamma_2, f, \Gamma_1](\lambda; \tau_2, x_2; \tau_1, x_1) \\ := \int_{\tau_1}^{\tau_2} \int_{\mathbb{R}} \Gamma_2(\lambda; \tau_2, x_2; u, y) e^{y-x_1} f(\lambda, u, y) \Gamma_1(\lambda; u, y; \tau_1, x_1) dy du, \end{aligned}$$

and, using (2.5) and (2.6), we obtain the estimate:

$$|\partial_{\tau_2} K| + \sum_{j=0}^2 \left| \partial_{x_2}^j K \right| \leq c_8 \|f(\lambda)\| (\tau_2 - \tau_1)^{-3/2} \exp\left(-c_9 \frac{(x_2 - x_1)^2}{\tau_2 - \tau_1}\right), \quad (2.8)$$

where the constants  $c_8, c_9$  depend on  $\Gamma_1$  and  $\Gamma_2$ , but not on  $\lambda$ .

We now proceed with the proof of the proposition. Writing the initial value problem (2.1) twice, first with  $e^h$ , and then with  $e^{h+h'}$ , and subtracting one from another, we can, formally, apply the Feynman-Kac formula and obtain

$$\begin{aligned} \mathbf{F}_{\varepsilon}(h + h')(\tau, x) &= \mathbf{F}_{\varepsilon}(h)(\tau, x) \\ &+ \frac{1}{2} \int_0^{\tau} \int_{\mathbb{R}} p(e^{h+h'}; \tau, x; u, y) e^{h(u,y)+y} (e^{h'(u,y)} - 1) q(e^h; 0, 0; u, y) dy du. \end{aligned} \quad (2.9)$$

This representation follows from the uniqueness of weak solution of (2.1), see, for example, [7] for details. Applying the same technique to the fundamental solution  $p$ , we get

$$\begin{aligned} \Delta p(\tau, x; u, y) &:= p(e^{h+h'}; \tau, x; u, y) - p(e^h; \tau, x; u, y) \\ &= \int_u^{\tau} \int_{\mathbb{R}} p(e^{h+h'}; \tau, x; v, z) e^{h(v,z)} (e^{h'(v,z)} - 1) D_z p(e^h; v, z; u, y) dz dv \\ &= I \left[ \left\{ p(e^{h+h'}) \right\}_{h' \in U(0)}, \left\{ e^h (e^{h'} - 1) \right\}_{h' \in U(0)}, \left\{ p(e^h) \right\}_{h' \in U(0)} \right] (h'; \tau, x; u, y) \end{aligned} \quad (2.10)$$

Since all the families of functions considered in this part of the proof are parameterized by  $h' \in U(0)$ , we use the shorter notation  $f(h')$  instead of  $\{f(h')\}_{h' \in U(0)}$ , for the arguments of operator  $I$ .

We define  $\Delta q$  in a similar way. Next we rewrite (2.8) as

$$\mathbf{F}_{\varepsilon}(h + h') = \mathbf{F}_{\varepsilon}(h) + \mathbf{F}_{\varepsilon}'(h)[h'] + r_1 + r_2,$$

with

$$\begin{aligned} r_1(\tau, x) &= \frac{1}{2} \int_0^{\tau} \int_{\mathbb{R}} p(e^{h+h'}; \tau, x; u, y) e^{h(u,y)+y} \left( e^{h'(u,y)} - 1 - h'(u, y) \right) \cdot \\ &\quad q(e^h; 0, 0; u, y) dy du \\ &= \frac{1}{2} K \left[ p(e^{h+h'}), e^h (e^{h'} - 1 - h'), q(e^h) \right] (h'; \tau, x; 0, 0), \end{aligned}$$

and

$$\begin{aligned}
r_2(\tau, x) &= \frac{1}{2} \int_0^\tau \int_{\mathbb{R}} \Delta p(\tau, x; u, y) e^{h(u, y) + y} h'(u, y) \cdot \\
&\quad q(e^h; 0, 0; u, y) dy du \\
&= \frac{1}{2} K \left[ I \left[ p(e^{h+h'}), e^h(e^{h'} - 1), p(e^h) \right], e^h h', q(e^h) \right] (h'; \tau, x; 0, 0).
\end{aligned}$$

Because of the properties of the operator  $I$  derived earlier, it is easy to see that the function  $I \left[ p(e^{h+h'}), e^h(e^{h'} - 1), p(e^h) \right]$  belongs to  $(e^{h'} - 1) \cdot \mathcal{G}^3$ . Therefore, using estimate (2.7), we have immediately that for  $i = 1, 2$ ,

$$\|r_i\|_{\tilde{\mathcal{W}}_\varepsilon} \leq c_{10} \|h'\|_{\tilde{\mathcal{B}}}^2$$

and this implies that  $\mathbf{F}_\varepsilon$  is Fréchet differentiable, with Fréchet derivative as given in the statement of the proposition. The fact that  $\mathbf{F}_\varepsilon'(h)[\cdot]$  is bounded on the unit ball of  $\tilde{\mathcal{B}}$  follows, again, from (2.7).

We now compute the Fréchet derivative of  $\mathbf{F}_\varepsilon'(\cdot)$  using the same technique as in the first part of the proof.

We fix  $h \in \tilde{\mathcal{B}}$  and we consider families of functions parameterized by  $(h', h'') \in \Lambda := \tilde{\mathcal{B}} \times U(0)$ . We redefine  $\Delta p$ , using  $h''$  instead of  $h'$  in (2.9). Then we have

$$\begin{aligned}
(\mathbf{F}_\varepsilon'(h + h'') - \mathbf{F}_\varepsilon'(h)) [h'](\tau, x) &= \\
&\frac{1}{2} \int_0^\tau \int_{\mathbb{R}} h'(u, y) e^{h(u, y) + y} \Delta p(\tau, x; u, y) q(e^h; 0, 0; u, y) dy du \\
&+ \frac{1}{2} \int_0^\tau \int_{\mathbb{R}} h'(u, y) e^{h(u, y) + y} p(e^h; \tau, x; u, y) \Delta q(0, 0; u, y) dy du \quad (2.11) \\
&+ \frac{1}{2} \int_0^\tau \int_{\mathbb{R}} h'(u, y) e^{h(u, y) + y} \Delta p(\tau, x; u, y) \Delta q(0, 0; u, y) dy du \\
&+ \frac{1}{2} \int_0^\tau \int_{\mathbb{R}} h'(u, y) e^{h(u, y) + y} \left( e^{h''(u, y)} - 1 \right) p(e^{h+h''}; \tau, x; u, y) \cdot \\
&\quad q(e^{h+h''}; 0, 0; u, y) dy du
\end{aligned}$$

Next, we decompose the first integral in (2.10)

$$\begin{aligned}
&\int_0^\tau \int_{\mathbb{R}} h'(u, y) e^{h(u, y) + y} \Delta p(\tau, x; u, y) q(e^h; 0, 0; u, y) dy du \\
&= K \left[ I \left[ p(e^h), h'' e^h, p(e^h) \right], e^h h', q(e^h) \right] (h', h''; \tau, x; 0, 0) \\
&\quad + K \left[ I \left[ p(e^h), e^h(e^{h''} - 1 - h''), p(e^h) \right], e^h h', q(e^h) \right] (h', h''; \tau, x; 0, 0) \\
&\quad + K \left[ I \left[ I \left[ p(e^{h+h''}), e^h(e^{h''} - 1), p(e^h) \right], e^h(e^{h''} - 1), p(e^h) \right], \right. \\
&\quad \quad \left. e^h h', q(e^h) \right] (h', h''; \tau, x; 0, 0).
\end{aligned}$$

The first term in the right hand side of the above expression is linear in  $h''$ . It is the first component of  $\mathbf{F}_\varepsilon''$ . Using the properties of the operator  $I$ , we conclude

that

$$\begin{aligned} & I \left[ p(e^h), e^h(e^{h''} - 1 - h''), p(e^h) \right] \\ & + I \left[ I \left[ p(e^{h+h''}), e^h(e^{h''} - 1), p(e^h) \right], e^h(e^{h''} - 1), p(e^h) \right] = \|h'\|_{\tilde{\mathcal{B}}} \|h''\|_{\tilde{\mathcal{B}}}^2 \Gamma, \end{aligned}$$

where  $\Gamma \in \mathcal{G}^2$ . Therefore, using estimate (2.7), we conclude that the  $\|\cdot\|_{\tilde{\mathcal{W}}_\varepsilon}$  norms of the last two terms in the right hand side of (2.11) are bounded by a constant times  $\|h'\|_{\tilde{\mathcal{B}}} \|h''\|_{\tilde{\mathcal{B}}}^2$ .

A similar decomposition holds true for the second integral in the right hand side of (2.10) provided the operator  $I$  is replaced by  $J$ . Moreover, the  $\|\cdot\|_{\tilde{\mathcal{W}}_\varepsilon}$  - norms of last two integrals in (2.10) are also bounded by a constant times  $\|h'\|_{\tilde{\mathcal{B}}} \|h''\|_{\tilde{\mathcal{B}}}^2$ : to see this, recall (2.9) and write its analog for  $\Delta q$ , then apply the properties operators  $I$  and  $J$ , and use estimate (2.7). This yields the existence of  $\mathbf{F}_\varepsilon''(h)$ , as given in the proposition.

To show the continuity of the second derivative, fix any  $h'$  and  $h''$  in  $\tilde{\mathcal{B}}$  and consider any  $\Delta h \in U(0)$ . We only show the continuity of the first component of  $\mathbf{F}_\varepsilon''(\cdot)[h', h'']$  at  $h$ , uniformly over  $h'$  and  $h''$  in a bounded set. The proof for the second component is the same. We introduce the difference

$$\begin{aligned} \Delta K & := K \left[ I \left[ p(e^{h+\Delta h}), h''e^{h+\Delta h}, p(e^{h+\Delta h}) \right], e^{h+\Delta h}h', q(e^h + \Delta h) \right] \\ & \quad - K \left[ I \left[ p(e^h), h''e^h, p(e^h) \right], e^h h', q(e^h) \right] = \\ K & \left[ I \left[ I \left[ p(e^{h+\Delta h}), e^h(e^{\Delta h} - 1), p(e^h) \right], h''e^{h+\Delta h}, p(e^{h+\Delta h}) \right], e^{h+\Delta h}h', q(e^h + \Delta h) \right] \\ & \quad + K \left[ I \left[ p(e^h), h''e^h(e^{\Delta h} - 1), p(e^{h+\Delta h}) \right], e^{h+\Delta h}h', q(e^h + \Delta h) \right] \\ & \quad + K \left[ I \left[ p(e^h), h''e^h, I \left[ p(e^{h+\Delta h}), e^h(e^{\Delta h} - 1), p(e^h) \right] \right], e^{h+\Delta h}h', q(e^h + \Delta h) \right] \\ & \quad + K \left[ I \left[ p(e^h), h''e^h, p(e^h) \right], h'e^h(e^{\Delta h} - 1), q(e^h + \Delta h) \right] \\ & \quad + K \left[ I \left[ p(e^h), h''e^h, p(e^h) \right], h'e^h, -J \left[ q(e^{h+\Delta h}), e^h(e^{\Delta h} - 1), q(e^h) \right] \right]. \end{aligned}$$

And, as before, using the properties of  $I$ ,  $J$  and  $K$ , we conclude that

$$\|\Delta K\|_{\tilde{\mathcal{W}}_\varepsilon} \leq c_{11} \|h'\|_{\tilde{\mathcal{B}}} \|h''\|_{\tilde{\mathcal{B}}} \|\Delta h\|_{\tilde{\mathcal{B}}},$$

which completes the proof of the proposition.  $\blacksquare$

Recall that the price  $C_t(T, x)$  at time  $t$  of an European call option is given by  $w(\tilde{a}^2; T - t, x + \log S_t)$ , where  $w(\tilde{a}^2; \cdot, \cdot)$  is the solution of (2.1). Therefore, in order to get to the Fréchet differentiability of the price of a call option from the above result, we will need to compose  $\mathbf{F}_\varepsilon$  with another mapping. This justifies the introduction, for each  $T \in (\bar{\varepsilon}, \bar{\tau}]$  and  $x \in \mathbb{R}$  of the mapping

$$\delta_{T,x} : [0, T - \bar{\varepsilon}] \times \tilde{\mathcal{W}}_\varepsilon \times \mathbb{R} \hookrightarrow \mathbb{R}$$

defined by

$$\delta_{T,x}(t, w, y) = w(T - t, x + y).$$

We have:

- Proposition 2.4.** (1) For each  $(w, y) \in \tilde{\mathcal{W}}_\varepsilon \times \mathbb{R}$ ,  $\delta_{T,x}(\cdot, w, y)$  is continuously differentiable, and the partial derivative  $\partial\delta_{T,x}/\partial t$  is a continuous functional on  $[0, T - \varepsilon] \times \tilde{\mathcal{W}}_\varepsilon \times \mathbb{R}$ .
- (2) For each  $t \in [0, T - \varepsilon]$ ,  $\delta_{T,x}(t, \cdot, \cdot)$  is twice Fréchet differentiable and for any  $w, w', w'' \in \tilde{\mathcal{W}}_\varepsilon$  and  $y, y, y'' \in \mathbb{R}$ , its derivatives satisfy

$$\delta'_{T,x}(t, w, y)[w', y'] = w'(T - t, x + y) + y' \partial_x w(T - t, x + y)$$

and

$$\begin{aligned} \delta''_{T,x}(t, w, y)[(w', y'), (w'', y'')] &= y'' \partial_x w'(T - t, x + y) + y' \partial_x w''(T - t, x + y) \\ &\quad + y' y'' \partial_{x^2}^2 w(T - t, x + y). \end{aligned}$$

Moreover,  $\delta'_{T,x}$  and  $\delta''_{T,x}$  are continuous operators from  $[0, T - \varepsilon] \times \tilde{\mathcal{W}}_\varepsilon \times \mathbb{R}$  into  $\tilde{\mathcal{W}}_\varepsilon^* \times \mathbb{R}$  and  $L(\tilde{\mathcal{W}}_\varepsilon \times \mathbb{R}, \tilde{\mathcal{W}}_\varepsilon^* \times \mathbb{R})$  respectively.

*Proof.* Let us fix  $(w, y) \in \tilde{\mathcal{W}}_\varepsilon \times \mathbb{R}$ . Then, for any  $t \in [0, T - \varepsilon]$ , we have

$$\frac{\partial}{\partial t} \delta_{T,x}(t, w, y) = -\partial_\tau w(T - t, x + y).$$

We first show that this functional is continuous in  $(t, w, y) \in [0, T - \varepsilon] \times \tilde{\mathcal{W}}_\varepsilon \times \mathbb{R}$ . Consider any  $(t', w', y') \in [0, T - \varepsilon] \times \tilde{\mathcal{W}}_\varepsilon \times \mathbb{R}$ , then

$$\begin{aligned} &|\partial_\tau w(T - t, x + y) - \partial_\tau w'(T - t', x + y')| = \\ &\quad |\partial_\tau w(T - t, x + y) - \partial_\tau w(T - t', x + y')| \quad (2.12) \\ &\quad + |\partial_\tau w(T - t', x + y') - \partial_\tau w'(T - t', x + y')| \end{aligned}$$

The first difference in the right hand side above can be made as small as we want by choosing  $(t, x)$  and  $(t', x')$  close enough. The second difference is bounded by  $\|w - w'\|_{\tilde{\mathcal{W}}_\varepsilon}$ . This implies continuity of the partial derivative  $\partial\delta_{T,x}/\partial t$ , proving the first statement of the proposition.

Let us now compute the derivatives of  $\delta_{T,x}$ . We will keep  $(t, w, y) \in [0, T - \varepsilon] \times \tilde{\mathcal{W}}_\varepsilon \times \mathbb{R}$  fixed, and consider  $(w', y') \in U(0) \subset \tilde{\mathcal{W}}_\varepsilon \times \mathbb{R}$ , where  $U(0)$  is a neighborhood of zero. Notice that

$$\begin{aligned} &\delta_{T,x}(t, w + w', y + y') - \delta_{T,x}(t, w, y) = \\ &\quad w(T - t, x + y + y') - w(T - t, x + y) + w'(T - t, x + y + y') = \\ &\quad y' \partial_x w(T - t, x + y) + \bar{o}(y') + w'(T - t, x + y) + y' \partial_x w'(T - t, x + y + \xi y'), \end{aligned}$$

for some  $\xi \in [0, 1]$ , and that

$$|\bar{o}(y') + y' \partial_x w'(T - t, x + y + \xi y')| = \bar{o}\left(\sqrt{|y'|^2 + \|w'\|_{\tilde{\mathcal{W}}_\varepsilon}^2}\right),$$

Therefore, we have obtained the expression for  $\tilde{\mathbf{C}}'_{T,x}$ , as given in the proposition.

Now consider  $(w', y') \in \tilde{\mathcal{W}}_\varepsilon \times \mathbb{R}$  and  $(w'', y'') \in U(0) \subset \tilde{\mathcal{W}}_\varepsilon \times \mathbb{R}$ , the rest of parameters being fixed. Then:

$$\begin{aligned} & (\delta'_{T,x}(t, w + w'', y + y'') - \delta'_{T,x}(t, w, y)) [w', y'] = \\ & w'(T - t, x + y + y'') - w'(T - t, x + y) + y' \partial_x w''(T - t, x + y + y'') \\ & \quad + y' \partial_x w(T - t, x + y + y'') - y' \partial_x w(T - t, x + y) = \\ & y'' \partial_x w'(T - t, x + y) + (y'')^2 \partial_{x^2}^2 w'(T - t, x + y + \xi y'') + y' \partial_x w''(T - t, x + y) \\ & \quad + y' y'' \partial_{x^2}^2 w''(T - t, x + y + \xi' y'') + y' (y'' \partial_{x^2}^2 w(T - t, x + y) + \bar{o}(y'')), \end{aligned}$$

for some  $\xi, \xi' \in [0, 1]$ . Again, noticing that

$$\begin{aligned} & |(y'')^2 \partial_{x^2}^2 w'(T - t, x + y + \xi y'') + y' y'' \partial_{x^2}^2 w''(T - t, x + y + \xi' y'') + y' \bar{o}(y'')| \\ & \leq \sqrt{|y'|^2 + \|w'\|_{\tilde{\mathcal{W}}_\varepsilon}^2} \bar{o} \left( \sqrt{|y''|^2 + \|w''\|_{\tilde{\mathcal{W}}_\varepsilon}^2} \right). \end{aligned}$$

we get the desired expression for  $\delta''_{T,x}$ .

In order to show the continuity of  $\delta'_{T,x}$  and  $\delta''_{T,x}$ , we fix  $(w', y'), (w'', y'') \in \tilde{\mathcal{W}}_\varepsilon \times \mathbb{R}$ , and we prove the continuity of  $\delta'_{T,x}(\cdot, \cdot, \cdot)[w', h']$  and  $\delta''_{T,x}(\cdot, \cdot, \cdot)[(w', h'), (w'', h'')]$  by, essentially, repeating the argument of (??). Finally, notice that the continuity is uniform over  $(w', y'), (w'', y'')$  when they are restricted to a bounded set. ■

Now, consider the composition of the two operators introduced above. For each  $T \in [\bar{\varepsilon}, \bar{\tau}]$  and  $x \in \mathbb{R}$ , we have

$$\begin{aligned} \mathbf{C}_{T,x} &: [0, T - \bar{\varepsilon}] \times \tilde{\mathcal{B}} \times \mathbb{R} \rightarrow \mathbb{R} \\ \mathbf{C}_{T,x}(t, h, y) &= \delta_{T,x}(t, \mathbf{F}_\varepsilon(h), y) \end{aligned}$$

As a composition of twice Fréchet differentiable operators,  $\mathbf{C}_{T,x}(t, \cdot, \cdot)$  is, clearly twice Fréchet differentiable, for each  $t \in [0, T - \bar{\varepsilon}]$ . Due to the continuity of  $\mathbf{F}_\varepsilon''(\cdot)$ ,  $\delta'_{T,x}(\cdot, \cdot, \cdot)$  and  $\delta''_{T,x}(\cdot, \cdot, \cdot)$ , the Fréchet derivatives of  $\mathbf{C}_{T,x}(t, h, y)$  are also continuous in  $(t, h, y)$ . Finally,  $\mathbf{C}_{T,x}$ , clearly, satisfies the first statement of Proposition 2.4. Thus, applying the chain rule we obtain the following

**Proposition 2.5.** *For each  $t \in [0, T - \bar{\varepsilon}]$ , functional  $\mathbf{C}_{T,x}(t, \cdot, \cdot)$  is twice Fréchet differentiable, such that, for any  $h, h', h'' \in \tilde{\mathcal{B}}$  and  $y, y, y'' \in \mathbb{R}$ , we have*

$$\mathbf{C}'_{T,x}(t, h, y)[h', y'] = \mathbf{F}_\varepsilon'(h)[h'](T - t, x + y) + y' \partial_x \mathbf{F}_\varepsilon(h)(T - t, x + y),$$

and

$$\begin{aligned} \mathbf{C}''_{T,x}(t, h, y)[(h', y'), (h'', y'')] &= \mathbf{F}_\varepsilon''(h)[h', h''](T - t, x + y) \\ & \quad + y'' \partial_x \mathbf{F}_\varepsilon'(h)[h'](T - t, x + y) + y' \partial_x \mathbf{F}_\varepsilon'(h)[h''](T - t, x + y) \\ & \quad + y' y'' \partial_{x^2}^2 \mathbf{F}_\varepsilon(h)(T - t, x + y), \end{aligned}$$

and  $\mathbf{C}'_{T,x}$ ,  $\mathbf{C}''_{T,x}$  are continuous operators from  $[0, T - \bar{\varepsilon}] \times \tilde{\mathcal{B}} \times \mathbb{R}$  into  $\tilde{\mathcal{B}}^* \times \mathbb{R}$  and  $L(\tilde{\mathcal{B}} \times \mathbb{R}, \tilde{\mathcal{B}}^* \times \mathbb{R})$  respectively.

### 3. Using Itô's Formula in Infinite Dimension

The purpose of this section is to extend the proof of the semi-martingale property given in [3] to the case of infinitely many driving Wiener processes.

We denote by  $B$  the cylindrical Brownian motion constructed on the canonical cylindrical Gaussian measure of some separable Hilbert space  $\tilde{\mathcal{H}}$ . The reader can think of  $\tilde{\mathcal{H}} = l_2$  - the space of square - summable sequences but the specific form of this Hilbert space is totally irrelevant for what we are about to do.

The first step is to construct a Hilbert subspace of  $\tilde{\mathcal{B}}$ . For each functions  $f$  and  $g$  with enough derivatives square integrables and for each non-negative integers  $k$  and  $m$ , we define the scalar product

$$\begin{aligned} \langle f, g \rangle_{\tilde{\mathbf{W}}^{k,m}(\mathcal{S})} &= \sum_{i=0}^k \partial_{\tau^i} f(0,0) \partial_{\tau^i} g(0,0) + \sum_{j=0}^m \partial_{x^j} f(0,0) \partial_{x^j} g(0,0) \\ &+ \int_{\mathcal{S}} \nabla (\partial_{\tau^k} f(\tau, x)) \nabla (\partial_{\tau^k} g(\tau, x)) + \nabla (\partial_{x^m} f(\tau, x)) \nabla (\partial_{x^m} g(\tau, x)) dx d\tau. \end{aligned}$$

Now we fix a compact set  $\mathcal{K}$  contained in  $\mathcal{S}$  and containing the origin  $(0,0)$ , and we consider the space of functions on  $\mathcal{S}$  which are constant outside  $\mathcal{K}$ , namely whose derivatives vanish outside  $\mathcal{K}$ . For the sake of definiteness we will choose  $\mathcal{K} = [0, \tau] \times [-M, M]$  for a positive (large) number  $M$ . Equipped with the scalar product  $\langle \cdot, \cdot \rangle_{\tilde{\mathbf{W}}^{1,5}(\mathcal{S})}$ , defined above, this space of functions (more precisely of equivalence classes of functions) is a Hilbert space which we denote  $\mathcal{H}$ . It is clearly contained in  $\tilde{\mathcal{B}}$ . Define by  $\mathcal{B}$ , the completion of  $\mathcal{H}$  in the  $\|\cdot\|_{C^{1,5}(\mathcal{S})}$  norm. Thus, the pair  $(\mathcal{H}, \mathcal{B})$  forms a *conditional Banach Space*.

Clearly,  $\mathcal{B}$  is a subspace of  $\tilde{\mathcal{B}}$ , and therefore, Proposition 2.5 holds for the restriction of  $\mathbf{C}_{T,x}$  to  $\mathcal{B}$  as well.

For any given real separable Banach space  $\mathcal{G}$  we denote by  $\mathfrak{L}(\mathcal{G})$  the space of all non-anticipative random processes in  $\mathcal{G}$  (measurable mappings  $X : \Omega \times [0, \infty) \rightarrow \mathcal{G}$ ), such that

$$\mathbb{E} \int_0^t \|X_u\|_{\mathcal{G}}^2 du < \infty,$$

for all  $t \geq 0$ . Where  $\mathcal{G}$  is a Banach space. Also, we denote by  $L_2(\mathcal{H})$  the space of all Hilbert-Schmidt operators on  $\mathcal{H}$ .

Next, we choose  $\alpha \in \mathfrak{L}(\mathcal{B})$  and  $\beta \in \mathfrak{L}(L_2(\tilde{\mathcal{H}}, \mathcal{H}))$ , and we model dynamics of  $h_t$ , the logarithm of the squared local volatility at time  $t$ ,  $\tilde{\alpha}_t^2$ , by the infinite dimensional Itô's stochastic differential

$$dh_t = \alpha_t dt + \beta_t dB_t,$$

which together with an initial condition  $h_0 \in \mathcal{B}$ , defines a random process in  $\mathcal{B}$ . Also, we assume the following dynamics for the logarithm of the underlying

$$d \log S_t = -\frac{1}{2} \sigma_t^2 dt + \sigma_t \langle e_1, dB_t \rangle, \quad \log S_0, \quad (3.1)$$

where  $\sigma$  is  $\mathbb{R}$ -valued random process with  $\mathbb{E} \int_0^t \sigma_u^2 du < \infty$  almost surely, for any  $t \geq 0$ , and  $e_1 \in \tilde{\mathcal{H}}$  is a fixed unit vector.

Now, thanks to Proposition 2.5, we can apply Itô's formula (see, for example, [13], p. 200) to  $(\mathbf{C}_{T,x}(t, h_t, \log S_t))_{t \in [0, T-\bar{\varepsilon}]}$ . We get that for any  $T \in (\bar{\varepsilon}, \bar{\tau}]$  and  $x \in \mathbb{R}$ , we have, almost surely, for all  $t \in [0, T - \bar{\varepsilon}]$ ,

$$\begin{aligned} \mathbf{C}_{T,x}(t, h_t, \log S_t) &= \mathbf{C}_{T,x}(0, h_0, \log S_0) \\ &+ \int_0^t \left( \frac{\partial}{\partial t} \mathbf{C}'_{T,x}(u, h_u, \log S_u) + \mathbf{C}'_{T,x}(u, h_u, \log S_u) [\alpha_u, -\frac{1}{2} \sigma_u^2] \right. \\ &\quad \left. + \frac{1}{2} \text{Tr} \left( (\beta_u, \sigma_u e_1)^* \circ \mathbf{C}''_{T,x}(u, h_u, \log S_u) \circ (\beta_u, \sigma_u e_1) \right) \right) du \\ &\quad + \int_0^t \mathbf{C}'_{T,x}(u, h_u, \log S_u) \circ (\beta_u, \sigma_u e_1) dB_u \end{aligned}$$

where  $\mathbf{C}'_{T,x}$  and  $\mathbf{C}''_{T,x}$  are given in Proposition 2.5.

*Remark 3.1.* Since  $\bar{\varepsilon}$  can be made as small as we want, the above representation holds for any  $T \in (0, \bar{\tau}]$ , and all  $t \in [0, T)$ . Then, since we choose  $\bar{\tau}$  as large as we want, the above representation holds for any  $T > 0$ , and all  $t \in [0, T)$ .

We now restate the above result after choosing a complete orthonormal basis  $\{e_n\}_n$  of  $\tilde{\mathcal{H}}$ . Notice that without any loss of generality we can assume that the first element  $e_1$  of this basis is in fact the unit vector entering the equation for the dynamics (3.1) of the logarithm of the underlying spot price. As it should be clear, fixing a basis is essentially assuming that  $\tilde{\mathcal{H}} = l_2$ . If we consider that  $\beta_t$  is given by the sequence  $\{\beta_t^n(\cdot, \cdot) \in \mathcal{H}\}_{n=1}^\infty$  of its components on the basis vectors then we have:

Then, we have

**Theorem 3.2.** *For any  $T > 0$  and  $x \in \mathbb{R}$ , we have, almost surely, for all  $t \in [0, T)$ ,*

$$\begin{aligned} \mathbf{C}_{T,x}(t, h_t, \log S_t) &= \mathbf{C}_{T,x}(0, h_0, \log S_0) \\ &+ \int_0^t \left[ \mathbf{F}'_{\bar{\varepsilon}}(h_u) [\alpha_u] - \frac{1}{2} \sigma_u^2 \partial_x \mathbf{F}_{\bar{\varepsilon}}(h_u) - \partial_\tau \mathbf{F}_{\bar{\varepsilon}}(h_u) + \frac{1}{2} \sigma_u^2 \partial_{x^2}^2 \mathbf{F}_{\bar{\varepsilon}}(h_u) \right. \\ &\quad \left. + \sigma_u \partial_x \mathbf{F}'_{\bar{\varepsilon}}(h_u) [\beta_u^1] + \frac{1}{2} \sum_{n=1}^\infty \mathbf{F}''_{\bar{\varepsilon}}(h_u) [\beta_u^n, \beta_u^n] \right] (T - u, x + \log S_t) du \\ &+ \int_0^t \left[ \sum_{n=1}^\infty \mathbf{F}'_{\bar{\varepsilon}}(h_u) [\beta_u^n] + \sigma_u \partial_x \mathbf{F}_{\bar{\varepsilon}}(h_u) \right] (T - u, x + \log S_u) dB_u^n, \end{aligned}$$

if we use the notation  $\{B^n\}_n$  for the sequence of independent standard one-dimensional Brownian motions  $B_t^n = \langle e_n, B_t \rangle$ .  $\mathbf{F}'_{\bar{\varepsilon}}$  and  $\mathbf{F}''_{\bar{\varepsilon}}$  are given in Proposition 2.1.

This is the infinite dimensional version of the semi-martingale result of [3].

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