The Clean Development Mechanism and CER Price Formation in the Carbon Emission Markets

René Carmona and Max Fehr

Abstract. In this paper, we propose an equilibrium model for the joint price formation of allowances issued by regulators in the framework of a cap-and-trade scheme such as the European Union Emissions Trading Scheme (EU ETS) and offset certificates such as CERs generated within the framework of the Clean Development Mechanism (CDM) or the Joint Implementation (JI) of the Kyoto Protocol. The main thrust of the paper is to derive equilibrium price formulas which explain the spreads observed historically.

Keywords. Environmental risk, energy economics, cap-and-trade, carbon markets, Kyoto protocol.

1. Introduction

The Kyoto protocol offers three flexible mechanisms to meet pollution targets. The first is emission trading. Based on the success of the SO_x and NO_x markets set up in the US in the 1980s and the subsequent acid rain program, several regional voluntary markets have sprouted with various degrees of success. The most ambitious of these attempts is the recent Regional Greenhouse Gas Initiative (RGGI,). While limited to electric power plants in Northeastern and Mid-Atlantic states, it is the first mandatory market-based effort in the United States to reduce greenhouse gas emissions: its goal is to reduce CO_2 emissions from these installations by 10% by 2018. However, the European Union Emission Trading Scheme (EU ETS) is the largest mandatory market of emission allowances. It was set up by *Directive 2003/87/ec* of the European parliament as a market mechanism to help its participants meet the Green House Gas (GHG) emission reduction targets set

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within the Kyoto protocol signed by its members. The gory details of EU ETS can be found in the original directive [1] and a comprehensive presentation can be found in the edited volume [8]. A number of alternative approaches to GHG mitigation are under consideration in the United States and a measure proposing a national cap-and-trade system (the American Clean Energy and Security Act of 2009 also known and the Waxman-Makey bill) was recently voted by the US House of Representatives, and is soon to be considered by the Senate. If such a legislation is voted, it is highly likely that countries like Canada, Japan, Australia, New Zeland, etc. will follow suit and the *carbon markets* could become some of the largest and most active financial markets in the near future.

The other flexible mechanisms proposed by the Kyoto protocol are the Clean Development Mechanism (CDM) and the Joint Implementation (JI). They differ in that they apply to different geographic regions and are governed by different rules and different bodies. For example, JI status can be given to projects located in economies in transition while CDM status is granted to projects in developing countries. However, because of their strong similarities, we will only refer to the Clean Development Mechanism in this paper.

The CDM allows emission-reduction (or emission removal) projects in developing countries to earn Certified Emission Reduction (CER) credits, each equivalent to one ton of CO_2. These CERs can be traded and sold, and used by industrialized countries to a meet a part of their emission reduction targets under the Kyoto Protocol. The mechanism stimulates sustainable development and emission reductions, while giving industrialized countries some flexibility in how they meet their emission reduction targets.

The projects must qualify through a rigorous and public registration and issuance process designed to ensure real, measurable and verifiable emission reductions that are additional to what would have occurred without the project. The mechanism is overseen by the CDM Executive Board which ultimately reports to the countries that have ratified the Kyoto Protocol. In order to be considered for registration, a project must first be approved by the Designated National Authorities (DNA). Operational since the beginning of 2006, the mechanism has already registered more than 1,000 projects and is anticipated to produce CERs amounting to more than 2.7 billion tonnes of CO_2 equivalent in the first commitment period of the Kyoto Protocol, $2008-2012$.

A general description of the framework of JI and CDM can be found in [13] and [14] and the practical elements of their financial implications in [9]., Equilibrium models for simple forms of cap-and-trade schemes not including the trading of offsets generated by mechanisms like the CDM or JI have been studied by many authors since the groundbreaking work of Montgomery [10] in the deterministic case. See for example [3, 5, 4] for example. The specific issues related to banking of allowances from one compliance period to the next was already studied in [7, 12] and [11] for example.

We close this introduction with a short survey of the contents of the paper.

Section 2 presents our mathematical model of the economy. The economic agents we consider are firms or installations covered by cap-and-trade regulations. The firms are involved in different markets, and these markets are subject to regulations with different (non-overlaping) compliance periods. They produce and sell goods. They are risk neutral as they aim at maximizing their expected terminal wealth, using linear utility. They face an inelastic demand. This assumption may be restrictive for some markets, but it will come handy with our equilibrium analysis. Production processes are the source of an externality, say emissions of GHGs, and market mechanisms in the form of cap-and-trade regulations are imposed to control and possibly reduce these emissions. Most inputs of our model, demands for goods, costs of production, etc. are given by stochastic processes. Already, equilibrium models have been proposed and used in stochastic frameworks to enlighten price formation for the pollution certificates issued by the regulators (see for example [5]) or for the joint formation of the prices of goods and emissions (see for example [4]). As a minor side effect, the present paper gives a generalization of the analysis of [5] to the multi-markets, multi-compliance periods framework. But most importantly, our new model accommodates different abatement strategies, say based on short term or long term abatement measures, and so doing, can be used to model emission reduction by means of projects covered by the Clean Development Mechanism of the Kyoto protocol. The main thrust of the paper is the joint price formation for pollution permits coming from two different sources: 1) standard emission cap-and-trade schemes, and 2) the Clean Development Mechanism. Prices appear in a competitive equilibrium based on a model of short and long term abatement strategies, emission trading involving physical and financial positions and regulatory compliance restrictions. The gory details are spelled out in Section 2 below. We choose to work in the framework of discrete time processes for the sake of convenience only. But even then, notations are rather involved and to help the reader follow the presentation, we collected most of the notations and the definitions in two short appendices at the end of the paper. These appendices play the role of an index of notation.

The competitive equilibrium set-up is given in Section 3. We first articulate the optimization problem faced by each individual firm, we give the definition of the notion of equilibrium appropriate for our model, and we give an equivalence result which reduces the equilibrium analysis to the study of a reduced form of equilibrium for a simpler model not involving trading. We then formulate the problem of an informed central planner (the so-called representative agent) and rewrite its optimization problem as a large linear program in function space. Note that, despite the fact that like in [4] we use properties of the weak* topologies of L^∞-spaces when in duality with L^1-spaces, the proof given here is quite different since it relies on established properties of the theory of linear programming in infinte dimensional topological vector spaces. Duality theory is used and the complementarity slackness conditions are spelled out carefully as they are the main source of information from which properties of the equilibrium prices are derived in the following Section 4.

An interesting phenomenon is illustrated in Figure 1. The price of a CER is not equal to the price of an EUA even though a CER, like an EUA, is a certificate which can be used to offset one ton-equivalent of CO_2 emissions. The spread between the prices of these two offsets is a source of risk for the emission market participants, and various forms of trading this spread have emerged as risk mitigation maneuvers. It is enlightening to see that the equilibrium prices produced by our model do exhibit a spread, and in some sense, its analysis is the main goal of Section 4. We give several formulas expressing this difference in price between regular allowances and CERs, and we give intuitive explanations for their existence.

In this paper, we use the following conventions: we restrict the word *allowance* or *allowance certificate* for the permits issued by the regulator of a given market, while we use CER for permits and certificates generated through the Clean Development Mechanism (CDM) and Joint Implementation (JI). The generic term *offset* will refer to either one of these types of certificates.

2. Joint Model for Multiple Emissions Markets

In this section we present the set-up of our mathematical analysis. We consider an economy with different emission markets $m \in M$. Each market covers a certain set of firms $I(m)$, with $I(m) \cap I(m') = \emptyset$ if $m \neq m'$. This assumption is justified if one thinks of national or regional markets whose coverages are naturally disjoint. We denote by I the set of all the agents, i.e. the union of all the $I(m)$, $I := \bigcup_{m \in M} I(m)$. We assume that each market is similar to EU ETS, and comprises a finite set $Q(m) = \{1, \dots, |Q(m)|\}$ of consecutive compliance periods. We denote by $T_0^m < T_1^m < \cdots < T_q^m$ the end points of the compliance periods. In other words, $[T_{q-1}^m, T_q^m]$ is the q-th compliance period in emissions market $m \in M$. In order to avoid unnecessary technical issues we assume that no two markets have compliance periods ending at the same time, i.e. for all $m \neq m' \in M$ it holds that $T_q^m \neq T_{q'}^{m'}$ for all $q \in Q(m)$ and $q' \in Q(m')$. Moreover we assume that emission trading stops at $T_{|Q(m)|}^m$. This could be either because transition to clean technologies is completed, or after time $T_{|Q(m)|}^m$ emission trading schemes do not couple to preceding periods (i.e. banking is not allowed and penalty is purely financial). Further, we denote by $T = \max\{T_{|Q(m)|}^m | m \in M\}$ the last time point of our model. Also for notational ease we introduce the set $P = \{1, \dots, \sum_{m \in M} |Q(m)|\}$ and denote by $(T_p)_{p \in P}$ the vector of subsequent compliance time points such that for each $m \in M$ and $q \in Q(m)$ there exists a $p \in P$ with $T_p = T_q^m$ and $T_p < T_{p'}$ if and only if $p < p'$.

The main thrust of the paper is to propose a model for the coupling of the different markets through the Clean Development mechanism (CDM) by allowing each firm $i \in I$ to use up to a certain amount κ^i of Certified Emission Reductions

(CER) for compliance. Notice that countries where CDM projects are carried out are usualy not covered by emission trading schemes. For notational convenience we consider these markets to be covered by emission trading schemes with zero penalty and without allocation.

In what follows $(\Omega, \mathcal{F}, \{\mathcal{F}_t,\ t \in \{0,1,\dots,T\}\}, \mathbb{P})$ is a filtered probability space. We assume that $\mathcal{F}$ is complete and that $\mathcal{F}_0$ contains all the sets of probability zero. We denote by $\mathbb{E}[.]$ the expectation operator under probability $\mathbb{P}$ and by $\mathbb{E}_t[.]$ the conditional expectation with respect to $\mathcal{F}_t$. The σ-field $\mathcal{F}_t$ represents the information available at time t. We will also make use of the notation $\mathbb{P}_t(.) := \mathbb{E}_t[\mathbf{1}_{\{.\}}]$ for the conditional probability with respect to $\mathcal{F}_t$. We will use the notation $\eta \in L^1(\mathcal{F}_t)$ or $\eta \in L^1_t$ (resp. $\eta \in L^\infty(\mathcal{F}_t)$ or $\eta \in L^\infty_t$) to mean that η is an integrable (rep. bounded) random variable which is $\mathcal{F}_t$ measurable (i.e. known at time t). For example, we denote by $\Gamma^{i,q} \in L^\infty(\mathcal{F}_{T^m_q})$ the emissions of firm $i \in I(m)$ of market $m \in M$ over the period $q \in Q(m)$.

2.1. Emission Reductions

In a cap -and-trade system, the allowance price is determined by the cap, namely the total number of emission certificates issued by the regulator, the penalty applied to emissions which are not offset by redeemed certificates, the existing abatement strategies, their flexibility and costs. Conceptually, we have to distinguish the abatement measures according to the time horizon which is required to return a profit. In this regard, abatement measures range from short-term measures (no initial investments, savings being returned within days) to long-term measures (high and irreversible investments, savings are returned over decades). Examples of long-term measures are optimization/substitution of high polluting production units, installation of scrubbers, investment in CDM and JI projects. On the contrary, typical short term abatement measures yield emission savings by switching fuels or skipping/re-scheduling the production.

For the purpose of this study, and for the sake of simplicity, we assume that each firm $i \in I$ has access to either a short term abatement measure or a CDM project. In our model, firms have access to both short and long term reduction measures. However, optimal CDM strategies are not necessarily indicative of optimal long term abatement strategies, so this assumption may need to be revisited in subsequent studies.

Short Term Abatement. At each time $0 \le t \le T-1$, firm $i \in I$ decides to reduce emission throughout the period $[t, t+1)$ by the amount ξ^i_t. Since the choice of the reduction level ξ^i_t is based only on present and past observations, the processes ξ^i are supposed to be adapted and, since reduction cannot exceed a maximum reduction level $\overline{\xi}^i$, we require that the inequalities

$$0 \le \xi^i_t \le \overline{\xi}^i, \qquad i \in I,\ t = 0, 1, \cdots, T-1, \tag{2.1}$$

hold almost surely. Here, $\overline{\xi}^i$ is a deterministic constant giving the maximum abatement level possible for firm $i \in I(m)$. The actual cumulative short term emission reduction of firm i during compliance period $q \in Q(m)$ when it uses the short term abatement strategy ξ^i reads

$$\Pi^{i,q}(\xi^i) := \sum_{t=T^m_{q-1}}^{T^m_q - 1} \xi^i_t. \tag{2.2}$$

CDM / Long Term Abatement. At each time $0 \leq t \leq T-1$, firm $i \in I$ decides wether to exercise part of its CDM project or not. The amount that is exercised is given by a real number ζ^i_t. If $\zeta^i_t = 1$, the whole project is started at time point t. To avoid integer constraints altogether, we will assume that each CDM project can also be realized piece by piece. Especially for big CDM projects with several emission sources this is certainly a realistic assumption. For example, half of the project can be exercised at one time point and the other half at another point in time. Since the choice to exercise the project is based only on present and past observations, the processes ζ^i are also supposed to be adapted and, since a CDM project cannot be used for credit beyond its original scope, we require that the inequalities

$$0 \leq \sum_{t=0}^{T-1} \zeta^i_t \leq 1, \qquad i \in I \tag{2.3}$$

hold almost surely. Moreover for notational convenience we assume that CERs generated by CDM projects are issued right after their starting dates. Again, this is somehow an unrealistic assumption as it disregards the fact that an investment in a CDM can turn out to be a net loss if the project is not approved or rewarded with CERs. Let μ^i denote the emission reduction that is generated if agent $i \in I$ exercises his whole CDM project. The actual cumulative emission reduction of firm i during compliance period $p \in P$ when it uses reduction strategy ζ^i reads

$$\Pi^{i,p}(\zeta^i) := \mu^i \sum_{t=T_{p-1}}^{T_p - 1} \zeta^i_t. \tag{2.4}$$

Note that $\mu^i = 0$ simply means that firm i does not have access to CDM projects.

2.2. Emission Trading

We denote by $\pi^m_q \in [0, \infty)$ the *financial penalty applied in market $m \in M$ to each unit of pollutant* in compliance period $q \in Q(m)$. However we assume that it is only at the last time point $T^m_{|Q(m)|}$ that the penalty is actually paid. For the sake of simplicity, we assume that the entire period $[T^m_{q-1}, T^m_q]$ corresponds to one simple compliance period. Moreover, for periods $q < |Q(m)|$ *banking* of allowances and CERs to the next period is allowed. i.e. allowances that are not used for compliance, may be used for compliance in all subsequent periods up to time $T^m_{|Q(m)|}$.

In this economy, operators of installations that emit pollutants will have three fundamental choices in order to avoid unwanted penalties: 1) *reduce* emissions by producing with cleaner technologies, 2) *buy* allowances, 3) *buy* CERs.

At time T^m_{q-1}, i.e. at the beginning of the q-th compliance period of the market $m \in M$, each firm $i \in I(m)$ in this market is given an initial endowment of $\Theta^{i,q} \in L^\infty(\mathcal{F}_{T^m_{q-1}})$ allowances. Notice that $\Theta^{i,q}$ depends upon the market $m \in M$ through the participant i in this market. So if it were to hold on to this initial set of allowances until the end, it would be able to offset up to $\Theta^{i,q}$ units of emissions, and start paying penalty only if its actual cumulative emissions exceeded that level. This is the *cap* part of a cap-and-trade scheme. Depending upon their views on the demands for the various products and their risk appetites, firms may choose production schedules leading to cumulative emissions in excess of their caps. In order to offset expected penalties, they subsequently engage in buying allowances from firms which expect to meet demand with less emissions than their own cap. This is the *trade* part of a cap-and-trade schemes.

Allowances are physical in nature, since they are certificates which can be redeemed at time T^m_q to offset measured emissions. But, because of trading, these certificates change hands and they become financial instruments. In EU ETS, allowances are allocated in March each year, while the 5 year compliance period of the second phase started in January 2008. Therefore a significant amount of allowances are traded via forward contracts. Because compliance takes place at time T^m_q for $q \in Q(m)$ and $m \in M$, and only at these times will we restrict ourselves to the situation where trading of emission allowances is done via forward contracts settled at time T^m_q.

Remark 2.1. *Because compliance takes place at time T_p for $p \in P$, a simple no-arbitrage argument implies that the forward and spot prices both for allowances and CERs, differ only by a discounting factor, such that trading spot or forwards gives the same expected discounted payoff at time T_p. Therefore under the equilibrium definition that will be introduced in Section* 3*, considering only forward trading yields no loss of generality. For notational ease, we restrict ourselves to the case where all forwards are paid at time T and not T^m_q. Moreover allowing trading in forward contracts in our model provides a more flexible setting: it is more general than considering only spot trading, since it allows for trading even before these allowances are issued or allocated. This is an important feature when dealing with several compliance periods. In particular if at T_p all CERs are used for compliance it is not possible to trade CER spot before new CERs are issued.*

2.3. Financial and Physical Positions

We denote by $\tilde{A}^{q,m}_t$ the price at time $t = 0, \dots, T^m_q$ of a (q, m)-allowance forward contract guaranteeing delivery of one allowance certificate (that can be used for compliance in market m at T^m_q) at maturity T^m_q and payment at T. Moreover $\tilde{C}^p_t$ denotes the price at time $t = 0, \dots, T_p$ of a p-maturity CER forward contract (that can be used for compliance in all markets) at maturity T_p and payment at T.

Financial Positions. For simplicity we assume that agents can take positions only on their own allowance market, and we denote for $m \in M$ and $q \in Q(m)$ by $\tilde{\theta}_t^{i,q}$ the number of (q,m)-allowance forward contracts held by firm $i \in I(m)$ at the beginning of the time interval $[t, t+1)$. Similarly we denote for all $p \in P$ by $\tilde{\varphi}_t^{i,p}$ the number of p-maturity CER forward contracts held by firm $i \in I$ at the beginning of the time interval $[t, t+1)$.

We define a trading strategy $(\tilde{\theta}^i, \tilde{\varphi}^i)$ for firm $i \in I(m)$ as a couple of vector valued adapted stochastic processes $\tilde{\theta}^i = (\tilde{\theta}^{i,q})_{q \in Q(m)}$ and $\tilde{\varphi}^{i,p} = (\tilde{\varphi}^{i,p})_{p \in P}$ where $\tilde{\theta}^{i,q} = (\tilde{\theta}_t^{i,q})_{t=0,\cdots,T_q^m - 1}$ and $\tilde{\varphi}^{i,p} = (\tilde{\varphi}_t^{i,p})_{t=0,\cdots,T_p^m - 1}$ are scalar adapted processes. The net cash position at time T resulting from this trading strategy is:

$$R_T^{(A,C)}(\theta^i, \varphi^i) \; := \sum_{q \in Q(m)} \sum_{t=T_{q-1}^m}^{T_q^m - 1} \tilde{\theta}_t^{i,q} (\tilde{A}_{t+1}^{q,m} - \tilde{A}_t^{q,m}) + \sum_{p \in P} \sum_{t=T_{p-1}}^{T_p - 1} \tilde{\varphi}_t^{i,p} (\tilde{C}_{t+1}^p - \tilde{C}_t^p). \tag{2.5}$$

Physical Positions. We denote by $A_{T_q^m}^{q,m}$ the price at time T_q^m of a (q,m)-allowance which is paid at time T. Similarly, $C_{T_p}^p$ denotes the price at time T_p of a p-maturity CER forward contract with payment at T.

For simplicity we assume again that agents can take positions only on their own forward allowance market, and we denote for each $m \in M$ and $q \in Q(m)$ by $\gamma_{T_q}^{i,q}$ and $\theta_{T_q}^{i,q}$ the number of (q,m)-allowances banked and used for compliance by firm $i \in I(m)$ at T_q^m respectively. Similarly we denote for each $p \in P$ by $\phi_{T_p}^{i,p}$ and $\varphi_{T_p}^{i,p}$ the number of p-maturity CERs banked and used for compliance by firm $i \in I$ at T_p respectively. Clearly $\phi_{T_p}^{i,p} = 0$ if T_p is not a compliance date T_q^m.

We define a banking strategy (γ^i, ϕ^i) of firm $i \in I(m)$ by adapted processes $\gamma^i = (\gamma_{T_q}^{i,q})_{q \in Q(m)}$ and $\phi^i = (\phi_{T_p}^{i,p})_{p \in P}$. Similarly we define a compliance strategy (θ^i, φ^i) of firm $i \in I(m)$ by adapted processes $\theta^i = (\theta_{T_q}^{i,q})_{q \in Q(m)}$ and $\varphi^i = (\varphi_{T_p}^{i,p})_{p \in P}$. The random variables $\varphi_{T_p}^{i,p}$ will have to satisfy a constraint of the upper bound type since because of regulation, a firm can only use a limited amount of CERs toward its excess emissions.The costs at time T of these strategies read

$$\sum_{q \in Q(m)} A_{T_q^m}^{q,m} \left(\theta_{T_q^m}^i + \gamma_{T_q^m}^i - \gamma_{T_{q-1}^m}^i - \Theta_{T_q^m}^i \right) \tag{2.6}$$

for allowance trading, and

$$\sum_{p \in P} C_{T_p}^p \left(\varphi_{T_p}^i + \phi_{T_p}^i - \phi_{T_{p-1}}^i - \Pi^{i,p}(\zeta) \right). \tag{2.7}$$

for CER trading.

Penalties. We denote by $\pi_q^m \in [0, \infty)$ the *financial penalty per unit of pollutant* not covered by emission offset, whether in the form of allowance certificates or CERs in compliance period $q \in Q(m)$. This penalty is only paid at the last time point $T^m_{|Q(m)|}$. For compliance periods $q < |Q(m)|$ ending at T^m_q this penalty has two components: not only does it include the payment of π^m_q times the numbers of emission units not covered by redeemed offsets, but it also includes the transfer to the current period of the number of missing offsets from the next trading period. For each firm $i \in I(m)$ in market $m \in M$, the penalty is paid for each ton of net cummulative emission $\beta^i_{T^m_q}$ for period $q \in Q(m)$. It is computed at time T^m_q as the difference between the total amount $\Gamma^{i,q} - \Pi^{i,q}(\xi^i)$ of pollutants emitted over the entire period $[T^m_{q-1}, T^m_q]$ plus the short position $\beta^i_{T^m_{q-1}}$ from the preceding allowance period minus the number $\varphi^i_{T^m_q} + \theta^i_{T^m_q}$ of allowances and CERs submitted for compliance by the firm at time T^m_q. The net cumulative emission $\beta^i_{T^m_q}$ is this difference whenever positive, and 0 otherwise. Hence it fullfills

$$\beta^i_{T^m_q} = \left(\Gamma^{i,q} - \Pi^{i,q}(\xi) - \varphi^{i,p}_{T^m_q} - \theta^i_{T^m_q}\right)^+ \tag{2.8}$$

and the financial penalty at time T^m_q is given by $\pi^m_q \beta^i_{T^m_q}$ for all $q \in Q(m), i \in I(m)$ and $m \in M$.

Compliance Restrictions. Both for allowances and for CERs the amount that can be banked or used for compliance is restricted by their amount available in the market. Moreover, in the case of CERs, regulatory requirements impose further restrictions. For allowances, on any given period, the total number of allowances banked from the preceding periods and those resulting from compliance strategies should equal to the initial allocation for this period (vintage). Hence for each market $m \in M$, one should have

$$\sum_{i \in I(m)} [\theta^i_{T^m_q} + \gamma^i_{T^m_q} - \gamma^i_{T^m_{q-1}} + \beta^i_{T^m_{q-1}}] = \sum_{i \in I(m)} \Theta^{i,q}, \tag{2.9}$$

for each $q \in Q(m)$. For CERs, the amount that can be banked or redeemed for compliance is also restricted by the number of allowances available in the market. These are given by the amount banked from the previous period plus the amount of CERs generated since the last compliance event $p-1$ corrected by the number $\Xi^{i,p} \in L^\infty(\mathcal{F}_{T_p})$ of CERs that firm $i \in I$ decided to withdraw from the market, for example for voluntary offsets. These are not part of the strategy but an exgenously given random variable. Later we shall assume that there are no point masses in the distributions of these quantities. Hence balancing CER banking and compliance strategies at each date T_p for $p \in P$ gives

$$\sum_{i \in I} [\varphi^i_{T_p} + \phi^i_{T_p} - \phi^i_{T_{p-1}}] = \sum_{i \in I} [\Pi^{i,p}(\zeta) - \Xi^{i,p}].. \tag{2.10}$$

2.4. Costs

Despite the fact that we are jointly modeling markets with possibly different currencies, we purposedly ignore the risks and opportunities associated with fluctuations in foreign exchange rates. For the sake of simplicity, we assume that all the financial quantities are expressed in one single currency. Moreover as explained earlier, we express all cash flows, position values, firm wealth, and good prices in time T-currency. As a side fringe benefit, this avoids discounting in the computations. We use for numéraire the price $B_t(T)$ at time t of a Treasury (i.e. non defaultable) zero coupon bond maturing at T. We denote by $\{\tilde{S}^i_t\}_{t=0,1,\cdots,T}$ and $\{\tilde{L}^i_t\}_{t=0,1,\cdots,T}$ the adapted stochastic processes giving the short and long term abatement costs of firm $i \in I$, and according to the above convention, we find it convenient to work at each time t with the T-forward price

$$S^i_t = \tilde{S}^i_t / B_t(T) \ , \ L^i_t = \tilde{L}^i_t / B_t(T)$$

and we skip the dependence upon T from the notation of the T-forward prices. For us, a cash flow X_t at time t is equivalently valued as a cash flow $X_t/B_t(T)$ at maturity T. So if firm $i \in I$ follows the abatement policy $(\xi^i, \zeta^i) = (\xi^i_t, \zeta^i_t)_{t=0}^{T-1}$, its time T-forward costs are given by

$$\sum_{t=0}^{T-1} [S^i_t \xi^i_t + L^i_t \zeta^i_t \ ]. \tag{2.11}$$

Combining (2.11), (2.5), (2.6) and (2.7) together with (2.5), we obtain the following expression for the terminal cumulative costs $C^{\tilde{A},\tilde{C},A,C,i}$ of firm i:

$$\begin{aligned} C^{\tilde{A},\tilde{C},A,C,i} \quad = \quad & \sum_{t=0}^{T^m_{|Q|}} \zeta^i_t L^i_t + \sum_{t=0}^{T^m_{|Q|}} \xi^i_t S^i_t + R_T^{(\tilde{A},\tilde{C})}(\tilde{\theta}^i, \tilde{\varphi}^i) \\ & + \sum_{q \in Q(m)} A^{q,m}_{T^m_q} \left(\theta^i_{T^m_q} + \gamma^i_{T^m_q} - \gamma^i_{T^m_{q-1}} + \beta^i_{T^m_{q-1}} - \Theta^i_{T^m_q} \right) \\ & + \sum_{p \in P} C^p_{T_p} \left(\varphi^i_{T_p} + \phi^i_{T_p} - \phi^i_{T_{p-1}} - \Pi^{i,p}(\zeta) + \Xi^{i,p} \right) \\ & + \sum_{q \in Q(m)} \pi^q \left(\Gamma^{i,q} - \Pi^{i,q}(\xi) - \varphi^i_{T^m_q} - \theta^i_{T^m_q} \right)^+ . \end{aligned} \tag{2.12}$$

Recall that expected emissions and production costs change with time in a stochastic manner. The statistical properties of these processes are given exogenously, and are assumed to be known at time $t = 0$ by all firms. Moreover, we always assume that these processes satisfy the constraints (2.1) and (2.3) almost surely. Agents adjust their production and trading strategies in a non-anticipative manner to their observations of the fluctuations in demand and production costs. In turn, the production and trading strategies $(\xi^i, \zeta^i, \theta^i, \varphi^i, \gamma^i, \phi^i, \tilde{\theta}^i, \tilde{\varphi}^i)$ become a vector valued adapted stochastic processes on the stochastic base of the demand and production costs.

3. Equilibrium Analysis

We first consider the individual firm optimization problems.

3.1. The Individual Firm Optimization Problems

Clearly, each firm $i \in I$ tries to minimize its expected terminal cost, i.e. the expectation of $C^{\tilde{A},\tilde{C},A,C,i}$ defined above in (2.12). In this subsection, we define rigorously this optimization problem. Our strategy is to first linearize the objective function. Since the only non-linearities come from the positive parts in the last summation accounting for the penalty payments, we use the fact that for any integrable random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we have

$$\mathbb{E}\{X^+\} = \inf_{\beta \in \mathcal{F}, \beta \geq 0, \beta \geq X} \mathbb{E}\{\beta\}.$$

Using this identity, we can replace each expectation

$$\mathbb{E}\{\Big(\Gamma^{i,q} - \Pi^{i,q}(\xi) - \varphi^i_{T^m_q} - \theta^i_{T^m_q}\Big)^+\}$$

with the infimum of $\mathbb{E}\{\beta^i_q\}$ over $\beta^i_q \in L^\infty_{T^m_q}$ such that $\beta^i_q \geq 0$ and $\beta^i_q \geq \Gamma^{i,q} - \Pi^{i,q}(\xi^i) - \varphi^i_{T^m_q} - \theta^i_{T^m_q}$. So, for each given pair of price processes (A, C) and $(\tilde{A}, \tilde{C})$ the individual optimization problem of agent $i \in I$ is given by

$$\inf_{x^i \in \mathfrak{F}^i, (\tilde{\theta}^i, \tilde{\varphi}^i) \in \mathcal{H}^i} \mathbb{E}\{I^{\tilde{A},\tilde{C},A,C,i}\} \tag{3.1}$$

where $I^{\tilde{A},\tilde{C},A,C,i}$ is defined as:

$$\begin{aligned} I^{\tilde{A},\tilde{C},A,C,i} \quad = \quad & \sum_{t=0}^{T^m_{|Q|}} \zeta^i_t L^i_t + \sum_{t=0}^{T^m_{|Q|}} \xi^i_t S^i_t + R^{(\tilde{A},\tilde{C})}_T(\tilde{\theta}^i, \tilde{\varphi}^i) + \sum_{q \in Q(m)} \pi^q \beta^i_{T^m_q} \\ & + \sum_{q \in Q(m)} A^{q,m}_{T^m_q} \Big(\theta^i_{T^m_q} + \gamma^i_{T^m_q} - \gamma^i_{T^m_{q-1}} + \beta^i_{T^m_{q-1}} - \Theta^i_{T^m_q} \Big) \\ & + \sum_{p \in P} C^p_{T_p} \Big(\varphi^i_{T_p} + \phi^i_{T_p} - \phi^i_{T_{p-1}} - \Pi^{i,p}(\zeta) + \Xi^{i,p} \Big). \end{aligned}$$

and where the feasibility sets $\mathfrak{F}^i$ and $\mathcal{H}^i$ are defined as follows. First, we denote by x^i the physical strategy $(\beta^i, \xi^i, \zeta^i, \theta^i, \varphi^i, \gamma^i, \phi^i)$ which belongs to the following L^∞ space $\mathcal{L}^\infty_i$ which we write down as a product of individual L^∞ spaces in order to emphasize the respective measurability properties of the components of x^i.

$$\begin{aligned} \mathcal{L}^\infty_i \quad := \quad & \Big\{ x^i = (\beta^i, \xi^i, \zeta^i, \theta^i, \varphi^i, \gamma^i, \phi^i) \Big| \beta^i, \theta^i \in \prod_{q=1}^{|Q(m)|} L^\infty_{T^m_q}; \quad \xi^i, \zeta^i \in \prod_{t=0}^{T-1} L^\infty_t; \\ & \varphi^i \in \prod_{p=1}^{|P|} L^\infty_{T_p}; \quad \gamma^i \in \prod_{q=1}^{|Q(m)|-1} L^\infty_{T^m_q}; \quad \phi^i \in \prod_{p=1}^{|P|-1} L^\infty_{T_p} \Big\} \end{aligned}$$

and

$$\mathcal{H}^1_i := \Big\{ (\tilde{\theta}, \tilde{\varphi}) \Big| \tilde{\theta} \in \prod_{q=1}^{|Q(m)|} \prod_{t=0}^{T^m_q - 1} L^1_t; \quad \tilde{\varphi} \in \prod_{p=1}^{|P|} \prod_{t=0}^{T_p - 1} L^1_t \Big\} \tag{3.2}$$

where for $t = 0, \dots, T$ and $p = 1, \dots, \infty$, L_t^p denotes the space of equivalence classes of $\mathcal{F}_t$-measurable random variables in L^p. For notational convenience we also set:

$$\mathcal{L}^\infty = \prod_{i \in I} \mathcal{L}_i^\infty, \qquad \text{and} \qquad \mathcal{H}^1 = \prod_{i \in I} \mathcal{H}_i^1.$$

We already explicitly stated the individual constraints satisfied by some of the components of x^i, for example:

$$\beta^i_{T_q^m} + \varphi^i_{T_q^m} + \theta^i_{T_q^m} + \Pi^{i,q}(\xi^i) \geq \Gamma^{i,q} \text{ for all } q \in Q(m) \tag{3.3}$$

$$\varphi^i_{T_p} \leq \kappa^i \text{ for all } p \in P \tag{3.4}$$

$$\xi^i_t \leq \overline{\xi}^i \text{ for } t = 0, \dots, T-1 \tag{3.5}$$

$$\sum_{t=0}^{T-1} \zeta^i_t \leq 1, \tag{3.6}$$

where by convention we set:

$$\beta^i_{T_0^m} = 0, \quad \gamma^i_{T_0^m} = 0, \quad \text{and} \quad \phi^i_{T_0^m} = 0.$$

For the sake of notational convenience, we rewrite them (together with those we did not explicitly stated) in two different forms. We view x^i as a $m^i = 1+2(T+|Q(m)|+|P|)$-tuple of bounded random variables (with their own individual measurability properties which are irrelevant for the purpose of the present discussion), say $x^i = [x^{i,j}]_{j=1,\cdots,m_i}$. There exist m^i bounded random variables $\chi^{i,j}$ satisfying the constraints

$$0 \leq x^{i,j} \leq \chi^{i,j}, \qquad\qquad j = 1, \cdots, m^i. \tag{3.7}$$

According to our assumptions, most of the bounded random variables $\chi^{i,m}$ are in fact constants, and the other ones have the same measurability properties as the corresponding strategies $x^{i,j}$. We write this set of m^i constraints as

$$0 \leq x^i \leq \chi^i, \tag{3.8}$$

where we think of χ^i as the m^i-tuple $\chi^i = [\chi^{i,j}]_{j=1,\cdots,m_i}$ of bounded random variables. We singled out the constraints (3.7) because they will be at the heart of some of the compactness arguments providing existence of optima. Together with the set of remaining constraints, see for example constraints (3.3) and (3.6) above, as

$$F^i x^i \geq f^i \ , \ x^i \geq 0 \tag{3.9}$$

for an appropriate linear map $F^i : \mathcal{L}_i^\infty \mapsto \mathcal{K}_i^\infty$ and a vector $f^i \in \mathcal{K}_i^\infty$ with

$$\mathcal{K}_i^\infty = \left\{ (z^{j,i})_{j=1}^4 \middle| z^{1,i} \in \prod_{q=1}^{|Q(m)|} L^\infty_{T_q^m}, z^{2,i} \in \prod_{p=1}^{|P|} L^\infty_{T_p}, z^{3,i} \in \prod_{t=0}^{T-1} L^\infty_t, z^{4,i} \in L^\infty_{T-1} \right\}.$$

Hence, the set of feasible strategies for firm $i \in I(m)$ can be defined as:

$$\mathfrak{F}^i = \{ x^i \in \mathcal{L}_i^\infty |\ x^i \geq 0, \quad F^i x^i \geq f^i \} \tag{3.10}$$

and as usual, we set

$$\mathfrak{F} = \prod_{i\in I} \mathfrak{F}^i \quad \text{and} \quad \mathcal{K}^\infty = \prod_{i\in I} \mathcal{K}^i.$$

3.2. Equilibrium Definitions

In equilibrium, strategies must satisfy (2.9) and (2.10) for all $m \in M, q \in Q(m)$ and for all $p \in P$. In the following we write these constraints as

$$Gx = g, \tag{3.11}$$

for a linear map $G : \mathcal{L}^\infty \mapsto \mathcal{K}^\infty_G$ and an element $g \in \mathcal{K}^\infty_G$ with

$$\mathcal{K}^\infty_G = \left\{ (z_A, z_C) \middle| z_A \in \prod_{m\in M} \prod_{q=1}^{|Q(m)|} L^\infty_{T^m_q}; \quad z_C \in \prod_{p=1}^{|P|} L^\infty_{T_p} \right\}. \tag{3.12}$$

Using this notation the global feasible strategy set reads

$$\mathfrak{G} = \{x \in \mathcal{L}^\infty | Gx = g\}, \tag{3.13}$$

and if we define price spaces as

$$\mathcal{K}^1_G = \left\{ (A, C) \middle| A \in \prod_{m\in M} \prod_{q=1}^{|Q(m)|} L^1_{T^m_q}; \quad C \in \prod_{p=1}^{|P|} L^1_{T_p} \right\} \tag{3.14}$$

and

$$\mathcal{H}^\infty := \left\{ (\tilde{A}, \tilde{C}) \middle| \tilde{A} \in \prod_{m\in M} \prod_{q=1}^{|Q(m)|} \prod_{t=0}^{T^m_q} L^\infty_t; \quad \tilde{C} \in \prod_{p=1}^{|P|} \prod_{t=0}^{T_p} L^\infty_t \right\} \tag{3.15}$$

in order to emphasize once more the measurability properties of the constraints, the natural definition of a perfectly competitive equilibrium in the present set-up reads:

Definition 3.1. *The pair of forward price processes $(\tilde{A}^*, \tilde{C}^*) \in \mathcal{H}^\infty$ and spot prices processes $(A^*, C^*) \in \mathcal{K}^1_G$ form an equilibrium if for each $i \in I$ there exists $x^{*i} \in \mathfrak{F}^i$ and $(\tilde{\theta}^{*i}, \tilde{\varphi}^{*i}) \in \mathcal{H}^1_i$ such that: (i) All forward positions are in zero net supply, i.e. for all $m \in M$ and $q \in Q(m)$ it holds that*

$$\sum_{i\in I} \tilde{\theta}^{*i,q,m}_t = 0, \qquad t = 0, \ldots, T^m_q - 1 \tag{3.16}$$

and for all $p \in P$

$$\sum_{i\in I} \tilde{\varphi}^{*i,p}_t = 0, \qquad t = 0, \ldots, T_p - 1. \tag{3.17}$$

(ii) Strategies fullfill equilibrium constraints in the sense that $(x^{*i})_{i\in I} \in \mathfrak{G}$. (iii) Each firm $i \in I$ is satisfied by its own strategy in the sense that

$$\mathbb{E}[I^{\tilde{A}^*,\tilde{C}^*,A^*,C^*,i}(x^{*i}, \tilde{\theta}^{*i}, \tilde{\varphi}^{*i})] \leq \mathbb{E}[I^{\tilde{A}^*,\tilde{C}^*,A^*,C^*,i}(x^i, \tilde{\theta}^i, \tilde{\varphi}^i)] \quad \text{for all } x^i \in \mathfrak{F}^i \text{ and } (\tilde{\theta}^i, \tilde{\varphi}^i) \in \mathcal{H}^1_i. \tag{3.18}$$

Because trading only plays a marginal role in the construction of equilibriums, we introduce a somehow more restrictive notion of equilibrium without trading, and after proving that it is actually equivalent to the more general notion spelled out above, we use it in the existence proof. For each pair of price processes (A, C) and for each firm $i \in I(m)$ participating in market $m \in M$ with individual strategy $x^i \in X^i$ we define the individual physical utility (without forward trading) as:

$$\begin{aligned} L^{A,C,i}(x^i) &= \mathbb{E}\Big[\sum_{t=0}^{T-1} \zeta_t^i L_t^i + \sum_{t=0}^{T-1} \xi_t^i S_t^i + \sum_{q\in Q(m)} \pi^{q,m} \beta^i_{T_q^m} \\ &\quad + \sum_{q\in Q(m)} A^{q,m}_{T_q^m}\Big(\theta^i_{T_q^m} + \gamma^i_{T_q^m} - \gamma^i_{T^m_{q-1}} + \beta^i_{T^m_{q-1}} - \Theta^{i,q}\Big) \\ &\quad + \sum_{p\in P} C^p_{T_p}\Big(\varphi^i_{T_p} + \phi^i_{T_p} - \phi^i_{T_{p-1}} - \Pi^{i,p}(\zeta) + \Xi^{i,p}\Big)\Big]. \end{aligned}$$

Using this notation we define a reduced equilibrium (without forward trading) as follows.

Definition 3.2. *The spot prices $(A^*, C^*) \in \mathcal{K}^1_G$ form a reduced equilibrium of the market if there exists $x^* \in \mathfrak{F}$ such that: (i) Strategies fullfill equilibrium constraints $x^* \in \mathfrak{G}$. (ii) Each firm $i \in I$ is satisfied by its own strategy in the sense that*

$$L^{A^*,C^*,i}(x^{*i}) \le L^{A^*,C^*,i}(x^i) \text{ for all } x^i \in \mathfrak{F}^i. \tag{3.19}$$

The following equivalence result shows that there is no loss of generality in using this more restrictive notion of equilibrium.

Proposition 3.3. *Forward prices $(\tilde{A}^*, \tilde{C}^*) \in \mathcal{H}^\infty$ and spot prices $(A^*, C^*) \in \mathcal{K}^1_G$ with associated strategies $x^{*i} \in \mathcal{F}^i$ and $(\tilde{\theta}^{*,i}, \tilde{\varphi}^{*,i}) \in \mathcal{H}^1_i$ for all $i \in I$ form an equilibrium in the sense of Definition 3.1 if and only if the spot prices $(A^*, C^*) \in \mathcal{K}^1_G$ form an equilibrium in the sense of Definition 3.2 with associated strategies $x^{*i} \in \mathfrak{F}^i$ for all $i \in I$ and*

$$\tilde{A}^{*q,m}_t = \mathbb{E}[A^{*q,m}_{T_q^m}|\mathcal{F}_t] \text{ for all } t = 0, \dots, T_q^m, m \in M, q \in Q(m) \tag{3.20}$$

$$\tilde{C}^{*p}_t = \mathbb{E}[C^{*p}_{T_p}|\mathcal{F}_t] \text{ for all } t = 0, \dots, T^p, p \in \mathcal{P}. \tag{3.21}$$

Proof. We first show that for each $m \in M, q \in Q(m)$ the futures allowance price process $\tilde{A}^{*q,m}$ is a martingale for if not, there exists a time t and a set $\mathcal{A} \in \mathcal{F}_t$ of non-zero probability such that $\mathbb{E}_t[\tilde{A}^{*q,m}_{t+1}\mathbf{1}_{\mathcal{A}}] > \mathbf{1}_{\mathcal{A}}\tilde{A}^{*q,m}_t$ (or resp. <). Then for each firm $i \in I$ the trading strategy given by $\tilde{\theta}^{i,q,m}_s = \tilde{\theta}^{*i,q,m}_s$ for all $s \neq t$ and $\tilde{\theta}^{i,q,m}_t = \tilde{\theta}^{*i,q,m}_t + 1_{\mathcal{A}}$ (resp $\tilde{\theta}^{i,q,m}_t = \tilde{\theta}^{*i,q,m}_t - 1_{\mathcal{A}}$) outperforms the strategy $\tilde{\theta}^{*i}$, contradicting the property (3.16) of an equilibrium. Moreover the payoff of the forward is $A^{*q,m}_{T_q^m}$ and $\tilde{A}^{*q,m}_{T_q^m} = A^{*q,m}_{T_q^m}$ which proves (3.20). The same argument holds for $\tilde{C}^*$ together with property (3.17) of an equilibrium proving (3.21). Since both $\tilde{A}^*$ and $\tilde{C}^*$ are martingales it follows that

$$L^{A^*,C^*,i}(x^i) = \mathbb{E}[I^{\tilde{A}^*,\tilde{C}^*,A^*,C^*,i}(x^i,\tilde{\theta}^i,\tilde{\varphi}^i)] \tag{3.22}$$

for all $x^i \in \mathfrak{F}^i$, $(\tilde{\theta}^i,\tilde{\varphi}^i) \in \mathcal{H}^1_i$ and $i \in I$. Therefore (3.18) implies (3.19). Which proves that (A^*,C^*) form an equilibrium in the sense of Definition 3.2.
Conversely, if we assume that $(A^*,C^*) \in \mathcal{K}^1_G$ form an equilibrium in this sense with associated strategies $x^{*i} \in \mathfrak{F}^i$ for all $i \in I$, and $(\tilde{A}^*,\tilde{C}^*) \in \mathcal{H}^\infty$ are given by (3.20) and (3.21), then since $(\tilde{A}^*,\tilde{C}^*)$ are martingales it follows again that

$$L^{A^*,C^*,i}(x^i) = \mathbb{E}[I^{\tilde{A}^*,\tilde{C}^*,A^*,C^*,i}(x^i,\tilde{\theta}^i,\tilde{\varphi}^i)] \tag{3.23}$$

for all $x^i \in \mathfrak{F}^i$, $(\tilde{\theta}^i,\tilde{\varphi}^i) \in \mathcal{H}^1_i$ and $i \in I$. In particular this holds for x^{*i} together with $(\tilde{\theta}^{*i},\tilde{\varphi}^{*i}) = (0,0)$ which also satisfy conditions (3.16) and (3.17). Hence we conclude that (3.19) implies (3.18) proving that (A^*,C^*) and $(\tilde{A}^*,\tilde{C}^*)$ form an equilibrium with associated strategies x^{*i} and $(\tilde{\theta}^{*i},\tilde{\varphi}^{*i}) = (0,0)$ for all $i \in I$. □

3.3. Equilibrium and Global Optimality: Linear Programming Formulation

The space $\mathcal{L}^\infty$ of strategies was defined in the previous section. Now we set

$$\mathcal{K}^\infty_F = \left\{ (z^i)_{i\in I} \middle| z^i \in \mathcal{K}^\infty_i \text{ for all } i \in I \right\}. \tag{3.24}$$

and we define the space of constraints as

$$\mathcal{K}^\infty = \left\{ (z, z_A, z_B) \middle| z \in \mathcal{K}^\infty_F; \quad (z_A, z_B) \in \mathcal{K}^\infty_G \right\}. \tag{3.25}$$

We then define the linear map $F : \mathcal{L}^\infty \mapsto \mathcal{K}^\infty_F$ in a natural way as the *matrix of linear maps*

$$F := \begin{pmatrix} F^1 & & \\ & \ddots & \\ & & F^{|I|} \end{pmatrix}.$$

the vector f by $f = [f^i]_{i\in I}$ and the upper bound χ by $\chi = [\chi^i]_{i\in I}$. The space of feasible strategies can be rewritten as:

$$\mathfrak{F} \cap \mathfrak{G} = \{x \in \mathcal{L}^\infty |\ x \geq 0,\ Fx \geq f,\ Gx = g\}. \tag{3.26}$$

The above notations were introduced in order to reformulate the equilibrium existence problem as a linear program. The primal problem (P) of a *representative agent* (informed central planner) can be stated as:

$$P^* = \inf_{x\geq 0,\ Fx\geq f,\ Gx=g} \langle x, c\rangle_{\mathcal{L}} \tag{3.27}$$

if we use the notation

$$\langle x, c\rangle_{\mathcal{L}} = \mathbb{E}\Bigg[\sum_{i\in I}\left(\sum_{t=0}^{T-1}\zeta^i_t L^i_t + \sum_{t=0}^{T-1}\xi^i_t S^i_t\right) + \sum_{m\in M}\sum_{i\in I(m)}\sum_{q\in Q(m)}\pi^{m,q}\beta^i_{T^m_q}\Bigg]. \tag{3.28}$$

The sum of individual problems (SIP) can be written as

$$R(A,C) = \inf_{x\geq 0,\ Fx\geq f} \sum_{i\in I} L^{A,C,i}(x^i) \tag{3.29}$$

for all $(A,C)\in\mathcal{K}^1_G$, the Lagrange relaxation (LR) of the global constraints $Gx=g$ is given by

$$LR^* = \sup_{(A,C)\in\mathcal{K}^1_G} R(A,C) = \sup_{(A,C)\in\mathcal{K}^1_G} \inf_{0\leq x\leq\chi,\ Fx\geq f} \sum_{i\in I} L^{A,C,i}(x^i) \tag{3.30}$$

and the dual program (D) of the representative agent problem is given by

$$D^* = \sup_{w_F\in\mathcal{K}^1_F,\ w_F\geq 0,\ w_G\in\mathcal{K}^1_G,\ F^*w_F+G^*w_G+w_\chi\leq c} \langle (f,g),(w_F,w_G)\rangle_{\mathcal{K}}. \tag{3.31}$$

3.4. Existence of Optima and Relation with the Original Equilibrium Model

The main existence result is given in the following proposition.

Proposition 3.4. *There exist feasible solutions both for the primal linear program (P). The duality gap vanishes and the infimum P^* is attained for an optimal feasible solution $\overline{x}$.*

Proof. We first prove the feasibility claim by inspection.

The linear constraints $Fx\geq f$ translate for each market $m\in M$ and for the individual firm $i\in I(m)$ to inequalities (3.3)-(3.6), while the global equality constraints $Gx=g$ yield (2.9) and (2.10) almost surely. In order to prove primal feasibility, we need for each $i\in I$, to find x^i such that $x=(x^i)_{i\in I}$ satisfies $x\geq 0$, $Fx\geq f$ and $Gx=g$. Recall that $x^i=(\beta^i,\xi^i,\zeta^i,\theta^i,\varphi^i,\gamma^i,\phi^i)$, so that, if we choose $\xi^i=0$ (no abatement), $\zeta^i=0$ (no CDM project at all), $\theta^i=0$ (no physical allowance redeemed for compliance), $\varphi^i=0$ (no physical CER redeemed for compliance), then setting $\beta^i_{T^m_q}=\Gamma^{i,q}$ guarantees that inequalities (3.3)-(3.6) are satisfied, and finally we construct γ^i and ϕ^i recursively from the equalities (2.9) and (2.10) to guarantee that the equality constraints are satisfied as well.

The next part of the proof relies on standard arguments from the theory of convex optimization in infinite dimensional topological vector spaces, so we only outline the major steps, and for the reader's convenience, we give precise references to the classical functional analysis results which we use.

In order to solve the primal problem, we need to minimize the linear function $x\hookrightarrow\langle x,c\rangle_{\mathcal{L}}$ over the set

$$\mathcal{U}=\{x\in\mathcal{L}^\infty;\ x\geq 0,\ Fx\geq f,\ Gx=g\}.$$

Extracting almost surely convergent sequences if needed, on easily check that $\mathcal{U}$ is closed in the sense of the norm of $\mathcal{L}^1$. Moreover, $\mathcal{U}$ is weakly* closed in $\mathcal{L}^\infty$. Indeed, since $\mathcal{U}$ is a convex and a norm-closed subset of $\mathcal{L}^1$ it follows from the Hahn-Banach Theorem that $\mathcal{U}$ is the intersection of halfspaces $H_{x,c}=\{y\in\mathcal{L}^1|\langle y,x\rangle\leq c\}$ with $x\in\mathcal{L}^\infty$ and $c\in\mathbb{R}$ such that $\mathcal{U}\subseteq H$. Since $\mathcal{L}^\infty\subseteq\mathcal{L}^1$ it holds for each of these halfspaces $H_{x,c}$ that $x\in\mathcal{L}^1$. Thus we conclude that $H_{x,c}\cap\mathcal{L}^\infty=\{y\in\mathcal{L}^\infty|\langle y,x\rangle\leq$

$c\}$ is closed in $(\mathcal{L}^\infty, \sigma(\mathcal{L}^\infty, \mathcal{L}^1))$. Since by definition it holds that $\mathcal{U} \subseteq \mathcal{L}^\infty$ it follows that $\mathcal{U}$ is given by the intersection of the sets $H_{x,c} \cap \mathcal{L}^\infty$. Since any intersection of closed sets is closed we conclude that $\mathcal{U}$ is weakly* closed in $\mathcal{L}^\infty$.

Since $\mathcal{U}$ is bounded and weakly* closed, it follows from the theorem of Banach-Alaoglu that $\mathcal{U}$ is weakly* compact. Moreover since the objective function we try to minimize is continuous for the weak* topology, the proof is complete since any continuous function attains its minimum on a compact set.

The final claim, vanishing of the duality gap, follows from standard linear programming results. See for example [2] Chapter IV or [6] Chapter III. □

The following result highlights the correspondence between equilibrium prices and the optimization problems (P) and (LR).

Proposition 3.5. *Price processes $(\bar{A}, \bar{C})$ form an equilibrium with associated strategies $\bar{x}$ if and only if $(\bar{A}, \bar{C})$ and $\bar{x}$ are optimal solutions of (LR) and (P) respectivelly and the duality gap is zero, i.e. $LR^* = P^*$.*

Proof. Notice that for $x \in \mathfrak{G}$ (in particular when $Gx = g$), it holds that

$$\sum_{i \in I} L^{A,C,i}(x) = \langle x, c \rangle_{\mathcal{L}} \tag{3.32}$$

for all $(A, C) \in \mathcal{K}^1_G$ and hence

$$\sum_{i \in I} L^{A,C,i}(x^*) = \langle x^*, c \rangle_{\mathcal{L}} = P^* \tag{3.33}$$

for all $(A, C) \in \mathcal{K}^1_G$ for a primal optimal solution x^*. Now let us assume that (A^*, C^*) and x^* are optimal solutions of (LR) and (P) respectivelly. Then it holds that

$$\begin{aligned} LR^* &= \sup_{(A,C) \in \mathcal{K}^1_G} \sum_{i \in I} \inf_{x^i \in \mathfrak{F}^i} L^{A,C,i}(x^i) \\ &= \sum_{i \in I} \inf_{x^i \in \mathfrak{F}^i} L^{A^*,C^*,i}(x^i) \leq \sum_{i \in I} L^{A^*,C^*,i}(x^{i*}) = P^*. \end{aligned}$$

Since we have $LR^* = P^*$ by assumption, we conclude that the above inequality is in fact an equality and we obtain

$$L^{A^*,C^*,i}(x^{*i}) \leq L^{A^*,C^*,i}(x^i) \text{ for all } x^i \in \mathfrak{F}^i, i \in I \tag{3.34}$$

proving that (A^*, C^*) is an equilibrium with associated strategies x^*. Conversely, if we suppose that $(\bar{A}, \bar{C})$ is an equilibrium with associated strategies $\bar{x}$ then it follows that

$$\sum_{i \in I} \inf_{x^i \in \mathfrak{F}^i} L^{\bar{A},\bar{C},i}(x^i) = \sum_{i \in I} L^{\bar{A},\bar{C},i}(\bar{x}^i) \tag{3.35}$$

and $\bar{x} \in \mathfrak{G}$. The latter implies that the right hand side of (3.35) equals $\langle c, \bar{x}\rangle$ and we obtain

$$\begin{aligned} LR^* &= \sup_{(A,C)\in\mathcal{K}^1_G} inf_{0\le x\le \chi,\, Fx\ge f} \sum_{i\in I} \inf_{x^i\in\mathfrak{F}^i} L^{A,C,i}(x^i) \\ &= \sup_{(A,C)\in\mathcal{K}^1_G} \sum_{i\in I} \inf_{x^i\in\mathfrak{F}^i} L^{A,C,i}(x^i) \\ &\ge \sum_{i\in I} \inf_{x^i\in\mathfrak{F}^i} L^{\bar{A},\bar{C},i}(x^i) = \sum_{i\in I} L^{\bar{A},\bar{C},i}(\bar{x}^i) \\ &= \langle c, \bar{x}\rangle \ge \inf_{0\le x\in\mathfrak{F}\cap\mathfrak{G}} \langle c, x\rangle = P^*. \end{aligned}$$

Since weak duality implies equality between above terms, it follows that the optimal solution of the Lagrange relaxation problem (LR) is attained at $(\bar{A}, \bar{C})$, the primal optimal solution being attained at $\bar{x}$ and $LR^* = P^*$. □

It seems difficult to prove that the supremum in the dual problem (D) is attained in the full generality of this section. We will prove existence of a solution to the dual problem, and hence existence of equilibrium prices for allowances and CERs, essentially be inspection later in Subsection 4.4. However as we will see, this existence proof requires some technical assumptions.

3.5. Complementary Slackness Conditions

For the sake of convenience, we bundle the operators F and G providing the linear constraints into a single operator A defined by:

$$A : \mathcal{L}^\infty \ni x \mapsto Ax = (Fx, Gx) \in \mathcal{K}^\infty.$$

If we use L, a set with $|L| = \sum_{i\in I} m_i$, to label the scalar components x_l of $x = (x^i)_{i\in I}$, then for each $x \in \mathcal{L}^\infty$, Ax can be expressed as

$$Ax = \left(\sum_{l\in L} a_{k,l} x_l\right)_{k\in L}. \tag{3.36}$$

The specific forms of the constraints (3.3)-(3.6) and (2.9) and (2.10) give that $a_{k,l} \ne 0$ implies that $t_l \le s_k$. Hence each element of $x_l \in \mathcal{L}^\infty_{t_l}$ is also an element of $\mathcal{L}^\infty_{s_k}$ and A actually maps $\mathcal{L}^\infty$ into $\mathcal{K}^\infty$. Moreover if we write the canonical bilinear form giving the duality between $\mathcal{L}^\infty$ and $\mathcal{L}^1$ with this new notation, then

$$\langle x, y\rangle_{\mathcal{L}} = \sum_{l\in L} \mathbb{E}[x_l y_l] \text{ for all } x \in \mathcal{L}^\infty, y \in \mathcal{L}^1 \tag{3.37}$$

for the dual pair $(\mathcal{L}^\infty, \mathcal{L}^1)$ and

$$\langle z, w\rangle_{\mathcal{K}} = \sum_{k\in K} \mathbb{E}[z_k w_k] \text{ for all } z \in \mathcal{K}^\infty, w \in \mathcal{K}^1 \tag{3.38}$$

for the dual pair $(\mathcal{K}^\infty, \mathcal{K}^1)$ if we use K for the indexes of the components of the elements of $\mathcal{K}^\infty$ and $\mathcal{K}^1$.

Lemma 3.6. *The adjoint A^* of A is given by*

$$A^* : \mathcal{K}^1 \ni w \mapsto A^*(w) = \left(\sum_{k \in K} a_{k,l} \mathbb{E}[w_k | \mathcal{F}_{t_l}] \right)_{k \in K} \in \mathcal{L}^1 \tag{3.39}$$

Proof. Because $w_k \in \mathcal{L}^1_{s_k}$ it follows that $\mathbb{E}[w_k|\mathcal{F}_{t_l}] \in \mathcal{L}^1_{t_l}$ for all $k \in K$ hence $(A^*(w))_l \in \mathcal{L}^1_{t_l}$ for all $l \in L$ which proves that

$$A^* : \mathcal{K}^1 \mapsto \mathcal{L}^1. \tag{3.40}$$

Moreover for all $x \in \mathcal{L}^\infty$ and $w \in \mathcal{K}^1$ it holds that

$$\begin{aligned} \langle x, A^*(w) \rangle_{\mathcal{L}} &= \sum_{l \in L} \mathbb{E}\left[x_l \left(\sum_{k \in K} a_{k,l} \mathbb{E}[w_k|\mathcal{F}_{t_l}] \right) \right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\sum_{l \in L} \sum_{k \in K} x_l a_{k,l} w_k \middle| \mathcal{F}_{t_l} \right] \right] \\ &= \sum_{k \in K} \mathbb{E}\left[\left(\sum_{l \in L} a_{k,l} x_l \right) w_k \right] \\ &= \langle A(x), w \rangle_{\mathcal{K}}, \end{aligned} \tag{3.41}$$

where we used the property that $x_l \mathbb{E}[w_k|\mathcal{F}_{t_l}] = \mathbb{E}[x_l w_k|\mathcal{F}_{t_l}]$ for all $x_l \in \mathcal{L}^\infty_{t_l}$ and $w_k \in \mathcal{L}^1_{s_k}$. From (3.40) and (3.41) we conclude that A^* is the adjoint of A. □

Primal Feasibility

By the very definition of the operators F and G, the linear constraints $Fx \geq f$ translate for each market $m \in M$ and for the individual firm $i \in I(m)$ to the inequalities (3.3)-(3.6), while the global equality constraints $Gx = g$ yield (2.9) and (2.10) almost surely.

Dual Feasibility

The dual feasibility conditions

$$\begin{aligned} c - F^* \bar{w} - G^*(\bar{A}, \bar{C})^T &\geq 0 \\ \bar{w} &\geq 0 \end{aligned}$$

imply that for each market $m \in M$ and agent $i \in I(m)$, the following inequalities hold almost surely:

$$(\pi + \nu \mathbb{E}[w^{1,i}_{T^m_{q+1}} | \mathcal{F}_{T^m_q}] - w^{1,i}_{T^m_q}) \geq 0\,,\ (A^{q.m}_{T^m_q} - w^{1,i}_{T^m_q}) \geq 0 \tag{3.42}$$

$$(A^{q,m}_{T^m_q} - \mathbb{E}[A^{q+1,m}_{T^m_{q+1}} | \mathcal{F}_{T^m_q})] \geq 0 \tag{3.43}$$

$$(C^p_{T_p} - \mathbb{E}[C^{p+1}_{T_{p+1}} | \mathcal{F}_{T_p})] \geq 0\,,\ (C^p_{T_p} - w^{1,i}_{T_p} + w^{2,i}_{T_p}) \geq 0 \tag{3.44}$$

$$(S^i_t - w^{1,i}_t + w^{3,i}_t) \geq 0\,,\ (L^i_t - \mathbb{E}[w^{4,i}_T + rC_{T_p} | \mathcal{F}_t)] \geq 0, \tag{3.45}$$

as well as:

$$w^{1,i}_{T^m_q} \geq 0\,,\ w^{2,i}_{T^m_q} \geq 0 \text{ a.s. for all } i \in I(m) \tag{3.46}$$

Complementary Slackness

The complementary slackness condition

$$\langle \bar{x}, c - F^*\bar{w} - G^*(\bar{A},\bar{C})^T\rangle_{\mathcal{L}} = 0 \tag{3.47}$$

together with $\bar{x} \geq 0$ and $c - F^*\bar{w} - G^*(\bar{A},\bar{C})^T \geq 0$ give the following conditions for allowance trading

$$\begin{aligned} \langle \beta^i_{T^m_q}, \pi + \nu\mathbb{E}[A^{q+1,m}_{T^m_{q+1}}|\mathcal{F}_{T^m_q}] - w^{1,i}_{T^m_q}\rangle &= 0 \quad (3.48)\\ \langle \theta^i_{T^m_q}, A^{q,m}_{T^m_q} - w^{1,i}_{T^m_q}\rangle &= 0 \quad (3.49)\\ \langle \gamma^i_{T^m_q}, A^{q,m}_{T^m_q} - \mathbb{E}[A^{q+1,m}_{T^m_{q+1}}|\mathcal{F}_{T^m_q}]\rangle &= 0. \quad (3.50)\end{aligned}$$

for all $m \in M$, $q \in Q(m)$ and $i \in I(m)$ as well as the following conditions for CER trading

$$\begin{aligned} \langle \phi^i_{T_p}, C^p_{T_p} - \mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}]\rangle &= 0 \quad (3.51)\\ \langle \varphi^i_{T_p}, C^p_{T_p} - w^{1,i}_{T_p} + w^{2,i}_{T_p}\rangle &= 0 \quad (3.52)\end{aligned}$$

for all periods $p \in P$ and all agents $i \in I$. Moreover (3.47), together with dual feasibility give for each $i \in I$, the following reduction policy constraints:

$$\begin{aligned} \langle \xi^i_t, S^i_t - \mathbb{E}[w^{1,i}_{T^m_q}|\mathcal{F}_t] + w^{3,i}_t\rangle &= 0 \text{ for all } t = T^m_{q-1},\dots,T^m_q\\ \langle \zeta^i_t, L^i_t - \mathbb{E}[rC_{T_q}|\mathcal{F}_t] + \mathbb{E}[w^{4,i}_T|\mathcal{F}_t]\rangle &= 0 \text{ for all } t = T_{p-1},\dots,T_p.\end{aligned}$$

The other complementary slackness condition

$$\langle F\bar{x} - f, \bar{w}\rangle_{\mathcal{K}} = 0 \tag{3.53}$$

together with the primal feasibility condition $F\bar{x} - f \geq 0$ and the dual feasibility constraint $\bar{w} \geq 0$ imply:

$$\langle \beta^i_{T^m_q} + \theta^i_{T^m_q} + \varphi^i_{T^m_q} + \Pi^{iq}(\xi) - \Gamma^{i,q} - \beta^i_{T^m_{q-1}}, w^{1,i}_{T^m_q}\rangle = 0 \tag{3.54}$$

for all $m \in M$, $q \in Q(m)$ and $i \in I(m)$, as well as:

$$\langle \kappa^i - \varphi^i_{T_p}, w^{2,i}_{T_p}\rangle = 0. \tag{3.55}$$

for all periods $p \in P$ and all agents $i \in I$. Moreover they also give the following reduction policy constraints:

$$\begin{aligned} \langle w^{3,i}_t, \bar{\xi}^i - \xi^i_t\rangle &= 0 \text{ for all } t = 0,\dots,T-1\\ \langle w^{4,i}_T, 1 - \sum_{t=0}^{T-1}\zeta^i_t\rangle &= 0.\end{aligned}$$

4. Existence and Analysis of Equilibrium Prices

The goal of this section is to derive formulas for the equilibrium prices of allowance and CERs which explain the spread separating them. These formulas will come at the price of technical assumptions which we now formulate. As an added benefit, these formulas will make it possible to prove the existence results which we could not prove in the full generality of the abstract setting of last section.

4.1. Technical Assumptions

Notice that, because the complementary slackness conditions hold for all optimal solutions of the primal and dual problems, we choose to restrict ourselves to solutions which satisfy those conditions. In particular we only consider optimal strategies $\bar{x} \in \mathcal{L}^\infty$ where

$$\bar{\beta}^i_{T^m_q} = \left(\Gamma^{i,q} - \Pi^{i,q}(\bar{\xi}) - \bar{\varphi}^i_{T^m_q} - \bar{\theta}^i_{T^m_q}\right) \tag{4.1}$$

for all $m \in M$ and $q \in Q(m)$. This implies that the penalty does not exceed the short position and no firm uses more allowances/CERs than needed for compliance. Such a solution can be obtained from any optimal solution x^* by increasing/decreasing the amount of banked allowances and/or CERs.

Assumption 1. *Let for each market $m \in M$ the initial allocations fullfill*

$$\sum_{p=1}^{q} \sum_{i \in I(m)} \Theta^{i,p} > \sum_{p=1}^{q-1} \sum_{i \in I(m)} \Gamma^{i,p} \ \ a.s. \tag{4.2}$$

for all $q = 2, \dots, |Q(m)|$ while

$$\sum_{i \in I(M)} \Theta^{i,1} > 0 \ \ a.s.$$

This assumption guaranties that there is a positive amount of allowances remaining in each compliance period even tough the short position from one period can be withdrawn from next periods allocations. This is made precise in the following lemma. For the results in this paper to be true we need Assumption 4.1 so one could replace Lemma 1 by an other assumption yielding the same result.

Lemma 4.1. *For any strategy $\bar{x}$ that fullfills (4.1) it holds under Assumption 1 that*

$$\sum_{i \in I(M)} (\Theta^{i,q} - \bar{\beta}^i_{T^m_{q-1}} + \bar{\gamma}^i_{T^m_{q-1}}) > 0 \tag{4.3}$$

almost surely for all $m \in M$ and $q \in Q(m)$.

Proof. Due to

$$\sum_{i \in I(m)} (\bar{\theta}^i_{T^m_q} + \bar{\gamma}^i_{T^m_q} + \bar{\beta}^i_{T^m_{q-1}}) = \sum_{i \in I(m)} (\Theta^{i,q} + \bar{\gamma}^i_{T^m_{q-1}}) \tag{4.4}$$

we have

$$
\begin{aligned}
&\sum_{i\in I(M)} (\bar\beta^i_{T^m_q} - \bar\gamma^i_{T^m_q}) \\
&\quad = \sum_{i\in I(M)} (\Gamma^{i,q} - \Pi^{i,q}(\bar\xi) - \bar\theta^i_{T^m_q} - \bar\varphi^i_{T^m_q} - \bar\gamma^i_{T^m_q}) \\
&\quad = \sum_{i\in I(M)} (\Gamma^{i,q} - \Pi^{i,q}(\bar\xi) - \bar\varphi^i_{T^m_q} - \Theta^{i,q} + (\bar\beta^i_{T^m_{q-1}} - \bar\gamma^i_{T^m_{q-1}})). \qquad (4.5)
\end{aligned}
$$

Hence

$$
\sum_{i\in I(M)} (\bar\beta^i_{T^m_q} - \bar\gamma^i_{T^m_q}) = \sum_{p=1}^{q} \sum_{i\in I(M)} (\Gamma^{i,p} - \Pi^{i,p}(\bar\xi) - \bar\varphi^i_{T^m_p} - \Theta^{i,p,m})
$$

and consequently

$$
\sum_{i\in I(M)} (\bar\beta^i_{T^m_q} - \bar\gamma^i_{T^m_q}) \le \sum_{p=1}^{q-1} \sum_{i\in I(M)} (\Gamma^{i,p} - \Theta^{i,p,m})
$$

and

$$
\sum_{i\in I(M)} (\Theta^{i,q} - \bar\beta^i_{T^m_{q-1}} + \bar\gamma^i_{T^m_{q-1}}) \ge \sum_{i\in I(M)} \Theta^{i,q} - \sum_{p=1}^{q-1} \sum_{i\in I(M)} (\Gamma^{i,q} - \Theta^{i,q}) > 0.
$$

□

Lemma 4.2. *Assuming the conclusion (4.3) of Lemma 4.1, for each market $m \in M$, $i \in I(m)$ and $q \in Q(m)$, it holds that*

$$
\left\{\bar\beta^i_{T^m_q} > 0\right\} \subseteq \left\{A^{q,m}_{T^m_q} = \pi^{q,m} + \mathbb{E}[A^{q,m}_{T^m_q} | \mathcal{F}_{T^m_q}]\right\} \qquad (4.6)
$$

up to zero sets.

Proof.

$$
\begin{aligned}
\left\{\bar\beta^i_{T^m_q} > 0\right\} &\subseteq \left\{w^{1,i}_{T^m_q} = \pi^{q,m} + \mathbb{E}[A^{q,m}_{T^m_q} | \mathcal{F}_{T^m_q}]\right\} \cap \left\{A^{q,m}_{T^m_q} \ge w^{1,i}_{T^m_q}\right\} \\
&\subseteq \left\{A^{q,m}_{T^m_q} \ge \pi^{q,m} + \mathbb{E}[A^{q,m}_{T^m_q} | \mathcal{F}_{T^m_q}]\right\}
\end{aligned}
$$

Moreover on

$$
\begin{aligned}
\left\{A^{q,m}_{T^m_q} > \pi^{q,m} + \mathbb{E}[A^{q,m}_{T^m_q} | \mathcal{F}_{T^m_q}]\right\} &\subseteq \bigcap_{i\in I(m)} \left\{\gamma^i_{T^m_q} = 0\right\} && (4.7) \\
&\subseteq \bigcap_{i\in I(m)} \left\{\theta^i_{T^m_q} > 0\right\} && (4.8) \\
&\subseteq \bigcap_{i\in I(m)} \left\{w^{1,i}_{T^m_q} = A^{q,m}_{T^m_q}\right\} && (4.9)
\end{aligned}
$$

where (4.8) follows from Lemma 4.1 This is a zero set due to primal feasibility

$$\pi^{q,m} + \mathbb{E}[A^{q,m}_{T^m_q} | \mathcal{F}_{T^m_q}] \geq w^{1,i}_{T^m_q} \tag{4.10}$$

which concludes the proof. □

In the sequel let for each agent $i \in I(m)$ we let

$$\bar{\Delta}^{i,q}_{T^m_q} = \Gamma^{i,q} - \Pi^{i,q}(\bar{\xi}). \tag{4.11}$$

denote the effective emissions of agent i in period $q \in Q(m)$. In the next lemma we prove that if the amount of CERs used for compliance is not extremal then allowance price and CER price are equal up to zero sets.

Lemma 4.3. *Assuming the conclusion* (4.3) *of Lemma* 4.1*, for any optimal strategy* $\bar{x} \in \mathcal{L}^\infty$ *fullfilling restriction* (4.1) *it holds that*

$$\Big\{ \sum_{i \in I(m)} \bar{\varphi}^i_{T^m_q} \in \big(0, \sum_{i \in I(m)} \min(\Delta^{i,q}_{T^m_q}, \kappa^i)\big) \Big\} \subseteq \Big\{ A^{q,m}_{T^m_q} = C^p_{T^m_q} \Big\} \tag{4.12}$$

Proof. First we notice that

$$\begin{aligned}
&\Big\{\bar{\varphi}^i_{T^m_q} < \min(\bar{\Delta}^{i,q}_{T^m_q}, \kappa^i)\Big\} \cap \Big\{\bar{\beta}^i_{T^m_q} = 0\Big\} && (4.13)\\
&\quad \subseteq \Big(\Big\{\bar{\varphi}^i_{T^m_q} < \bar{\Delta}^{i,q}_{T^m_q}\Big\} \cap \Big\{\bar{\beta}^i_{T^m_q} = 0\Big\}\Big) && (4.14)\\
&\qquad \cup \Big(\Big\{\bar{\varphi}^i_{T^m_q} < \kappa^i\Big\} \cap \Big\{\bar{\Delta}^{i,q}_{T^m_q} \geq \kappa^i\Big\} \cap \Big\{\bar{\beta}^i_{T^m_q} = 0\Big\}\Big)\\
&\quad \subseteq \Big\{\bar{\theta}^i_{T^m_q} > 0\Big\} \subseteq \Big\{A^{q,m}_{T^m_q} = w^{1,i}_{T^m_q}\Big\}
\end{aligned}$$

where the last inclusion follows from (3.49). Moreover it holds that

$$\begin{aligned}
\Big\{\bar{\beta}^i_{T^m_q} > 0\Big\} &\subseteq \Big\{\bar{\beta}^i_{T^m_q} > 0\Big\} \cap \Big\{A^{q,m}_{T^m_q} = \pi^{q,m} + \mathbb{E}[A^{q,m}_{T^m_q} | \mathcal{F}_{T^m_q}]\Big\}\\
&\subseteq \Big\{A^{q,m}_{T^m_q} = w^{1,i}_{T^m_q}\Big\}
\end{aligned}$$

where the last inclusion follows from (3.48) and Lemma 4.2. Also,

$$\begin{aligned}
&\Big\{\bar{\varphi}^i_{T^m_q} < \min(\bar{\Delta}^{i,q}_{T^m_q}, \kappa^i)\Big\} && (4.15)\\
&\quad \subseteq \Big(\Big\{\bar{\varphi}^i_{T^m_q} < \min(\bar{\Delta}^{i,q}_{T^m_q}, \kappa^i)\Big\} \cap \Big\{\bar{\beta}^i_{T^m_q} = 0\Big\}\Big)\\
&\qquad \cup \Big(\Big\{\bar{\varphi}^i_{T^m_q} < \min(\bar{\Delta}^{i,q}_{T^m_q}, \kappa^i)\Big\} \cap \Big\{\bar{\beta}^i_{T^m_q} > 0\Big\}\Big)\\
&\quad \subseteq \Big\{\bar{\varphi}^i_{T^m_q} < \min(\bar{\Delta}^{i,q}_{T^m_q}, \kappa^i)\Big\} \cap \Big\{A^{q,m}_{T^m_q} = w^{1,i}_{T^m_q}\Big\} \subseteq \Big\{A^{q,m}_{T^m_q} \leq C^p_{T^m_q}\Big\}.
\end{aligned}$$

Hence we conclude that

$$\begin{aligned}
&\Big\{ \sum_{i\in I(m)} \bar\varphi^i_{T^m_q} \in \big(0, \sum_{i\in I(m)} \min(\Delta^{i,q}_{T^m_q}, \kappa^i)\big) \Big\} \\
&\quad \subseteq \Big(\bigcup_{i\in I(m)} \Big\{ \bar\varphi^i_{T^m_q} > 0 \Big\} \Big) \cap \Big(\bigcup_{i\in I(m)} \Big\{ \bar\varphi^i_{T^m_q} < \min(\bar\Delta^{i,q}_{T^m_q}, \kappa^i) \Big\} \Big) \\
&\quad \subseteq \Big\{ A^{q,m}_{T^m_q} \geq C^p_{T^m_q} \Big\} \cap \Big\{ A^{q,m}_{T^m_q} \leq C^p_{T^m_q} \Big\} \subseteq \Big\{ A^{q,m}_{T^m_q} = C^p_{T^m_q} \Big\}
\end{aligned}$$

where the second to last inclusion follows from (3.52). □

Assumption 2. *For each $m \in M$ and $q \in Q(m)$ there is an agent $i^* \in I(m)$ satisfying*

$$\Gamma^{i^*,q} - \Pi^{i^*,q}(\bar\xi^{i^*}) > \kappa^{i^*} \tag{4.16}$$

almost surely and the $\mathcal{F}_{T^m_q-1}$-conditional distribution of $\Gamma^{i^*,q}$ has almost surely no point mass, or equivalently

$$\mathbb{P}\Big[\Big\{ \Gamma^{i^*,q} + \sum_{i\in\tilde I} \Gamma^{i,q} = Z \Big\}\Big] = 0 \tag{4.17}$$

for all $\mathcal{F}_{T^m_q-1}$-measurable random variables Z and $\tilde I \subseteq I(m) \setminus i^*$.

This assumption is reasonable for the 2008-2012 phase of EU ETS. There it is well known that the electricity sector could use significantly more CERs than allowed. Therefore in EU ETS agents in the electricity sector need to convince other industries to sell allowances and comply with more CERs. Only in that way the total CER limit for EU ETS $\sum_{i\in I(m)} \varphi^i_{T^m_q} = \sum_{i\in I(m)} \kappa^i$ can be reached. Moreover the electricity sector faces uncontroled emissions (from the primary and secondary markets) which satisfy the no point mass condition.

Lemma 4.4. *Under Assumption 2 it holds that for any strategy $\bar x$ that fullfills (4.1) we have for all $m \in M$ and $q \in Q(m)$*

$$\mathbb{P}\Big[\Big\{ \sum_{i\in I(m)} \bar\beta^i_{T^m_q} = 0 \Big\} \cap \Big\{ \sum_{i\in I(m)} \bar\gamma^i_{T^m_q} = 0 \Big\} \cap \Big\{ \sum_{i\in I(m)} \bar\varphi^i_{T^m_q} = \sum_{i\in I(m)} \min(\bar\Delta^{i,q}_{T^m_q}, \kappa^i) \Big\}\Big] = 0 \tag{4.18}$$

and

$$\mathbb{P}\Big[\Big\{ \sum_{i\in I(m)} \bar\beta^i_{T^m_q} = 0 \Big\} \cap \Big\{ \sum_{i\in I(m)} \bar\gamma^i_{T^m_q} = 0 \Big\} \cap \Big\{ \sum_{i\in I(m)} \bar\varphi^i_{T^m_q} = 0 \Big\}\Big] = 0. \tag{4.19}$$

Proof. Notice first that for all $m \in M$ and $q \in Q(m)$ Assumption 2 implies that

$$\mathbb{P}\Big[\Big\{ \Gamma^{i^*,q} + \sum_{i\in I(m)\setminus i^*} (\Gamma^{i,q} - Z^i)\mathbf{1}_{\{\mathcal{A}^i\}} = Z \Big\}\Big] = 0 \tag{4.20}$$

for all $\mathcal{F}_{T^m_q-1}$-measurable random variables Z, Z^i and sets $\mathcal{A}^i \in \mathcal{F}_{T^m_q}$.

$$
\begin{aligned}
&\Big\{\sum_{i\in I(m)} \bar\beta^i_{T^m_q}=0\Big\}\cap\Big\{\sum_{i\in I(m)} \bar\gamma^i_{T^m_q}=0\Big\}\cap\Big\{\sum_{i\in I(m)} \bar\varphi^i_{T^m_q}=\sum_{i\in I(m)}\min(\bar\Delta^{i,q}_{T^m_q},\kappa^i)\Big\}\\
&\subseteq\Big(\bigcap_{i\in I(m)}\Big\{\Gamma^{i,q}-\Pi^{i,q}(\bar\xi)-\bar\theta^i_{T^m_q}-\bar\varphi^i_{T^m_q}=0\Big\}\Big)\\
&\qquad\cap\Big\{\sum_{i\in I(m)}\bar\theta^i_{T^m_q}=\sum_{i\in I(m)}\big(\Theta^{i,q}-\bar\beta^i_{T^m_{q-1}}+\bar\gamma^i_{T^m_{q-1}}\big)\Big\}\\
&\qquad\qquad\cap\Big(\bigcap_{i\in I(m)}\Big\{\bar\varphi^i_{T^m_q}=\min(\bar\Delta^{i,q}_{T^m_q},\kappa^i)\Big\}\Big)\\
&\subseteq\Big\{\Gamma^{i^*,q}+\sum_{i\in I(m)\setminus i^*}(\Gamma^{i,q}-\Pi^{i,q}(\bar\xi)-\kappa^i)\mathbf{1}_{\{\mathcal{A}^i\}}=\sum_{i\in I(m)}\big(\Theta^{i,q}-\bar\beta^i_{T^m_{q-1}}+\bar\gamma^i_{T^m_{q-1}}\big)\Big\}
\end{aligned}
$$

where we used (4.16) for the last inclusion and $\mathcal{A}^i=\{\bar\varphi^i_{T^m_q}=\bar\Delta^{i,q}_{T^m_q}\}\in\mathcal{F}_{T^m_q}$ for all $i\in I(m)$. Since $Z^i:=\Pi^{i,q}(\bar\xi)+\kappa^i$ and $Z:=\sum_{i\in I(m)}\big(\Theta^{i,q}-\bar\beta^i_{T^m_{q-1}}+\bar\gamma^i_{T^m_{q-1}}\big)$ are $\mathcal{F}_{T^m_q-1}$-measurable which together with (4.20) implies (4.18). Similarly it holds that

$$
\begin{aligned}
&\Big\{\sum_{i\in I(m)} \bar\beta^i_{T^m_q}=0\Big\}\cap\Big\{\sum_{i\in I(m)} \bar\gamma^i_{T^m_q}=0\Big\}\cap\Big\{\sum_{i\in I(m)} \bar\varphi^i_{T^m_q}=0\Big\}\\
&\subseteq\Big(\bigcap_{i\in I(m)}\Big\{\Gamma^{i,q}-\Pi^{i,q}(\bar\xi)-\bar\theta^i_{T^m_q}-\bar\varphi^i_{T^m_q}=0\Big\}\Big)\\
&\qquad\cap\Big\{\sum_{i\in I(m)}\bar\theta^i_{T^m_q}=\sum_{i\in I(m)}\big(\Theta^{i,q}-\bar\beta^i_{T^m_{q-1}}+\bar\gamma^i_{T^m_{q-1}}\big)\Big\}\cap\Big(\bigcap_{i\in I(m)}\Big\{\bar\varphi^i_{T^m_q}=0\Big\}\Big)\\
&\subseteq\Big\{\Gamma^{i,q}+\sum_{i\in I(m)\setminus i^*}\Gamma^{i,q}=\sum_{i\in I(m)}\big(\Theta^{i,q}+\Pi^{i,q}(\bar\xi))-\bar\beta^i_{T^m_{q-1}}+\bar\gamma^i_{T^m_{q-1}}\big)\Big\}.
\end{aligned}
$$

Since $Z:=\sum_{i\in I(m)}\big(\Theta^{i,q}+\Pi^{i,q}(\bar\xi))-\bar\beta^i_{T^m_{q-1}}+\bar\gamma^i_{T^m_{q-1}}\big)$ is $\mathcal{F}_{T^m_q-1}$-measurable (4.19) follows from (4.17). □

Assumption 3. *For all $m\in M$, $q\in Q(m)$ and $p\in P$ with $T^m_q=T_p$, the $\mathcal{F}_{T^m_q-1}$-conditional distribution of the sum of $\sum_{i\in I(m)}\Gamma^{i,q}$ and $\sum_{i\in I}\Xi^{i,p}$ possesses almost surely no point mass, or equivalently, for all $\mathcal{F}_{T_p-1}$-measurable random variables Z*

$$
\mathbb{P}\Big[\Big\{\sum_{i\in I(m)}\Gamma^{iq}+\sum_{i\in I}\Xi^{i,p}=Z\Big\}\Big]=0. \tag{4.21}
$$

Lemma 4.5. *Under Assumption* 3 *it holds that*

$$\mathbb{P}\Big[\Big\{\sum_{i\in I(m)}\bar\beta^i_{T^m_q}=0\Big\}\cap\Big\{\sum_{i\in I(m)}\bar\gamma^i_{T^m_q}=0\Big\}\cap\Big\{\sum_{i\in I}\bar\phi^i_{T^m_q}=0\Big\}\Big]=0 \tag{4.22}$$

and hence

$$\Big\{\sum_{i\in I(m)}\bar\beta^i_{T^m_q}=0\Big\}\cap\Big\{\sum_{i\in I(m)}\bar\gamma^i_{T^m_q}=0\Big\}\subseteq\Big\{C_{T_p}=\mathbb{E}[C_{T_{p+1}}|\mathcal{F}_{T_p}]\Big\} \tag{4.23}$$

up to zero sets.

Proof. Notice that

$$\begin{aligned}
&\Big\{\sum_{i\in I(m)}\bar\beta^i_{T^m_q}=0\Big\}\cap\Big\{\sum_{i\in I(m)}\bar\gamma^i_{T^m_q}=0\Big\}\cap\Big\{\sum_{i\in I}\bar\phi^i_{T^m_q}=0\Big\}\\
&\subseteq\Big\{\sum_{i\in I(m)}\Big(\Gamma^{i,q}-\Pi^{i,q}(\bar\xi)-\bar\theta^i_{T^m_q}-\bar\varphi^i_{T^m_q}\Big)=0\Big\}\\
&\qquad\cap\Big\{\sum_{i\in I(m)}\bar\theta^i_{T^m_q}=\sum_{i\in I(m)}\Big(\Theta^{i,q}-\bar\beta^i_{T^m_{q-1}}+\bar\gamma^i_{T^m_{q-1}}\Big)\Big\}\\
&\qquad\cap\Big\{\sum_{i\in I}\bar\varphi^i_{T^m_q}=\sum_{i\in I}\Big(\bar\phi^i_{T^m_q}+\Pi^{i,p}(\bar\zeta)-\Xi^{i,p}\Big)\Big\}\\
&\subseteq\Big\{\sum_{i\in I(m)}\Big(\Gamma^{i,q}-\Pi^{i,q}(\bar\xi)-\bar\theta^i_{T^m_q}\Big)-\sum_{i\in I}\Big(\bar\phi^i_{T^m_q}+\Pi^{i,p}(\bar\zeta)-\Xi^{i,p}\Big)=0\Big\}\\
&\qquad\cap\Big\{\sum_{i\in I(m)}\bar\theta^i_{T^m_q}=\sum_{i\in I(m)}\Big(\Theta^{i,q}-\bar\beta^i_{T^m_{q-1}}+\bar\gamma^i_{T^m_{q-1}}\Big)\Big\}.
\end{aligned}$$

Since moreover the random variables

$$\sum_{i\in I(m)}\Big(\Pi^{i,q}(\bar\xi)+\bar\theta^i_{T^m_q}\Big)\ \text{and}\ \sum_{i\in I}\Big(\bar\phi^i_{T^m_q}+\Pi^{i,p}(\bar\zeta)\Big) \tag{4.24}$$

are $\mathcal{F}_{T^m_q-1}$-measurable, the lemma follows from Assumption 3.

□

Corollary 4.6. *Assuming the conclusion (*4.3*) of Lemma* 4.1*, Assumptions* 2 *and* 3*, for any strategy $\bar x$ that fullfills (*4.1*), it holds that for each $m\in M$ and $q\in Q(m)$*

$$\Big\{\sum_{i\in I(m)}\bar\beta^i_{T^m_q}=0\Big\}\cap\Big\{\sum_{i\in I(m)}\bar\gamma^i_{T^m_q}=0\Big\}\subseteq\Big\{A_{T^m_q}=\mathbb{E}[C_{T_{p+1}}|\mathcal{F}_{T^m_q}]\Big\} \tag{4.25}$$

up to zero sets.

Assumption 4. *For all $p\in P$ it holds that*

$$\sum_{i\in I}-\Xi^{i,p}>0 \tag{4.26}$$

almost surely. Moreover for all $\mathcal{F}_{T_p-1}$-measurable random variables Z

$$\mathbb{P}\left[\left\{\sum_{i\in I}\Xi^{i,p}+\sum_{i\in\tilde{I}}\Gamma^{i,q}=Z\right\}\right]=0. \tag{4.27}$$

for all $\tilde{I}\subseteq I(m)$.

$\Xi^{i,p}$ models randomness in the number of CERs and offsets produced. Assumption 4 says that the random amount of CERs that enters the market is almost surely greater than the amount of offsets. (resulting for example from projects that were started before $t=0$). It could as well be assumed that the optimal strategy ζ always produces more CERs than used for offsetting purposes.

Lemma 4.7. *Under Assumption 4 it holds that*

$$\mathbb{P}\left[\left\{\sum_{i\in I(m)}\bar{\varphi}^i_{T_p}=0\right\}\cap\left\{\sum_{i\in I}\bar{\phi}^i_{T_p}=0\right\}\right]=0, \tag{4.28}$$

and

$$\mathbb{P}\left[\left\{\sum_{i\in I(m)}\bar{\varphi}^i_{T_p}=\sum_{i\in I(m)}\min(\bar{\Delta}^{i,q}_{T^m_q},\kappa^i)\right\}\cap\left\{\sum_{i\in I}\bar{\phi}^i_{T_p}=0\right\}\right]=0, \tag{4.29}$$

and consequently

$$\left\{\sum_{i\in I(m)}\bar{\varphi}^i_{T_p}=0\right\}\cup\left\{\sum_{i\in I(m)}\bar{\varphi}^i_{T_p}=\sum_{i\in I(m)}\min(\bar{\Delta}^{i,q}_{T^m_q},\kappa^i)\right\}\subseteq\left\{C_{T_p}=\mathbb{E}[C_{T_{p+1}}|\mathcal{F}_{T_p}]\right\} \tag{4.30}$$

up to zero sets.

Proof.

$$\left\{\sum_{i\in I(m)}\bar{\varphi}^i_{T_p}=0\right\}\cap\left\{\sum_{i\in I}\bar{\phi}^i_{T_p}=0\right\}\subseteq\left\{\sum_{i\in I}\bar{\phi}^i_{T_{p-1}}+\Pi^{i,p}(\bar{\zeta})-\Xi^{i,p}=0\right\} \tag{4.31}$$

this together with (4.26) implies (4.28). Due to (4.27) it holds that

$$\mathbb{P}\left[\left\{\sum_{i\in I}\Xi^{i,p}+\sum_{i\in I(m)}(\Gamma^{i,q}-Z^i)\mathbf{1}_{\{\mathcal{A}^i\}}+\kappa^i\mathbf{1}_{\{(\mathcal{A}^i)^c\}}=Z\right\}\right]=0. \tag{4.32}$$

for all $\mathcal{F}_{T_p-1}$-measurable Z,Z^i and $\mathcal{A}^i\in\mathcal{F}_{T_p-1}$. This together with

$$\begin{aligned}&\left\{\sum_{i\in I(m)}\bar{\varphi}^i_{T_p}=\sum_{i\in I(m)}\min(\bar{\Delta}^{i,q}_{T^m_q},\kappa^i)\right\}\cap\left\{\sum_{i\in I}\bar{\phi}^i_{T_p}=0\right\}\\&\quad\subseteq\left\{\sum_{i\in I}\bar{\phi}^i_{T_{p-1}}+\Pi^{i,p}(\bar{\zeta})-\Xi^{i,p}=\sum_{i\in I(m)}\min(\Gamma^{i,q}-\Pi^{i,q}(\bar{\xi}),\kappa^i)\right\}\\&\quad\subseteq\left\{\sum_{i\in I}\bar{\phi}^i_{T_{p-1}}+\Pi^{i,p}(\bar{\zeta})-\Xi^{i,p}=\sum_{i\in I(m)}(\Gamma^{i,q}-\Pi^{i,q}(\bar{\xi}))\mathbf{1}_{\{\mathcal{A}^i\}}+\kappa^i\mathbf{1}_{\{(\mathcal{A}^i)^c\}})\right\}\end{aligned}$$

implies (4.29) with $\mathcal{A}^i = \{\bar{\varphi}^i_{T_{p-1}} = \Gamma^{i,q} - \Pi^{i,q}(\bar{\xi})\}$ and $\mathcal{F}_{T_p-1}$-measurable $Z^i = \Pi^{i,q}(\bar{\xi})$, $Z = \sum_{i\in I}\bar{\phi}^i_{T_{p-1}} + \Pi^{i,p}(\bar{\zeta})$.

□

4.2. First Equilibrium Price Formulas

In the sequel we assume that Lemma 4.1, Assumption 2, 3 and 4 are fullfilled.

Proposition 4.8. *Let $p \in P$, $m \in M$ and $q \in Q(m)$ such that $T^m_q = T_p$. Moreover let (A, C) be an equilibrium with corresponding strategies $x \in X$ then it holds for that*

$$\begin{aligned} A^{q,m}_{T^m_q} &= (\pi + \nu\mathbb{E}[A^{q+1,m}_{T^m_{q+1}}|\mathcal{F}_{T^m_q})]\mathbf{1}_{\{\beta_{T^m_q}>0\}} \\ &\quad + \left(\mathbb{E}[A^{q+1,m}_{T^m_{q+1}}|\mathcal{F}_{T^m_q}]\mathbf{1}_{\{\gamma_{T^m_q}>0\}} + \mathbb{E}[C^{p+1}_{T_{q+1}}|\mathcal{F}_{T^m_q})\mathbf{1}_{\{\gamma_{T^m_q}=0\}}\right)\mathbf{1}_{\{\beta_{T^m_q}=0\}} \end{aligned} \tag{4.33}$$

with

$$\beta_{T^m_q} = \sum_{i\in I(m)} \beta^i_{T^m_q}\,,\ \gamma_{T^m_q} = \sum_{i\in I(m)} \gamma^i_{T^m_q}. \tag{4.34}$$

The intuitive meaning of this formula is the following. On the event $\{\beta_{T^m_q} > 0\}$ that the economy at large is short of allowances despite the usage of CERs, then the price of the allowance is given by the penalty π plus the cost of the allowances from the next period which need to be used for compliance, appearing in the formula as the conditional expectation of the price of the next period. Alternatively, on the event $\{\beta_{T^m_q} = 0\}$ that the economy is not short of allowances at time of compliance, then the price of the allowance is either the expected value of the an allowance the next period on the event $\{\gamma_{T^m_q} > 0\}$ that the allowances are banked for use in the next period, or the expected value of a CER the next period on the event $\{\gamma_{T^m_q} = 0\}$ that the allowances are not banked and we use CERs for compliance.

Proof. Since we clearly have:

$$\mathbb{P}\left[\{\beta_{T^m_q} > 0\} \cup \left(\{\beta_{T^m_q} = 0\} \cap \left(\{\gamma_{T^m_q} > 0\} \cup \{\gamma_{T^m_q} = 0\}\right)\right)\right] = 1 \tag{4.35}$$

the rest follows from Lemma 4.2, Corollary 4.6 and condition (3.50). □

Proposition 4.9. *Let $p \in P$, $m \in M$ and $q \in Q(m)$ such that $T^m_q = T_p$. Moreover let (A, C) an equilibrium with corresponding strategies $x \in X$ then it holds for each $q \in Q$ that*

$$C^p_{T_p} = \mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}]\mathbf{1}_{\{\varphi_{T_p}\in\{0,\kappa_p\}\}} + A^{q,m}_{T_p}\mathbf{1}_{\{\varphi_{T_p}\in(0,\kappa_p)\}} \tag{4.36}$$

with

$$\varphi_{T_p} = \sum_{i\in I(m)} \varphi^i_{T_p} \text{ and } \kappa_p = \sum_{i\in I(m)} \min(\Delta^{i,p}, \kappa^i) \text{ for all } p \in P. \tag{4.37}$$

The intuitive meaning of this formula is the following. As defined κ_p is the maximum amount of CERs which can be used at time T_p and φ_{T_p} is the total amount of CERs used for compliance at time T_p. So on the event that φ_{T_p} is in the open interval $(0, \kappa_p)$, allowances and CERs can be used interchangeably for compliance, so the price $C^p_{T_p}$ of a CER has to be the same as the price of an allowance $A^{q,m}_{T_p}$. On the other end, when φ_{T_p} is either 0 or κ_p, only CERs from the next period can be used and the price $C^p_{T_p}$ becomes the conditional expectation of the price of a CER of the next period.

Proof. This follows directly from Lemma 4.7 and Lemma 4.3. □

4.3. Equilibrium Abatement Strategies

Proposition 4.10. *Fix a market $m \in M$ an associated compliance period $q \in Q(m)$ and time $t \in \{T^m_{q-1}, \ldots, T^m_q - 1\}$. Let $\bar{A}$ be an equilibrium allowance price process and ξ^* the corresponding equilibrium short term abatement policy then it holds that*

$$\begin{aligned} \{S^i_t - A^{q,m}_t < 0\} &\subseteq \{\xi^{*i}_t = \overline{\xi}^i\} \\ \{S^i_t - A^{q,m}_t > 0\} &\subseteq \{\xi^{*i}_t = 0\}. \end{aligned}$$

The intuitive meaning of this result is the following. On the event $\{S^i_t - A^{q,m}_t < 0\}$ that *fuel switch* is cheaper than the cost of an allowance, the production is maximal, while on the event $\{S^i_t - A^{q,m}_t > 0\}$ that *fuel switch* is more expensive than the cost of an allowance, the equilibrium production is 0.

Proof. The complementary slackness conditions read

$$\langle \xi^i_t, S^i_t - A^{q,m}_t + w^{3,i}_t \rangle = 0 \text{ for all } i \in I(m) \tag{4.38}$$

$$\langle \overline{\xi}^i - \xi^i_t, w^{3,i}_t \rangle = 0 \text{ for all } i \in I(m). \tag{4.39}$$

While dual feasibility implies

$$S^i_t - A^{q,m}_t + w^{3,i}_t \geq 0 \text{ for all } i \in I(m) \tag{4.40}$$

$$w^{3,i}_t \geq 0 \text{ for all } i \in I(m). \tag{4.41}$$

On $\{S_t - A^{q,m}_t < 0\}$ condition (4.40) implies that $w^{3,i}_t > 0$ and hence $\xi^*_t = \overline{\xi}^i$ with condition (4.39). On the other hand on $\{S^i_t - A^{q,m}_t > 0\}$ we have $S^i_t - A^{q,m}_t + w^{3,i}_t > 0$ due to (4.41) and hence $\xi^{*i}_t = 0$ due to (4.38). □

In the next proposition we show how long term projects such as CER projects are exercised in contrast to short term reductions such as fuel switches. In contrast to these are only exercised if the spread between reduction cost and CER price exceeds a price proces B^i_t given by

$$B^i_t = \mathbb{E}[L^i_{\tau^i} - \hat{C}_\tau | \mathcal{F}_t] \tag{4.42}$$

for all $t = 0, \ldots, T-1$ where $\tau^i = \inf\{t;\ \zeta^i_t > 0\}$ and $\hat{C}_t = \sum_{p \in P} \tilde{C}_t \mathbf{1}_{\{T_{p-1} \leq t < T_p\}}$. In the case that we have only one market and no restriction on CER compliance in this market CDM projects can be seen as normal irreversible emission reduction projects and this result can be applied by setting $A = C$.

Proposition 4.11. *Fix $p \in P$ and let $t \in \{T_{p-1}, \ldots, T_p - 1\}$ while C is an equilibrium CER price process and ζ^* the corresponding equilibrium long term abatement policy. Then it holds that*

$$\begin{aligned} \{L_t^i - C_t^p + B_t^i < 0\} &\subseteq \{\zeta_t^{*i} = 1\} \\ \{L_t^i - C_t^p + B_t^i > 0\} &\subseteq \{\zeta_t^{*i} = 0\}. \end{aligned}$$

Proof.

$$\begin{aligned} \langle \zeta_t^i, L_t^i - \mathbb{E}[rC_{T_q}|\mathcal{F}_t] + \mathbb{E}[w_T^{4,i}|\mathcal{F}_t] \rangle &= 0 \text{ for all } t \le T_q, q \in Q, i \in I \\ \langle 1 - \sum_{t=0}^{T} \zeta_t^i, w_T^{4,i} \rangle &= 0 \text{ for all } q \in Q, i \in I. \end{aligned}$$

Dual feasibility implies

$$\begin{aligned} L_t^i - \mathbb{E}[rC_{T_q}|\mathcal{F}_t] + \mathbb{E}[w_T^{4,i}|\mathcal{F}_t] &\ge 0 \text{ for all } i \in I, t = 0, \ldots, T-1 \\ w_T^{4,i} &\ge 0 \text{ for all } i \in I \end{aligned}$$

□

4.4. Existence of Equilibrium Prices

For the sake of convenience, we restate the contents of the two propositions of Subsection 4.2 in the form of a necessary condition for the expression of equilibrium prices.

Proposition 4.12. *Let q' denote the last compliance period $|Q(m)|$ of market $m \in M$. And $p' = |P|$ be the last period where CERs can be used for compliance. For each $x \in \mathcal{L}^\infty$, we denote by $A(x)$ and $C(x)$ the price processes defined recursively by (4.33) and (4.36) through the backward induction given by these formulas and starting at $\mathbb{E}[A^{q'+1,m}_{T^m_{q'+1}}|\mathcal{F}_{T^m_{q'}}] = 0$ and $\mathbb{E}[C^{p'}_{T_{p'+1}}|\mathcal{F}_{T_{p'}}] = 0$. Then if (A, C) is an equilibrium with corresponding strategies $x \in \mathcal{L}^\infty$ then $A = A(x)$ and $C = C(x)$.*

We now revisit the existence problem for the dual problem. Recall that we assume that the conclusion of Lemma 4.1, and Assumptions 2, 3 and 4 hold true.

Proposition 4.13. *The dual optimal solution is attained and the duality gap is zero.*

Proof. The dual objective function reads

$$\mathbb{E}\Bigg[\sum_{i \in I}\bigg(- w_T^{4,i} - \sum_{p \in P}\Big(\kappa^i w_{T_p}^{2,i} + \Xi^{i,p} C_{T_p}^p\Big)\bigg) + \sum_{m \in M}\sum_{i \in I(m)}\sum_{q \in Q(m)}\Big(\Gamma^{i,q} w_{T_q^m}^{1,i} - \Theta^{i,q} A_{T_q^m} - \sum_{t=T_{q-1}^m}^{T_q^m - 1} \chi^i w_t^{3,i}\Big)\Bigg]$$

Now let us prove that A and C as defined in (4.8) and (4.9) together with

$$
\begin{aligned}
w^{1,i}_{T^m_q} &= A^{q,m}_{T^m_q} && (4.43)\\
w^{2,i}_{T_p} &= (A^{q(p),m(p)}_{T_p} - C^p_{T_p})^+ && (4.44)\\
w^{3,i}_{t} &= (A^{q,m}_{t} - S^i_t)^+ \quad t \in \{T^m_{q-1}, \dots, T^m_q - 1\} && (4.45)\\
w^{4,i}_{T} &= \hat{C}_{\tau^i} - L^i_{\tau^i} && (4.46)
\end{aligned}
$$

give a dual optimal solution by proving that its value equals the primal optimal solution. Here $q(p)$ and $m(p)$ are defined such that $T^m_q = T_p$ while the stopping time τ^i is given by $\tau^i = \inf\{t;\ \zeta^i_t > 0\}$ and $C_{\tau^i} := \sum_{p \in P} C^p_{\tau^i} \mathbf{1}_{\{T_{p-1} \le \tau^i \le T_P - 1\}}$. Using (4.8), (4.9) and (3.3)-(3.6), the dual objective value can be rewritten as:

$$
\mathbb{E}\left[\sum_{i \in I}\left(- (\hat{C}_{\tau^i} - L^i_{\tau^i}) - \sum_{p \in P}\left(\kappa^i (A^{q(p),m(p)}_{T_p} - C^p_{T_p})^+ + \Xi^{i,p} C^p_{T_p}\right)\right) + \sum_{m \in M} \sum_{i \in I(m)} \sum_{q \in Q(m)} \left(\Gamma^{i,q} A^{q,m}_{T^m_q} - \Theta^{i,q} A^{q,m}_{T^m_q} - \sum_{t=T^m_{q-1}}^{T^m_q - 1} \chi^i (A^{q,m}_t - S^i_t)^+\right)\right].
$$

Using the primal feasibility conditions (2.9) and (2.10) as well as Propositions 4.10, 4.11 and the definition of τ^i this translates to

$$
\begin{aligned}
\mathbb{E}\Bigg[&\sum_{i \in I}\Bigg(- \sum_{t=0}^{T^m_{|Q|}-1} \overline{\zeta}_t (\hat{C}_t - L^i_t)\\
&- \sum_{p \in P}\left(\overline{\varphi}^i (A^{q(p),m(p)}_{T_p} - C^p_{T_p}) + \big(\overline{\varphi}^i_{T_p} + \overline{\phi}^i_{T_p} - \overline{\phi}^i_{T_{p-1}} - \Pi^{i,p}(\overline{\zeta})\big) C^p_{T_p}\right)\Bigg)\\
&+ \sum_{m \in M} \sum_{i \in I(m)} \sum_{q \in Q(m)} \Big(\Gamma^{i,q} A^{q,m}_{T^m_q} - \big(\overline{\theta}^i_{T^m_q} + \overline{\gamma}^i_{T^m_q} - \overline{\gamma}^i_{T^m_{q-1}} + \overline{\beta}^i_{T^m_{q-1}}\big) A_{T^m_q}\\
&- \sum_{t=T^m_{q-1}}^{T^m_q - 1} \overline{\xi}^i_t (A^{q,m}_t - S^i_t)\Bigg],
\end{aligned}
$$

and simple algebraic manipulations lead to

$$\mathbb{E}\Bigg[\sum_{i\in I}\Bigg(-\sum_{t=0}^{T^m_{|Q|}-1}\overline{\zeta}_t(\hat{C}_t-L^i_t)+\sum_{t=0}^{T^m_{|Q|}-1}\overline{\xi}^i S^i_t$$

$$-\sum_{p\in P}\Big(\overline{\varphi}^i_{T_p}(A^p_{T_p}-C^p_{T_p})+\big(\overline{\varphi}^i_{T_p}-\Pi^{i,p}(\overline{\zeta})\big)C^p_{T_p}\Big)+\overline{\phi}^i_{T_p}(C^p_{T_p}-C^p_{T_{p+1}})\Bigg)$$

$$+\sum_{m\in M}\sum_{i\in I(m)}\sum_{q\in Q(m)}\Big(\Gamma^{i,q}A^{q,m}_{T^m_q}-\overline{\theta}^i_{T^m_q}A_{T^m_q}-\overline{\gamma}^i_{T^m_q}(A_{T^m_q}-A_{T^m_{q+1}})-\overline{\beta}^i_{T^m_q}A^{q+1,m}_{T^m_q}\Big)$$

$$-\sum_{t=T^m_{q-1}}^{T^m_q-1}\overline{\xi}^i_t A^{q,m}_t\Bigg].$$

Using $\mathbb{E}[\overline{\phi}^i_{T_p}(C^p_{T_p}-C^p_{T_{p+1}})]=0$ and $\mathbb{E}[\overline{\gamma}^i_{T^m_q}(A_{T^m_q}-A_{T^m_{q+1}})]=0$ we easily get:

$$\mathbb{E}\Bigg[\sum_{i\in I}\Bigg(\sum_{t=0}^{T^m_{|Q|}-1}\overline{\zeta}_t L^i_t+\sum_{t=0}^{T^m_{|Q|}-1}\overline{\xi}^i S^i_t\Bigg)$$

$$+\sum_{m\in M}\sum_{i\in I(m)}\sum_{q\in Q(m)}\Big(\Big(\Gamma^{i,q}-\overline{\varphi}^i_{T_p}-\overline{\theta}^i_{T^m_q}-\Pi^{i,q}(\overline{\xi})\Big)A^{q,m}_{T^m_q}-\overline{\beta}^i_{T^m_q}A^{q+1,m}_{T^m_q}\Big)\Bigg],$$

and because of Lemma 4.2 this reduces to

$$\mathbb{E}\Bigg[\sum_{i\in I}\Bigg(\sum_{t=0}^{T^m_{|Q|}-1}\overline{\zeta}_t L^i_t+\sum_{t=0}^{T^m_{|Q|}-1}\overline{\xi}^i S^i_t\Bigg)$$

$$+\sum_{m\in M}\sum_{i\in I(m)}\sum_{q\in Q(m)}\Big(\Big(\Gamma^{i,q}-\overline{\varphi}^i_{T_p}-\overline{\theta}^i_{T^m_q}-\Pi^{i,q}(\overline{\xi})\Big)^+(\pi+A^{q+1,m}_{T^m_q})-\overline{\beta}^i_{T^m_q}A^{q+1,m}_{T^m_q}\Big)\Bigg],$$

and hence

$$\mathbb{E}\Bigg[\sum_{i\in I}\Bigg(\sum_{t=0}^{T^m_{|Q|}-1}\overline{\zeta}_t L^i_t+\sum_{t=0}^{T^m_{|Q|}-1}\overline{\xi}^i S^i_t\Bigg)$$

$$+\sum_{m\in M}\sum_{i\in I(m)}\sum_{q\in Q(m)}\Big(\Big(\Gamma^{i,q}-\overline{\varphi}^i_{T_p}-\overline{\theta}^i_{T^m_q}-\Pi^{i,q}(\overline{\xi})\Big)^+\pi\Big)\Bigg]$$

which corresponds to the primal optimal solution. Hence the duality gap is zero and the dual optimal solution is attained. □

The next proposition shows that a dual solution can be obtained by completing the shadow prices A and C defined as the Lagrange multipliers given by solution of the Lagrange relaxation problem (LR).

Proposition 4.14. *For any optimal solution* $(\overline{A}, \overline{C})$ *of the Lagrange relaxation problem (LR), there exist dual multipliers* w *such that* $(\overline{A}, \overline{C}, w)$ *is an optimal solution of the dual problem (D). Moreover, the equilibrium allowance and CER prices are almost surely unique.*

Proof. The Lagrange relaxation problem (LR) reads:

$$\sup_{(A,C)\in\mathcal{K}_G^1} \inf_{0\le x\le\chi,\, Fx\ge f} \mathbb{E}\Bigg[\sum_{i\in I}\Bigg(\sum_{t=0}^{T_{|Q|}^m-1} \zeta_t L_t^i + \sum_{t=0}^{T_{|Q|}^m-1} \xi^i S_t^i\Bigg)$$
$$+ \sum_{m\in M}\sum_{i\in I(m)}\sum_{q\in Q(m)} \Big(\Big(\Gamma^{i,q} - \varphi_{T_p}^i - \theta_{T_q^m}^i - \Pi^{i,q}(\xi)\Big)^+ \pi\Big)$$
$$+ \sum_{m\in M}\sum_{i\in I(m)}\sum_{q\in Q(m)} \big(\theta_{T_q^m}^i + \gamma_{T_q^m}^i - \gamma_{T_{q-1}^m}^i + \beta_{T_{q-1}^m}^i - \Theta^{i,q}\big) A_{T_q^m}^{q,m}$$
$$+ \sum_{i\in I}\sum_{p\in P} \big(\varphi_{T_p}^i + \phi_{T_p}^i - \phi_{T_{p-1}}^i - \Pi^{i,p}(\zeta) + \Xi^{i,p}\big) C_{T_p}^p\Bigg]$$

This implies that

$$\overline{C}_{T_p}^p \ge \mathbb{E}[\overline{C}_{T_{p+1}}^{p+1}|\mathcal{F}_{T_p}] \quad \text{and} \quad \overline{A}_{T_q^m}^{q,m} \ge \mathbb{E}[\overline{A}_{T_{q+1}^m}^{q+1,m}|\mathcal{F}_{T_q^m}] \tag{4.47}$$

almost surely for all $p \in P$, $m \in M$ and $q \in Q(m)$. Moreover for any optimal solution $(\overline{A}, \overline{C})$ of (LR) and any optimal solution $\overline{x}$ of (P) it holds that

$$\begin{aligned}
D^* \le LR^* &= \sup_{(A,C)\in\mathcal{K}_G^1} \inf_{0\le x\le\chi,\, Fx\ge f} \sum_{i\in I} L^{A,C,i}(x^i)\\
&= \inf_{0\le x\le\chi,\, Fx\ge f} \sum_{i\in I} L^{\overline{A},\overline{C},i}(x^i)\\
&\le \sum_{i\in I} L^{\overline{A},\overline{C},i}(\overline{x}^i)\\
&\le \sup_{(A,C)\in\mathcal{K}_G^1} \sum_{i\in I} L^{A,C,i}(\overline{x}^i)\\
&= \inf_{0\le x\le\chi,\, Fx\ge f} \sup_{(A,C)\in\mathcal{K}_G^1} \sum_{i\in I} L^{A,C,i}(x^i) = P^*.
\end{aligned}$$

Since strong duality holds between (P) and (D) all inequalities are fullfilled with equality and in particular we conclude that

$$\sum_{i\in I} L^{\overline{A},\overline{C},i}(\overline{x}^i) = \inf_{0\le x\le\chi,\, Fx\ge f} \sum_{i\in I} L^{\overline{A},\overline{C},i}(x^i) \tag{4.48}$$

for any primal optimal solution $\overline{x}$. This together with (4.47) implies that

$$0 = \mathbb{E}[\overline{\phi}^i_{T_p}(\overline{C}^p_{T_p} - \overline{C}^{p+1}_{T_{p+1}})] \quad \text{for all } p \in P, i \in I$$
$$0 = \mathbb{E}[\overline{\gamma}^i_{T^m_q}(\overline{A}^{q,m}_{T^m_q} - \overline{A}^{q+1,m}_{T^m_{q+1}})] \quad \text{for all } q \in Q(m), i \in I(m), m \in M.$$

Moreover we derive from (4.48) that

$$\overline{\zeta}_t(\overline{C}^p_t - L^i_t) = \kappa^i(\overline{C}^p_t - L^i_t)^+$$

holds for all $t \in \{T_{p-1}, \ldots, T_p - 1\}$, $p \in P$, $i \in I$ while

$$q\overline{\xi}^i_t(\overline{A}^{q,m}_t - S^i_t) = \overline{\xi}^i(\overline{A}^{q,m}_t - S^i_t)^+$$

holds for all $t \in \{T^m_{q-1}, \ldots, T^m_q - 1\}$, $q \in Q(m)$, $i \in I(m)$ and $m \in M$. If for each market $m \in M$ Lemma 4.1 is fulfilled then we conclude from (4.48) that

$$\left\{\bar{\beta}^i_{T^m_q} > 0\right\} \subseteq \left\{\overline{A}^{q,m}_{T^m_q} = \pi^{q,m} + \mathbb{E}[\overline{A}^{q+1,m}_{T^m_{q+1}}|\mathcal{F}_{T^m_q}]\right\}$$

holds up to zero sets for all $i \in I(m)$ and $q \in Q(m)$ and $m \in M$. Hence all requirements for the proof of Proposition 4.13 are fullfilled if moreover w is choosen as in (4.43)-(4.46). Hence $(\overline{A}, \overline{C}, w)$ is an optimal solution of (D).

We now prove uniqueness of the equilibrium prices. Let us assume that we have two different pairs $\widehat{P} = (\widehat{A}, \widehat{C})$ and $\tilde{P} = (\tilde{A}, \tilde{C})$ of equilibria with strategies $\widehat{x}$ and $\tilde{x}$. From the first part it follows that there exist $\widehat{w} \in \mathcal{K}^1_F \times \mathcal{L}^1$ such that $(\hat{A}, \hat{C}, w)$ is an optimal solution of (D), and since the equilibrium strategy $\tilde{x}$ is an optimal solution of P. It follows that $\tilde{x}$ and $\widehat{w}$ must fullfill the complemtary slackness conditions. Following the argumentation of the last subsection it follows that $\widehat{P} = (A(\tilde{x}), C(\tilde{x}))$ up to zero sets. Due to (ii) it holds also that $\widehat{P} = (A(\widehat{x}, C(\widehat{x}))$ up to zero sets implying that

$$(A(\tilde{x}), C(\tilde{x})) = (A(\widehat{x}), C(\widehat{x})) \tag{4.49}$$

up to zero sets which concludes the proof. □

4.5. More Equilibrium Price Formulas

For the sake of simplicity we assume $\nu = 1$ from now on. We first revisit the derivation of formulas for the allowance prices. The following notation will simplify some expressions.

$$\Upsilon^{q,m} = \sum_{i \in I(m)} [\Theta^{i,q} - \bar{\beta}^i_{T^m_{q-1}} + \bar{\gamma}^i_{T^m_{q-1}}] \tag{4.50}$$

$\Upsilon^{q,m}$ gives the effective amount of allowances present in the market, including the banked allowances and those withdrawn for use for compliance in the previous period.

Lemma 4.15.

$$\{\gamma_{T^m_q} = 0\} \cap \{\beta_{T^m_q} = 0\} \cap \{\mathbb{E}[C^{p+1}_{T_{q+1}}|\mathcal{F}_{T^m_q}] > \mathbb{E}[A^{q+1,m}_{T^m_{q+1}}|\mathcal{F}_{T^m_q}]\}$$
$$= \{\Delta^{q,m} > \Upsilon^{q,m}\} \cap \{\beta_{T^m_q} = 0\} \cap \{\mathbb{E}[C^{p+1}_{T_{q+1}}|\mathcal{F}_{T^m_q}] > \mathbb{E}[A^{q+1,m}_{T^m_{q+1}}|\mathcal{F}_{T^m_q}]\}$$

up to zero sets.

The intuition behind this equality is that on the event $\{\beta_{T_q^m} = 0\} \cap \{\mathbb{E}[C_{T_{q+1}}^{p+1} | \mathcal{F}_{T_q^m}] > \mathbb{E}[A_{T_{q+1}^m}^{q+1,m} | \mathcal{F}_{T_q^m}]\}$ that the market is not short of allowance (i.e. $\beta_{T_q^m} = 0$) and allowance prices are expected to be cheaper than CERs, emissions exceed the total number of allowances (i.e. $\Delta^{q,m} > \Upsilon^{q,m}$) exactly when banking is not needed (i.e. $\beta_{T_q^m} = 0$).

Proof. Lemma 4.4 and primal feasibility imply that

$$\begin{aligned}
&\{\gamma_{T_q^m} = 0\} \cap \{\beta_{T_q^m} = 0\} \\
&\subseteq \{\sum_{i \in I(m)} \varphi_{T_q^m}^i > 0\} \cap \left\{ \sum_{i \in I(m)} \theta_{T_q^m}^i = \sum_{i \in I(m)} \Theta^{i,q} - \bar{\beta}_{T_{q-1}^m}^i + \bar{\gamma}_{T_{q-1}^m}^i \right\} \\
&\subseteq \left\{ \sum_{i \in I(m)} \Gamma^{i,q} - \Pi^{i,q}(\bar{\xi}) > \sum_{i \in I(m)} \Theta^{i,q} - \bar{\beta}_{T_{q-1}^m}^i + \bar{\gamma}_{T_{q-1}^m}^i \right\} \\
&= \{\Delta^{q,m} > \Upsilon^{q,m}\}.
\end{aligned}$$

Moreover it holds that

$$\begin{aligned}
&\{\Delta^{q,m} > \Upsilon^{q,m}\} \cap \{\beta_{T_q^m} = 0\} \cap \{\mathbb{E}[C_{T_{q+1}}^{p+1} | \mathcal{F}_{T_q^m}] > \mathbb{E}[A_{T_{q+1}^m}^{q+1,m} | \mathcal{F}_{T_q^m}]\} \\
&\subseteq \left\{ \sum_{i \in I(m)} \Gamma^{i,q} - \Pi^{i,q}(\bar{\xi}) > \sum_{i \in I(m)} \Theta^{i,q} - \bar{\beta}_{T_{q-1}^m}^i + \bar{\gamma}_{T_{q-1}^m}^i \right\} \\
&\qquad \cap \{\beta_{T_q^m} = 0\} \cap \{\mathbb{E}[C_{T_{q+1}}^{p+1} | \mathcal{F}_{T_q^m}] > \mathbb{E}[A_{T_{q+1}^m}^{q+1,m} | \mathcal{F}_{T_q^m}]\} \\
&\subseteq \{\varphi_{T_q^m} > 0\} \cap \{\mathbb{E}[C_{T_{q+1}}^{p+1} | \mathcal{F}_{T_q^m}] > \mathbb{E}[A_{T_{q+1}^m}^{q+1,m} | \mathcal{F}_{T_q^m}]\} \\
&\subseteq \{A_{T_q^m}^{q,m} = C_{T_p}^p\} \cap \{\mathbb{E}[C_{T_{q+1}}^{p+1} | \mathcal{F}_{T_q^m}] > \mathbb{E}[A_{T_{q+1}^m}^{q+1,m} | \mathcal{F}_{T_q^m}]\} \\
&\subseteq \{A_{T_q^m}^{q,m} > \mathbb{E}[A_{T_{q+1}^m}^{q+1,m} | \mathcal{F}_{T_q^m}]\} \\
&\subseteq \{\gamma_{T_q^m} = 0\}
\end{aligned}$$

which concludes the proof. □

Proposition 4.16.

$$\begin{aligned}
A_{T_q^m}^{q,m} \quad = \quad & \mathbb{E}[A_{T_{q+1}^m}^{q+1,m} | \mathcal{F}_{T_q^m}] + \pi \mathbf{1}_{\{\beta_{T_q^m} > 0\}} \\
& + \left(\mathbb{E}[C_{T_{q+1}}^{p+1} | \mathcal{F}_{T_q^m}] - \mathbb{E}[A_{T_{q+1}^m}^{q+1,m} | \mathcal{F}_{T_q^m}] \right)^+ \mathbf{1}_{\{\beta_{T_q^m} = 0\}} \mathbf{1}_{\{\Delta^{q,m} > \Upsilon^{q,m}\}}
\end{aligned}$$

This new form of the price of an allowance states that the spread between the allowance price and its forward for the next period is equal to the penalty when the market is short (i.e. $\beta_{T_q^m} > 0$) or the positive part of the spread between the forward CER price and the forward allowance price when compliance has to be met with the use of CERs (i.e. when $\beta_{T_q^m} = 0$ and $\Delta^{q,m} > \Upsilon^{q,m}$).

Proof.

$$\begin{aligned}\tilde{A}^{q,m}_{T^m_q} &= \mathbb{E}[A^{q+1,m}_{T^m_{q+1}}|\mathcal{F}_{T^m_q}] + \pi \mathbf{1}_{\{\beta_{T^m_q}>0\}} \\ &\quad + \left(\mathbb{E}[C^{p+1}_{T_{q+1}}|\mathcal{F}_{T^m_q}] - \mathbb{E}[A^{q+1,m}_{T^m_{q+1}}|\mathcal{F}_{T^m_q}]\right)^+ \mathbf{1}_{\{\beta_{T^m_q}=0\}}\mathbf{1}_{\{\Delta^{q,m}>\Upsilon^{q,m}\}} \\ &= \mathbb{E}[A^{q+1,m}_{T^m_{q+1}}|\mathcal{F}_{T^m_q}] + \pi \mathbf{1}_{\{\beta_{T^m_q}>0\}} \\ &\quad + \left(E(C^{p+1}_{T_{q+1}}|\mathcal{F}_{T^m_q}) - \mathbb{E}[A^{q+1,m}_{T^m_{q+1}}|\mathcal{F}_{T^m_q}]\right) \mathbf{1}_{\{\beta_{T^m_q}=0\}}\mathbf{1}_{\{\Delta^{q,m}>\Upsilon^{q,m}\}} \\ &\qquad\qquad \mathbf{1}_{\{\mathbb{E}[C^{p+1}_{T_{q+1}}|\mathcal{F}_{T^m_q}]>\mathbb{E}[A^{q+1,m}_{T^m_{q+1}}|\mathcal{F}_{T^m_q}]\}}\end{aligned}$$

Moreover

$$\begin{aligned}A^{q,m}_{T^m_q} &= (\pi + \mathbb{E}[A^{q+1,m}_{T^m_{q+1}}|\mathcal{F}_{T^m_q}]\mathbf{1}_{\{\beta_{T^m_q}>0\}} \\ &\quad + \left(\mathbb{E}[A^{q+1,m}_{T^m_{q+1}}|\mathcal{F}_{T^m_q}]\mathbf{1}_{\{\gamma_{T^m_q}>0\}} + \mathbb{E}[C^{p+1}_{T_{q+1}}|\mathcal{F}_{T^m_q}]\mathbf{1}_{\{\gamma_{T^m_q}=0\}}\right)\mathbf{1}_{\{\beta_{T^m_q}=0\}} \\ &= E(A^{q+1,m}_{T^m_{q+1}}|\mathcal{F}_{T^m_q})) + \pi \mathbf{1}_{\{\beta_{T^m_q}>0\}} \\ &\quad + \left(E(C^{p+1}_{T_{q+1}}|\mathcal{F}_{T^m_q}) - E(A^{q+1,m}_{T^m_{q+1}}|\mathcal{F}_{T^m_q})\right)^+ \mathbf{1}_{\{\gamma_{T^m_q}=0\}}\mathbf{1}_{\{\beta_{T^m_q}=0\}} \\ &= E(A^{q+1,m}_{T^m_{q+1}}|\mathcal{F}_{T^m_q})) + \pi \mathbf{1}_{\{\beta_{T^m_q}>0\}} \\ &\quad + \left(\mathbb{E}[C^{p+1}_{T_{q+1}}|\mathcal{F}_{T^m_q}] - \mathbb{E}[A^{q+1,m}_{T^m_{q+1}}|\mathcal{F}_{T^m_q}]\right) \mathbf{1}_{\{\gamma_{T^m_q}=0\}}\mathbf{1}_{\{\beta_{T^m_q}=0\}} \\ &\qquad\qquad \mathbf{1}_{\{\mathbb{E}[C^{p+1}_{T_{q+1}}|\mathcal{F}_{T^m_q}]>\mathbb{E}[A^{q+1,m}_{T^m_{q+1}}|\mathcal{F}_{T^m_q}]\}}\end{aligned}$$

where the second last equality holds due to dual feasibility. The proposition follows from Lemma 4.15. □

We now consider the equilibrium CER prices, and we introduce the notation

$$\Lambda^p = \sum_{i\in I} \bar{\phi}^i_{T_{p-1}} + \Pi^{i,p}(\bar{\zeta}) - \Xi^{i,p} = \sum_{i\in I} \bar{\phi}^{i,p}_{T_p} + \sum_{i\in I(m)} \bar{\varphi}^{i,p}_{T_p} \tag{4.51}$$

denoting the effective amount of CERs in the market.

Lemma 4.17. *It holds almost surely that*

$$\begin{aligned}&\{\Lambda^p < \kappa^p\} \cap \{A^{q,m}_{T_p} > \mathbb{E}[C^{p+1}_{T_p}|\mathcal{F}_{T_p}]\} \\ &\qquad = \{\varphi_{T_p} \in (0,\kappa^p)\} \cap \{\phi_{T_p} = 0\} \cap \{A^{q,m}_{T_p} > \mathbb{E}[C^{p+1}_{T_p}|\mathcal{F}_{T_p}]\}\end{aligned}$$

Proof. Due to primal feasibility it holds that

$$\{\varphi_{T_p} \in (0,\kappa^p)\} \cap \{\phi_{T_p} = 0\} \subseteq \{\Lambda^p < \kappa^p\} \tag{4.52}$$

up to zero sets.

$$
\begin{aligned}
&\{\Lambda^p < \kappa^p\} \cap \{A^{q,m}_{T_p} > \mathbb{E}[C^{p+1}_{T_p}|\mathcal{F}_{T_p}]\} \\
&\subseteq \{\bar{\phi}_{T_p} + \bar{\varphi}_{T_p} < \kappa^p\} \cap \{A^{q,m}_{T_p} > \mathbb{E}[C^{p+1}_{T_p}|\mathcal{F}_{T_p}]\} \\
&\subseteq \left(\bigcup_{i \in I(m)} \{\bar{\varphi}_{T_p} < \min(\Delta^{i,p}, \kappa^i)\} \right) \cap \{A^{q,m}_{T_p} > \mathbb{E}[C^{p+1}_{T_p}|\mathcal{F}_{T_p}]\} \\
&\subseteq \{A^{q,m}_{T_p} \le C^p_{T_p}\} \cap \{A^{q,m}_{T_p} > \mathbb{E}[C^{p+1}_{T_p}|\mathcal{F}_{T_p}]\} \qquad (4.53) \\
&\subseteq \{C^p_{T_p} > \mathbb{E}[C^{p+1}_{T_p}|\mathcal{F}_{T_p}]\} \subseteq \{\phi_{T_p} = 0\} \subseteq \{\phi_{T_p} = 0\} \cap \{\varphi_{T_p} > 0\}
\end{aligned}
$$

up to zero sets where we used (4.15) for (4.53). Hence

$$
\begin{aligned}
&\{\Lambda^p < \kappa^p\} \cap \{A^{q,m}_{T_p} > \mathbb{E}[C^{p+1}_{T_p}|\mathcal{F}_{T_p}]\} \\
&\subseteq \{\varphi_{T_p} \in (0, \kappa^p)\} \cap \{\phi_{T_p} = 0\} \cap \{A^{q,m}_{T_p} > \mathbb{E}[C^{p+1}_{T_p}|\mathcal{F}_{T_p}]\}
\end{aligned}
$$

holds up to zero sets. □

Proposition 4.18. *It holds almost surely that*

$$
\begin{aligned}
C^p_{T_p} \quad = \quad & \mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}] + (\pi + \mathbb{E}[A^{m,q+1}_{T^m_{q+1}}|\mathcal{F}_{T_p}] - \mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}])^+ \mathbf{1}_{\{\beta_{T_p} > 0\}} \mathbf{1}_{\{\Lambda^p < \kappa^p\}} \\
& + (\mathbb{E}[A^{m,q+1}_{T^m_{q+1}}|\mathcal{F}_{T_p}] - \mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}])^+ \mathbf{1}_{\{\beta_{T_p} = 0\}} \mathbf{1}_{\{\Lambda^p < \kappa^p\}}
\end{aligned}
$$

almost surely.

One particular case of the above formula has a clear intuitive interpretation. It says that the spread between a CER price and its forward is 0 whenever the total amount of CERs in the market is greater than the maximum amount which can be used in the market (i.e. $\Lambda^p \ge \kappa^p$).

Proof. Let

$$
\begin{aligned}
\tilde{C}^p_{T_p} \quad = \quad & \mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}] + (\pi + \mathbb{E}[A^{m,q+1}_{T^m_{q+1}}|\mathcal{F}_{T_p}] - \mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}])^+ \mathbf{1}_{\{\beta_{T_p} > 0\}} \mathbf{1}_{\{\Lambda^p < \kappa^p\}} \\
& + (\mathbb{E}[A^{m,q+1}_{T^m_{q+1}}|\mathcal{F}_{T_p}] - \mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}])^+ \mathbf{1}_{\{\beta_{T_p} = 0\}} \mathbf{1}_{\{\Lambda^p < \kappa^p\}} \\
= \quad & \mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}] \\
& + (\pi + \mathbb{E}[A^{m,q+1}_{T^m_{q+1}}|\mathcal{F}_{T_p}] - \mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}]) \mathbf{1}_{\{\beta_{T_p} > 0\}} \mathbf{1}_{\{\Lambda^p < \kappa^p\}} \\
& \qquad \mathbf{1}_{\{\pi + \mathbb{E}[A^{m,q+1}_{T^m_{q+1}}|\mathcal{F}_{T_p}] - \mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}] > 0\}} \\
& + (\mathbb{E}[A^{m,q+1}_{T^m_{q+1}}|\mathcal{F}_{T_p}] - \mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}]) \mathbf{1}_{\{\beta_{T_p} = 0\}} \mathbf{1}_{\{\Lambda^p < \kappa^p\}} \\
& \qquad \mathbf{1}_{\{\mathbb{E}[A^{m,q+1}_{T^m_{q+1}}|\mathcal{F}_{T_p}] - \mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}] > 0\}}.
\end{aligned}
$$

Dual feasibility implies that

$$
\{\mathbb{E}[A^{m,q}_{T^m_q}|\mathcal{F}_{T_p}] > \mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}]\} \subseteq \{A^{m,q}_{T^m_q} > \mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}]\}.
$$

Further Proposition 4.8 implies that

$$\{\beta_{T_p}=0\}\cap\{A^{m,q}_{T^m_q}>\mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}]\}\subseteq\{\beta_{T_p}=0\}\cap\{\mathbb{E}[A^{m,q}_{T^m_q}|\mathcal{F}_{T_p}]>\mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}]\}$$

Because moreover $A^{m,q}_{T^m_q}=\pi+\mathbb{E}[A^{m,q+1}_{T^m_{q+1}}|\mathcal{F}_{T_p}]$ on $\{\beta_{T_p}>0\}$ this translates to

$$\begin{aligned}\tilde{C}^p_{T_p} &= \mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}]+(A^{m,q}_{T^m_q}-\mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}])\mathbf{1}_{\{\beta_{T_p}>0\}}\mathbf{1}_{\{\Lambda^p<\kappa^p\}} \\ &\qquad \mathbf{1}_{\{A^{m,q}_{T^m_q}>\mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}]\}} \\ &\quad +(A^{m,q}_{T^m_q}-\mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}])\mathbf{1}_{\{\beta_{T_p}=0\}}\mathbf{1}_{\{\Lambda^p<\kappa^p\}}\mathbf{1}_{\{A^{m,q}_{T^m_q}>\mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}]\}} \qquad (4.54) \\ &= \mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}]+(A^{q,m}_{T^m_q}-\mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}])\mathbf{1}_{\{\Lambda^p<\kappa^p\}}\mathbf{1}_{\{A^{q,m}_{T^m_q}>\mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}]\}} \qquad (4.55)\end{aligned}$$

almost surely. Proposition 4.9 implies that

$$C^p_{T_p} = \mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}]\mathbf{1}_{\{\varphi_{T_p}\in\{0,\kappa^p\}\}}+A^{q,m}_{T^m_q}\mathbf{1}_{\{\varphi_{T_p}\in(0,\kappa^p)\}}.$$

Due to dual feasibility it holds that $A^{q,m}_{T^m_q}\geq\mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}]$ on $\{\varphi_{T_p}\in(0,\kappa^p)\}$. Hence

$$\begin{aligned}C^p_{T_p} &= \mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}]+(A^{q,m}_{T^m_q}-\mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}])^+\mathbf{1}_{\{\varphi_{T_p}\in(0,\kappa^p)\}} \\ &= \mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}]+(A^{q,m}_{T^m_q}-\mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}])\mathbf{1}_{\{\varphi_{T_p}\in(0,\kappa^p)\}}\mathbf{1}_{\{A^{q,m}_{T_p}>\mathbb{E}[C^{p+1}_{T_p}|\mathcal{F}_{T_p}]\}} \\ &= \mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}]+(A^{q,m}_{T^m_q}-\mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}])\mathbf{1}_{\{\varphi_{T_p}\in(0,\kappa^p)\}}\mathbf{1}_{\{\phi_{T_p}=0\}}\mathbf{1}_{\{A^{q,m}_{T_p}>\mathbb{E}[C^{p+1}_{T_p}|\mathcal{F}_{T_p}]\}} \\ &= \mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}]+(A^{q,m}_{T^m_q}-\mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}])\mathbf{1}_{\{\Lambda^p<\kappa^p\}}\mathbf{1}_{\{A^{q,m}_{T^m_q}>\mathbb{E}[C^{p+1}_{T_{p+1}}|\mathcal{F}_{T_p}]\}} \qquad (4.56)\end{aligned}$$

holds almost surely where the last equality holds due to Lemma 4.17. Consequently (4.55) and (4.56) imply that $C^p_{T_p}=\tilde{C}^p_{T_p}$ almost surely. □

5. Appendix I: Index of Acronyms

CDM	Clean Development Mechanism
JI	Joint Implementation
CER	Certified Emission Reduction
ERU	Emission Reduction Unit
EU ETS	European Union Emission Trading Scheme
RGGI	Regional Greenhouse Gas Initiative

6. Appendix II: Table of Notation

M	Set of emission markets
I	Set of all the firms in the economy
m	Typical emission market
$I(m)$	Set of firms involved in emission market m
$Q(m)$	Set of compliance periods in emission market m
$\|Q(m)\|$	Number of compliance periods in emission market m
$[T^m_{q-1}, T^m_q]$	q-th compliance period in emission market m
T	Horizon of the model
κ_i	Maximum number of CERs firm i is allowed to use
$\Gamma^{i,q}$	Emissions of firm $i \in I(m)$ during compliance period $q \in Q(m)$
π^m_q	Financial penalty for over-emission during compliance period $q \in Q(m)$ in market $m \in M$
ξ^i_t	Short term abatement by firm $i \in I$ for time period $[t, t+1]$
$\overline{\xi}^i$	deterministic constant giving the maximum abatement level possible for firm $i \in I(m)$
$\Pi^{i,q}(\xi^i)$	Short term abatement by firm $i \in I$ over compliance period $q \in Q(m)$ for strategy ξ^i
ζ^i_t	Proportion of CDM project, or long term abatement by firm $i \in I$ for time period $[t, t+1]$
$\Pi^{i,q}(\zeta^i)$	Long term abatement by firm $i \in I$ over compliance period $q \in Q(m)$ for strategy ξ^i
S^i_t	T-forward cost of unit short term abatement by firm $i \in I$ for time period $[t, t+1]$
L^i_t	T-forward cost of long term abatement by firm $i \in I$ for time period $[t, t+1]$
$C^{\tilde{A},\tilde{C},A,C,i}$	Terminal cumulative costs of firm i
$\Theta^{i,q}$	Initial allowance endowment of firm $i \in I(m)$ at time T^m_{q-1}
$\Xi^{i,p}$	Number of CERs that agent $i \in I$ voluntarily withdraws from the market at time T_p
$\tilde{A}^{q,m}_t$	price at time t of a (q,m)-allowance forward contract with maturity T^m_q
$\tilde{C}^p_t$	price at time t of a p-maturity CER forward contract with maturity T_p
$\tilde{\theta}^{i,q}_t$	number of (q,m) allowances held by firm $i \in I(m)$ at time t
$\tilde{\varphi}^{i,p}_t$	number of p-maturity CERs held by firm $i \in I(m)$ at time t
$R^{(\tilde{A},\tilde{C})}_T\tilde{(}\theta^i, \tilde{\varphi}^i)$	P&L from financial trading in allowances and CERs
$\gamma^{i,q}_{T_q}$	number of (physical) (q,m)-allowances banked by firm $i \in I(m)$ at time T^m_q
$\theta^{i,q}_{T_q}$	number of (physical) (q,m)-allowances used for compliance by firm $i \in I(m)$ at time T^m_q
$\phi^{i,p}_{T_p}$	number of (physical) p-maturity CERs banked by firm $i \in I(m)$ at time T^m_p
$\varphi^{i,p}_{T_p}$	number of (physical) p-maturity CERs used for compliance by firm $i \in I(m)$ at time T^m_p
$\beta^i_{T^m_q}$	Net cumulative emissions at time T^m_q of firm $i \in I(m)$
$I^{\tilde{A},\tilde{C},A,C,i}$	P&L of firm $i \in I(m)$
$\mathfrak{F}^i$	Admissible *physical* strategies for firm $i \in I(m)$
$\mathcal{H}^i$	Admissible *financial* strategies for firm $i \in I(m)$

References

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René Carmona
Department of Operations Research and Financial Engineering
Princeton University
Princeton, NJ 08544, USA
Also with the Bendheim Center for Finance and the Applied and Computational Mathematics Program
e-mail: `rcarmona@princeton.edu`

Max Fehr
Institute for Operations Research
ETH Zurich
CH-8092 Zurich, Switzerland
e-mail: maxfehr@ifor.math.ethz.ch

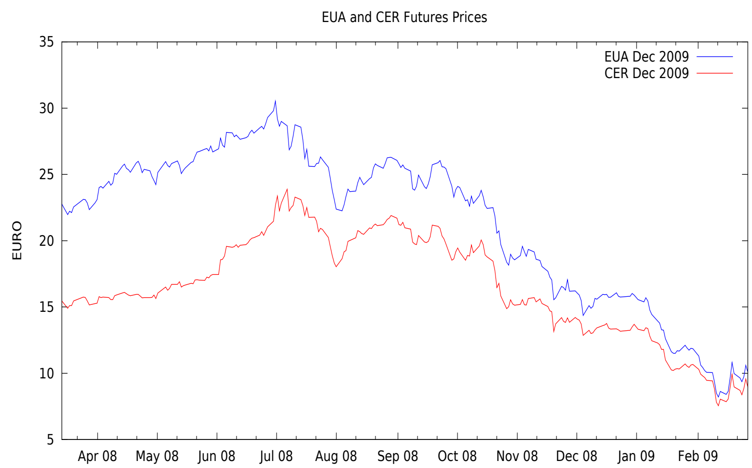

Figure 1. Prices of the December 2012 EUA futures contract (EU-ETS second phase), together with the price of the corresponding CER futures contract.