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Auctions and Relative Allocation Mechanisms for Cap-and-Trade Schemes

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The present paper is a contribution to the quantitative analysis of the cap-and-trade schemes touted by some as the regulation of choice in the market based approach to climate change. Its main thrust is twofold. Firstly we prove that the well known social optimality of cap-and-trade schemes, which is the main reason for these schemes to be implemented, can be extended to the multi-period stochastic setting, if the reduction targets are defined in an appropriate manner. As a side effect we obtain a new result on the asymmetry between taxes and cap-and-trade regulations. Secondly, we propose a new allocation procedure incorporating the advantages of the existing schemes while at the same time avoiding their documented shortcomings. The cap-and-trade scheme introduced in this paper retains the social optimality property enjoyed by the so-called standard cap-and-trade systems, and like them it can be calibrated to reach the emission target with as much statistical confidence as desired. But like the relative schemes introduced and studied earlier, and unlike the standard schemes, it provides a tight control of the windfall profits.

From a mathematical point of view, the main contributions of the paper are the proof of social optimality in a dynamic stochastic setting, and the proof of an equivalence between equilibria. This last abstract result identifies a one-to-one correspondence between economies based on different cap-and-trade schemes, and gives explicit formulae for the changes in equilibrium prices. In particular, it explains why, and shows how, the prices of goods are reduced by the new allocation scheme, demonstrating that an apparently innocuous small change in the design of the allowance allocation procedure can have a dramatic economic impact.

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1. Introduction

Climate change has been a great source of concern for economists but the treatment of this externality had to wait for the development of cap-and-trade schemes, e.g. the successful acid rain program in the US, the voluntary carbon dioxide markets, and most importantly, the mandatory European Union Emission Trading Scheme (EU ETS).

However, the quantitative analysis of the market mechanisms for the control of greenhouse gas emissions is a relatively recent trend in the economic literature, and even more so in the mathematical literature, and only through the development of environmental finance. The starting point is the seminal paper Montgomery (1972), proving that emission trading schemes are socially optimal in the sense that a given emission target is reached at the lowest possible costs. This result was proven in the static setting of a one period deterministic model where prices of the goods were assumed given exogenously, and only allowance price formation was obtained by an

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equilibrium analysis. In a deterministic setting, agents are not allowed to emit more than their allowance allocations, so emissions never exceed the cap. However emission trading schemes, e.g. EU ETS, are operating in an uncertain world, and realistic models should allow for excess emissions modulo a penalty payment which can be seen as a safety valve. In the first part of the paper, we prove that, if the emission target is formulated appropriately, this kind of design is still socially optimal in a dynamic stochastic setting even if both goods and allowances prices are derived from an equilibrium argument. As an aside, we obtain new evidence on the asymetry between taxes and cap-and-trade schemes. This topic was first addressed by Weitzman in Weitzman (1974) where the author discusses whether to fix price or quantity in order to optimize overall efficiency of costs and benefits of emission reduction. In contrast, we determine how emissions and reduction costs relate to each under different regulatory policies. As an extension we also consider mixing taxes and cap-and-trade schemes, as was done recently in the UK. The idea of mixing the schemes dates back to Robert and Spence (1976) who extend the work Weitzman (1974).

In the second part of the paper, we consider the most controversial component of a cap-and-trade scheme: the initial allocation of the emission certificates. We use the general framework of Carmona et al. (2010), and we propose a new allocation mechanism which preserves social optimality and at the same time, eliminates several of the shortcomings of the standard schemes. The overall initial allocation of emission allowances, also known as the cap, should be dictated by the regulatory emission target. While the initial allocation among the various installations does not influence the overall emission reduction, other impacts are not as clear, and too many half-truths can be found in the popular press and in some pseudo-scientific magazines. While we do not attempt to describe exhaustively what allocation schemes can or cannot do, we analyze both theoretically and numerically, the advantages of a natural extension of some of the schemes already implemented or considered in the scientific literature.

Obviously, costs of production are higher in the presence of regulation. Worse, as observed during the first phase of EU ETS, consumers costs can exceed the overall reduction costs: producers receive the allowances for free, price them into their costs of production, and take advantage of the trading scheme to make extra profit, the so-called *windfall profits*. Climate change regulation cannot afford to be one more reason for higher heating bills since independently, *fuel poverty* already increased dramatically in the wake of the recent economic crisis. Households are said to be in *fuel poverty* when they spend more than 10% of income on keeping homes warm.

Fueled by populist pressure, auctioning of allowances has been touted as the solution to this problem. While the measure which passed the US House of Representative last year included auctioning of a mere 15% of the initial allocation, the original proposal favoured by President Obama and ushered by Rep. Waxman and Markey was for 100% auctioning of the allowances. The rationale is very appealing: if producers have to pay for their allowances, regulators can return revenues to consumers, or invest these revenues in other emission reduction projects. However, we argue that auctions are not sufficient since its return is essentially equal to the monetary value of the auctioned allowances which is not enough to match the overall consumer burden, since the latter is not directly related to the cap, but instead to the quantities consumed within one compliance period. Using the example of Japan's electricity market we confirmed this effect numerically, by showing in Carmona et al. (2008) that auctioning of the total initial allocation of a standard cap-and-trade scheme can not reduce windfall profits to a reasonable level.

Motivated by the above discussion of the shortcomings of standard cap-and-trade schemes, the authors of Carmona et al. (2010) introduced relative allocation schemes and showed that these schemes can be used to control the level at which allowances are priced into products, and in this way, lead to lower equilibrium prices for the goods. The drawback of the relative scheme is the fact that the actual amount of allowances injected in the market is not known in advance since it depends upon the production and hence the demand for goods.

In the second part of this paper, we propose to study a mixture of relative allocations and auctions. Our goal is to reduce electricity prices by relative allocation while keeping the amount of allowances in the market fixed at the regulatory cap. To this end, part of the initial allocation is put into a *pot* from which allowances are withdrawn proportionally to production. If at the end of the compliance period the implementation of the relative allocation does not exceed the amount of allowances in the pot, the allowances that remain in the pot are auctioned, otherwise, some procedure described in detail below in Section 2 is used to guarantee that the cap is not exceeded. This procedure fixes the number of allowances in the market. It also reduces the marginal cost of production for each agent because, he/she obtains for each unit of good produced, a given number of allowances for free, relieving him/her from buying them later. This decrease in marginal production costs leads to reduced prices of goods, and to a tighter control of the windfall profits and fuel poverty.

Controlling the level at which allowances are priced into end products is not only interesting when fighting fuel poverty, but it can also be useful when combining cap-and-trade schemes with emission taxes as it can introduce a carbon price floor, as was done recently by the UK. This gives more price certainty to the end-consumers unable to react to quick carbon price changes anyway. It was shown in Carmona et al. (2010) (see also Section 5.2) that using a tax to regulate sectors with volatile reduction costs(e.g. electricity) leads to a statistical distribution for the cumulative emissions which is much wider than in the Business As Usual (BAU for short) case. This extra emission uncertainty can be reduced by combining the tax with a cap-and-trade scheme on the volatile sectors. And with the allocation mechanism we propose in this paper, this can be done without influencing the consumers carbon price.

We close this introduction with a short summary of the contents of the paper.

Section 2 presents the mathematical model of the economy in which we introduce cap-and-trade schemes to control externalities. We quickly review the basic building blocks introduced in Carmona et al. (2010) and recall the notion of market equilibrium used there.

Economic theory posits that the transfer of allowances by trading is the core principle that leads to the minimization of the costs caused by regulation. Section 3 examines this claim from the mathematical point of view in the context of our multi-period stochastic setting. Our social optimality results are presented in the form of three corollaries. Among other things, they show that, given any choice of emission target, one can find a penalty level so that the standard cap-and-trade regulation based on these choices has minimal costs of production and average excess emissions. We use this result to shed new light on the asymetry between taxes and cap-and-trade schemes.

In Section 4 we introduce the hybrid allocation procedure which we advocate in this paper. We prove a form of equivalence between standard and hybrid cap and trade schemes showing that any equilibrium production strategy of the standard cap-and-trade scheme is also an equilibrium production strategy for the hybrid cap-and-trade scheme. Hence the social optimality of the standard cap and trade scheme is preserved by the hybrid scheme. But on the other hand we prove that in contrast to the standard scheme the hybrid scheme can control the extent to which allowances are priced into products.

In Section 5 we illustrate our theoretical results with a case study. We use data from the Korean electricity market to compare numerically the impacts of a tax and cap-and-trade schemes with different allocation rules. We also show why and how the hybrid allocation mechanism can be used to combine an emission tax with a cap-and-trade scheme.

The paper ends with appendices collecting the technical definitions of the various costs used to compare regulations, the details of the case study of the Korean electricity market, as well as the proofs of five technical lemmas and the main theorem of the second part of the paper which are not given in the text.

2. Description of the Model and First Equilibrium Results

In this section we present the elements of our mathematical model. In what follows $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t,\ t \in \{0,1,\dots,T\}\}, \mathbb{P})$ is a filtered probability space. We denote by $\mathbb{E}[.]$ the expectation operator under the probability $\mathbb{P}$ and by $\mathbb{E}_t[.]$ the expectation operator conditional on the information available at time t as given by the σ-field $\mathcal{F}_t$. We will also make use of the notation $\mathbb{P}_t(.) := \mathbb{E}_t[\mathbf{1}_{\{.\}}]$ for the conditional probability with respect to $\mathcal{F}_t$.

2.1. Production, Trading and Profits

Production A finite set I of firms produce and sell a set K of different goods at times $0,1,\dots,T-1$. In order to produce good $k \in K$, each firm $i \in I$ has access to a set $J^{i,k}$ of different technologies which are sources of emissions (e.g. greenhouse gases).

We denote by $\widetilde{C}_t^{i,j,k}$ the marginal cost of producing one unit of good k at time t with technology $j \in J^{i,k}$, $e^{i,j,k} \geq 0$ measuring the volume of pollutants emitted per unit of good k produced by firm i with technology j, and $\kappa^{i,j,k}$ the production capacity. In our model, the instantaneous costs $\widetilde{C}_t^{i,j,k}$ are random. We assume that for each (i,j,k), they form an integrable adapted process.

At each time t, firm $i \in I$ decides to produce throughout the period $[t, t+1)$ the amount $\xi_t^{i,j,k}$ of good $k \in K$, using the technology $j \in J^{i,k}$. The choice of the production level $\xi_t^{i,j,k}$ is based only on present and past information (i.e. the processes $\xi^{i,j,k}$ are adapted), and $0 \leq \xi_t^{i,j,k} \leq \kappa^{i,j,k}$. We denote by $\mathcal{U}^i$ the set of adapted processes satisfying this capacity constraint. $\mathcal{U}^i$ is the set of admissible production strategies for firm $i \in I$.

We denote by D_t^k the demand at time t for good $k \in K$, and we assume that it does not exceed the capacity $\kappa_k = \sum_{i \in I} \sum_{j \in J^{i,k}} \kappa^{i,j,k}$. As for the costs, demand is random. We further assume that demand is *inelastic*. We denote by $\mathcal{U}$ the set of admissible production strategies:

$$\mathcal{U} = \left\{ \xi \in \prod_{i \in I} \mathcal{U}^i ; \sum_{i \in I} \sum_{j \in J^{i,k}} \xi_t^{i,j,k} = D_t^k \ \mathbb{P}-.a.s.,\ \ k \in K,\ \ t = 0, \cdots, T-1 \right\}. \tag{1}$$

Beyond the design of the allocation scheme, a cap-and-trade scheme is identified by two scalar parameters controlled by the regulator: the cap giving the emission target for the compliance period, and the penalty $\pi \in [0, \infty)$ applied to each unit of pollutant which is not offset by an allowance certificate. As in Carmona et al. (2010), we assume for the sake of simplicity that the entire period $[0,T]$ corresponds to a single compliance period, we do not allow firms to borrow allowances from a subsequent compliance period, and allowances become worthless if not used by time T, i.e. we do not allow for *banking* from one phase to the next. This was the case for the first phase of the European Union Emission Trading Scheme (EU ETS).

Trading In a *standard scheme*, each installation $i \in I$ is at time $t = 0$ provided with a free allocation Λ_0^i of allowances that may be traded on the market. Instead of considering the spot price of these contracts, we denote by A_t the price at time t of a forward contract guaranteeing either physical or financial settlement of one allowance certificate at maturity T. Firms take positions on the forward market, and we denote by θ_t^i the number of forward contracts held by firm i at time t. As usual, $\theta_t^i > 0$ when the firm is long θ_t^i contracts, and $\theta_t^i < 0$ when it is short $|\theta_t^i|$ contracts. The net cash position at time T resulting from the pure financial trading of forward contracts is given by:

$$\sum_{t=0}^{T-2} \theta_t^i (A_{t+1} - A_t) - \theta_{T-1}^i A_{T-1}. \tag{2}$$

Notice that even though firms can take very large long/short positions, each sale must be offset by a purchase and vice versa, so that the clearing constraint $\sum_{i \in I} \theta_t^i = 0$ must hold at each time $t = 0, \dots, T-1$. We now denote by θ_T^i the number of (physical) allowances settled at time T from the financial positions in the various forward contracts and we call the sequence $\{\theta_t^i\}_{i\in I, t=0,1,\cdots,T}$ a (financial) trading strategy. With this notation, we get the final form of the clearing condition

$$\sum_{i\in I}\theta_t^i = 0, \qquad\qquad t = 0,1,\cdots,T, \tag{3}$$

and the net cash position at time T resulting from trading the forward contracts and settling the final positions for physical allowances is given by:

$$R_T^A(\theta) = \sum_{t=0}^{T-1}\theta_t^i(A_{t+1} - A_t) - \theta_T^i A_T. \tag{4}$$

With this interpretation, $\Lambda_0^i + \theta_T^i$ is the number of (physical) allowance certificates that firm i surrenders for compliance, and it is the number used for the computation of the penalty given by formula (7). Of course this amount must be nonnegative, giving a lower bound for the trading at the last time point T of the compliance period, i.e. $\theta_T^i \geq -\Lambda_0^i$ must hold almost surely.

Profits Since T is the only horizon we consider in this work, we find it convenient to express all the prices (replacing if needed spot prices by T-forward prices), costs, wealths, etc in time T-currency, avoiding all forms of discounting in the process. So if we denote by S_t^k the T-forward price at time t of good $k \in K$, and by $C_t^{i,j,k}$ the T-forward cost incurred by firm $i \in I$ at time t for producing one unit of good $k \in K$ with technology $j \in J^{i,k}$, the total net gains of firm $i \in I$ from a production schedule $\xi^i = \{\xi_t^{i,j,k}\}_{j,k,t}$ are given by:

$$\sum_{t=0}^{T-1}\sum_{(,k}(S_t^k - C_t^{i,j,k})\xi_t^{i,j,k}, \tag{5}$$

the corresponding cumulative emissions being given by:

$$\Pi^i(\xi^i) := \sum_{t=0}^{T-1}\sum_{(j,k)\in M_i} e^{i,j,k}\xi_t^{i,j,k}. \tag{6}$$

We also include sources of emissions on which firm i has no control in the final cumulative emissions tally. They are given in the form of a random variable denoted by $\Delta^i \geq 0$ a.s. whose value is only known at time T. With the notation θ_T^i defined above, the penalty payment due by firm i at time T for using the strategy ξ^i is:

$$\pi\left(\Delta^i + \Pi^i(\xi^i) - \theta_T^i - \Lambda_0^i\right)^+. \tag{7}$$

Combining (5) – (7) together with (4), we obtain the following expression for the terminal wealth (profits and losses at time T) of firm i

$$L^{A,S,i}(\theta^i,\xi^i) := \sum_{t=0}^{T-1}\sum_{(j,k)\in M_i}(S_t^k - C_t^{i,j,k})\xi_t^{i,j,k} + \sum_{t=0}^{T-1}\theta_t^i(A_{t+1}-A_t) - \theta_T^i A_T - \pi\left(\Delta^i + \Pi^i(\xi^i) - \Lambda_0^i - \theta_T^i\right)^+ . \tag{8}$$

2.2. Equilibrium for Standard Cap-and-Trade Systems

Some of the theoretical results which we prove in this work rely on results proven in Carmona et al. (2010) for equilibria of standard cap-and-trade schemes with cap $\sum_{i\in I}\Lambda_0^i = \Lambda > 0$ for which all the allowances are given away for free as an upfront allocation at time $t=0$. We recall the notion of market equilibrium and the fundamental equilibrium result for standard schemesof Carmona et al. (2010). But first, we fix the spaces for the price processes and the agents strategies. To this end we introduce for any $1 \le p \le \infty$ and for any normed vector space F, the following space of adapted processes:

$$\mathcal{L}_t^p(F) := \left\{(X_s)_{s=0}^t;\ \mathbb{F}\text{-adapted},\ F\text{-valued}, \|X_s\| \in L^p(\mathcal{F}_s), s=0,\dots,t\right\}. \tag{9}$$

Based on these spaces we choose allowance and product price processes in the space $\mathcal{L}_T^1(\mathbb{R}) \times \mathcal{L}_{T-1}^1(\mathbb{R}^{|K|})$. Moreover, we define the following spaces of admissible strategies:

$$\mathcal{Q}^i(A) := \left\{(\theta^i,\xi^i)\,\middle|\,\xi^i \in \mathcal{U}^i\,, R_T^A(\theta^i) \text{ are integrable}\,, \theta_T^i \ge -\tilde{\Lambda}_0^i \text{ a.s.}\right\}.$$
$$\mathcal{Q}(A) := \prod_{i\in I}\mathcal{Q}^i(A).$$

Our first assumption guarantees existence of expected values.

Assumption 1. *we suppose that the uncontrolled emissions Δ^i and the production costs $C_t^{i,j,k}$ are integrable random variables.*

We will also use a technical assumption introduced in Carmona et al. (2010) on the nature of the uncontrolled emissions. This technical assumption states that up until the end of the compliance period, there is always uncertainty about the expected pollution level due to unpredictable events. More precisely, we shall assume that

Assumption 2. *conditionally on the information available at time $T-1$, the distribution of $\sum_{i\in I}\Delta^i$ possesses almost surely no point mass.*

As we already pointed out, these technical assumption help refine the statements of some of the results leading to the equilibria. Throughout this paper we also assume that the sets I and K are nonempty and finite. Moreover, we assume that for all agents $i \in I$ the set $J^{i,k}$ of different technologies to produce good $k \in K$ is finite and that for all $k \in K$ there is at least one $i \in I$ with $J^{i,k}$ nonempty.

In Theorem 1 we will see that the equilibrium allowance price process A^* is in $\mathcal{L}_T^\infty(\mathbb{R})$. However, our definition allows in principle for a rather general set $\mathcal{L}_T^1(\mathbb{R})$ of equilibrium prices. As we will prove uniqueness of the allowance price process, this leads to stronger results than if we had restricted A^* to $\mathcal{L}_T^\infty(\mathbb{R})$.

Following Carmona et al. (2010), we define the notion of equilibrium for a standard cap-and-trade scheme:

Definition 1. A pair of integrable price processes $(A^*,S^*) \in \mathcal{L}_T^1(\mathbb{R}) \times \mathcal{L}_{T-1}^1(\mathbb{R}^{|K|})$ form an equilibrium if there exist admissible trading and production strategies $(\theta^*,\xi^*) \in \mathcal{Q}(A^*)$ such that:

(i) All financial positions are in zero net supply (34)
(ii) Supply meets demand for each good (36)
(iii) Each firm $i \in I$ is satisfied by it's own strategy, in the sense that

$$\mathbb{E}[L^{A^*,S^*,i}(\theta^{*i},\xi^{*i})] \ge \mathbb{E}[L^{A^*,S^*,i}(\theta^i,\xi^i)] \tag{10}$$

for all $(\theta^i,\xi^i) \in \mathcal{Q}^i(A^*)$.

For standard schemes the equilibrium is closely related to the global optimal control problem

$$\inf_{\xi \in \mathcal{U}} \mathbb{E}\left[G^{\pi,\Lambda}(\xi)\right] \tag{11}$$

where the objective function is given by

$$G^{\pi,\Lambda}(\xi) = C(\xi) + \pi(\Delta + \Pi(\xi) - \Lambda)^{+} \tag{12}$$

where $\Delta = \sum_{i \in I} \Delta^i$ denotes the overall uncontrolled emissions,

$$\Pi(\xi) = \sum_{i \in I} \Pi^i(\xi^i) = \sum_{i,j,k,t} e^{i,j,k} \xi_t^{i,j,k} \tag{13}$$

denotes the total cumulative emissions from production, and where

$$C(\xi) = \sum_{i,j,k,t} \xi_t^{i,j,k} C_t^{i,j,k} \tag{14}$$

stands for the total costs of production in the economy. We will need the following result proven in Carmona et al. (2010).

THEOREM 1. *Under Assumptions 1 and 2 , the following hold:*
(i) If $\overline{\xi} \in \mathcal{U}$ is a solution of the global optimization problem (11), then the processes $(\overline{A}, \overline{S})$ defined by

$$\overline{A}_t = \pi \mathbb{P}_t(\Gamma + \Pi(\overline{\xi}) \geq 0), \qquad t = 0, \dots, T \tag{15}$$

and

$$\overline{S}_t^k = \max_{i \in I,\, j \in J^{i,k}} (C_t^{i,j,k} + e^{i,j,k} \overline{A}_t) 1_{\{\overline{\xi}_t^{i,j,k} > 0\}}, \qquad t = 0, \dots, T-1 \;\; k \in K, \tag{16}$$

is an equilibrium for which the associated production strategy is $\overline{\xi}$.
(ii) The equilibrium allowance price process is almost surely unique.
(iii) For each good $k \in K$, the price $\overline{S}^k$ is the smallest equilibrium price for good k in the sense that for any other equilibrium price process S^{*k}, we have $\overline{S}^k \leq S^{*k}$ almost surely.

3. Emission Targets and Social Optimality

In a ground breaking contribution, Montgomery proved in a one period deterministic setting that emission trading schemes are socially optimal in the sense that a given emission target is reached at the lowest possible cost. See Montgomery (1972). Because there is no randomness in the model, the emission target is a hard constraint, i.e. emissions in equilibrium have to stay below the cap. However, in the more realistic situation of random emissions, a stringent emission target can rapidly become prohibitive. Hence emission trading schemes, as e.g. EU ETS, allow for excess emissions modulo a penalty π which serves as a safety valve for the allowance price. In a stochastic setting, we need a new notion of emission target.

3.1. Definition of the Emission Target

A natural choice is to control the statistical distribution of the cumulative emissions at the end of the regulation period $[0,T]$ by a risk measure, as was done in Carmona et al. (2010) using a quantile measure. Like Value at Risk, the measure used there does a poor job at controlling the tail of the distribution since it controls only the frequencies of exceedances and not their actual sizes. In complete analogy with the expected shortfall used in financial applications, we propose to control the emissions by setting an upper bound η on the expected excess emissions $E^{\lambda}(\xi)$ above a threshold λ for a production strategy $\xi \in \mathcal{U}$, as defined by:

$$E^{\lambda}(\xi) = \mathbb{E}\left[\left(\Delta + \Pi(\xi) - \lambda\right)^{+}\right]. \tag{17}$$

Recall that $\Pi(\xi)$ defined in (13) represents the global cumulative emissions due to production strategy ξ. Due to demand inelasticity, there exists for every λ a lower bound on excess emissions given by

$$\tilde{a}(\lambda) = \inf_{\xi \in \mathcal{U}} E^{\lambda}(\xi). \tag{18}$$

This lower bound is attained for a specific production schedule $\tilde{\xi}$ whose existence is proven in Proposition 4.2 of Carmona et al. (2010).

3.2. Social Optimality of Standard Cap-and-Trade Schemes

In the sequel we say that a cap-and-trade scheme is socially optimal if given any choice of a reduction target, one can find control parameters (e.g. cap and penalty level) such that in equilibrium, the scheme reaches the emission target at minimal costs of production while the demands for goods are met. More precisely for the emission regulations discussed in this paper we define:

Definition 2. An emission regulation is said to be socially optimal if for every choice of the threshold $\lambda > 0$ and upper bound $\eta > \tilde{a}(\lambda)$ there exist regulatory parameters π and Λ such that (at least) one corresponding equilibrium production schedule ξ^* is a solution of

$$\inf_{\xi \in \mathcal{U} \text{ s.t.} E^{\lambda}(\xi) \leq \eta} \mathbb{E}[C(\xi)] \tag{19}$$

The meaning of equilibrium production schedule depends on the specific design of the emission regulation and is given by Definitions 3 and 1 respectively.

We now show that standard cap-and-trade schemes are socially optimal in the sense of Definition 2. To be more specific, we show that for every choice of emission target $\lambda > 0$ and for every upper bound $\eta > \tilde{a}(\lambda)$ ever so slightly greater than the minimum emission level possible $\tilde{a}(\lambda)$, there exists a standard cap-and-trade model whose equilibrium production schedule solves (19). The cap of this model is given by $\Lambda = \lambda$, and the penalty π is found as the Lagrange multiplier (shadow price) of the constraint appearing in (19) which is computed as the argument of

$$\sup_{\pi \geq 0} \left(\inf_{\xi \in \mathcal{U}} L^{\eta,\lambda}(\xi, \pi) \right) \tag{20}$$

where

$$L^{\eta,\lambda}(\xi, \pi) := \mathbb{E}[G^{\pi,\lambda}(\xi)] - \pi\eta = \mathbb{E}[C(\xi)] + \pi(E^{\lambda}(\xi) - \eta) \tag{21}$$

denotes the Lagrangian of problem (19) while

$$G^{\pi,\lambda}(\xi) = C(\xi) + \pi(\Delta - \lambda + \Pi(\xi))^{+} \tag{22}$$

denotes the costs to society incurred through production and penalty payments. This is the same as $G(\xi)$ with $\Lambda = \lambda$. We merely highlight its dependence on λ and π. Corollary 1 below states that under appropriate assumptions, the standard cap-and-trade scheme is socially optimal. For the proof we need following result.

THEOREM 2. *Under Assumption 1 it holds for every $\lambda \geq 0$ and $\eta > \tilde{a}(\lambda)$ that:*
(i) There exists both a solution $\overline{\xi} \in \mathcal{U}$ to the optimal control problem

$$\inf \left\{ \sup_{\pi \geq 0} L^{\eta,\lambda}(\xi,\pi) \middle| \xi \in \mathcal{U} \right\} \tag{23}$$

and a solution $\overline{\pi} \in [0,\infty)$ of problem (20).
(ii) The solution $(\overline{\pi},\overline{\xi})$ forms a saddle point of the Lagrangian, i.e.

$$L^{\eta,\lambda}(\xi,\overline{\pi}) \geq L^{\eta,\lambda}(\overline{\xi},\overline{\pi}) \geq L^{\eta,\lambda}(\overline{\xi},\pi) \quad \text{for all } \xi \in \mathcal{U},\ \pi \geq 0.$$

(iii) For a standard cap-and-trade scheme with penalty $\overline{\pi}$ and cap $\Lambda = \lambda$ the strategy $\overline{\xi}$ is a solution of the global optimal control problems (11) *and* (19)*.*

Proof of Theorem 2 (i) From Carmona et al. (2010), we know that the function $\xi \hookrightarrow L^{\eta,\lambda}(\xi,\pi) = \mathbb{E}[G^{\pi,\lambda}(\xi)] - \pi\eta$ is weak* lower semicontinuous. Since the pointwise supremum of a family of lower semicontinuous functions is also lower semicontinuous, the function $\xi \hookrightarrow \sup_{\pi \geq 0} L^{\eta,\lambda}(\xi,\pi)$ is lower semicontinuous in the weak* topology. Moreover, since $\mathcal{U}$ is weak* compact we conclude that its infimum is attained at a point $\overline{\xi} \in \mathcal{U}$.

Let us now prove that the supremum of $[0,\infty) \ni \pi \hookrightarrow \inf_{\xi \in \mathcal{U}} L^{\eta,\lambda}(\xi,\pi)$ is attained at some $\overline{\pi} < \infty$. Recall that $\tilde{\xi}$ denotes the cleanest production schedule from (18). Moreover assume momentarily that $\xi^* \in \mathcal{U}$ is a minimizer of $\xi \hookrightarrow \mathbb{E}[G^{\pi,\lambda}(\xi)]$, and that $E^\lambda(\xi^*) > \eta$ and $\pi = \mathbb{E}[C(\tilde{\xi})]/(\eta - \tilde{a}(\lambda))$. Then

$$\begin{aligned}\mathbb{E}[G^{\pi,\lambda}(\xi^*)] \geq \pi E^\lambda(\xi^*) > \pi\eta &= \mathbb{E}[C(\tilde{\xi})] + \pi\tilde{a}(\lambda) \\ &= \mathbb{E}[C(\tilde{\xi})] + \pi\mathbb{E}[(\Delta + \Pi(\tilde{\xi}) - \lambda)^+] = \mathbb{E}[G^{\pi,\lambda}(\tilde{\xi})]\end{aligned}$$

which contradicts the optimality of ξ^*, and proves that for any optimal ξ^* it holds that

$$E^\lambda(\xi^*) \leq \eta \quad \text{if} \quad \eta = \frac{\mathbb{E}[C(\tilde{\xi})]}{\pi} + \tilde{a}(\lambda).$$

Hence for all $\pi \geq 0$ it holds that

$$\inf_{\xi \in \mathcal{U}} L^{\eta,\lambda}(\xi,\pi) \leq \sup_{\xi \in \mathcal{U}} \mathbb{E}[C(\xi)] + \pi\big(\tilde{a}(\lambda) + \frac{C(\tilde{\xi})}{\pi} - \eta\big) = \sup_{\xi \in \mathcal{U}} \mathbb{E}[C(\xi)] + \pi(\tilde{a}(\lambda) - \eta) + C(\tilde{\xi}).$$

The same argument as before shows that the supremum is attained and it follows that

$$\limsup_{\pi \to \infty} \inf_{\xi \in \mathcal{U}} L^{\eta,\lambda}(\xi,\pi) = -\infty$$

whereas Proposition 4.2 in Carmona et al. (2010) implies that

$$\inf_{\xi \in \mathcal{U}} L^{\eta,\lambda}(\xi,\pi) > -\infty$$

for all $0 \leq \pi < \infty$, which proves that there exists a finite constant $b(\eta,\lambda) > 0$ such that

$$\sup_{\pi \geq 0} \inf_{\xi \in \mathcal{U}} L^{\eta,\lambda}(\xi,\pi) = \sup_{0 \leq \pi \leq b(\eta,\lambda)} \inf_{\xi \in \mathcal{U}} L^{\eta,\lambda}(\xi,\pi).$$

The function $\pi \hookrightarrow L^{\eta,\lambda}(\xi,\pi)$ is affine for each fixed $\xi \in \mathcal{U}$. Hence the infimum over $\xi \in \mathcal{U}$ of this family of functions is an upper semicontinuous function of π and its supremum over the compact set $[0, b(\eta,\lambda)]$ is attained at some $\overline{\pi} \in [0,\infty)$.

(ii) With these choices of $\bar{\xi}$ and $\bar{\pi}$ it holds that

$$\sup_{\pi\geq 0}\inf_{\xi\in\mathcal{U}} L^{\eta,\lambda}(\xi,\pi)=\inf_{\xi\in\mathcal{U}} L^{\eta,\lambda}(\xi,\bar{\pi}) \leq L^{\eta,\lambda}(\bar{\xi},\bar{\pi}) \leq \sup_{\pi\geq 0} L^{\eta,\lambda}(\bar{\xi},\pi)=\inf_{\xi\in\mathcal{U}}\sup_{\pi\geq 0} L^{\eta,\lambda}(\xi,\pi).$$

Since moreover $\mathcal{U}$ is convex and weak* compact and $\xi \hookrightarrow L^{\eta,\lambda}(\xi,\pi)$ is convex and weak* lower semicontinuous for all $\pi\geq 0$ while $\pi\hookrightarrow L^{\eta,\lambda}(\xi,\pi)$ is concave for all $\xi\in\mathcal{U}$ it holds due to Theorem 2.10.2 in Zalinescu (2002) that

$$\sup_{\pi\geq 0}\inf_{\xi\in\mathcal{U}} L^{\eta,\lambda}(\xi,\pi)=\inf_{\xi\in\mathcal{U}}\sup_{\pi\geq 0} L^{\eta,\lambda}(\xi,\pi).$$

Hence we conclude that

$$\inf_{\xi\in\mathcal{U}} L^{\eta,\lambda}(\xi,\bar{\pi})=L^{\eta,\lambda}(\bar{\xi},\bar{\pi})=\sup_{\pi\geq 0} L^{\eta,\lambda}(\bar{\xi},\pi)$$

and in particular $(\bar{\xi},\bar{\pi})$ is a saddle point of the Lagrangian $L^{\eta,\lambda}$ since

$$L^{\eta,\lambda}(\xi,\bar{\pi})\geq L^{\eta,\lambda}(\bar{\xi},\bar{\pi})\geq L^{\eta,\lambda}(\bar{\xi},\pi) \quad \text{for all } \xi\in\mathcal{U},\ \pi\geq 0.$$

(iii) The saddle point property implies that for all $\xi\in\mathcal{U}$ it holds that

$$\mathbb{E}[G^{\bar{\pi},\lambda}(\xi)]-\bar{\pi}\eta\geq \mathbb{E}[G^{\bar{\pi},\lambda}(\bar{\xi})]-\bar{\pi}\eta$$

or equivalently

$$\mathbb{E}[G^{\bar{\pi},\lambda}(\xi)]\geq \mathbb{E}[G^{\bar{\pi},\lambda}(\bar{\xi})]$$

proving that $\bar{\xi}$ is a solution of the global optimization problem (11) with cap $\Lambda=\lambda$ and hence an equilibrium strategy.

Now let us prove that $\bar{\xi}$ is also a solution to (19). Clearly for each $\xi\in\mathcal{U}$ we have:

$$\sup_{\pi\geq 0} L^{\eta,\lambda}(\xi,\pi)=\begin{cases}\mathbb{E}[C(\xi)] & \text{if } E^{\lambda}(\xi)\leq\eta\\ \infty & \text{if } E^{\lambda}(\xi)>\eta\end{cases}.$$

so that problem (19) rewrites

$$\inf_{\xi\in\mathcal{U},\,E^{\lambda}(\xi)\leq\eta}\mathbb{E}[C(\xi)]=\inf_{\xi\in\mathcal{U}}\left(\sup_{\pi\geq 0} L^{\eta,\lambda}(\xi,\pi)\right)$$

which completes the proof. □

From the result of this proposition it is straight forward to conclude that standard cap-and-trade schemes are socially optimal:

COROLLARY 1. *Under Assumptions 1 and 2 the standard cap-and-trade scheme is socially optimal.*

Proof of Corollary 1 For every $\lambda>0$ and $\eta>\tilde{a}(\lambda)$ Theorem 2 shows the existence of a penalty $\bar{\pi}$ and cap $\Lambda=\lambda$ such that there exists a strategy $\bar{\xi}$ which is a solution of the global optimal control problem (11) and (19). If Assumptions 1 and 2 are fulfilled Theorem 1 implies that $\bar{\xi}$ is also an equilibrium strategy, which concludes the proof. □

REMARK 1. The fact that $\Lambda=0$ makes no sense for a cap-and-trade scheme explains why we choose $\lambda>0$ in Definition 2.

The significance of the preceding result is to give a precise formulation of social optimality in a multi-period stochastic setting and to show that the emission target is actually reached at lowest expected cost by a cap-and-trade scheme with safety valve. Moreover we emphasize that as a side effect, it also gives a precise differentiation/comparison of emission taxes and cap-and-trade schemes.

3.3. Emission Tax versus Cap-and-Trade and Hybrid Schemes

While Corollary 1 states that the standard cap-and-trade scheme is an optimal policy to control expected shortfall emissions, we show now that a tax is the best policy to reduce emissions in average/expectation. The proof is similar to Corollary 1 and follows directly from Theorem 2 and Proposition 6.1 in Carmona et al. (2010). The precise result reads:

COROLLARY 2. *For any $\eta > \tilde{a}(0)$, there exists a tax level $\overline{\pi} \geq 0$ such that at least one equilibrium production schedule ξ^* is a solution to*

$$\inf_{\xi \in \mathcal{U} \text{ s.t. } \mathbb{E}[\Delta + \Pi(\xi)] \leq \eta} \mathbb{E}[C(\xi)] \tag{24}$$

Corollaries 1 and 2 give a new interpretation of the differences between standard cap-and-trade and tax schemes. If a regulator fixes a cap and does not want to overshoot the cap by too much in average, he should use a standard cap-and-trade scheme. If on the other hand, he is only interested in the average emissions, he should use a tax. However the results do not tell us how badly a tax performs if it is applied to reach a fixed cap or vice versa. We answer these questions numerically in Section 5.2 below. It is interesting to note that both regulations are equivalent in a deterministic setting like the one chosen by Montgomery in his groundbreaking work. This suggests that in markets where reduction costs are not very volatile, a tax may still be a good policy instrument to reach a fixed cap.

This result can not be interpreted in the framework proposed by Weitzman (1974) to find which scheme to use in order to optimize overall reduction costs and benefits from emission reduction, when reduction costs are uncertain. In contrast, we determine how emissions and reduction costs relate to each other under the different policies. For policy makers, emissions and costs might be more significant than benefits from emission reductions which need to be projected far in the future, and as a consequence, are difficult to asses.

A straightforward generalization of the above results is stated in the following corollary. If the goal is to reduce emissions in average to a certain level while at the same time controlling expected shortfall emissions, then a hybrid scheme combining tax and cap-and-trade is the best policy.

COROLLARY 3. *For every threshold λ, $\eta_1 > \tilde{a}(\lambda)$ and $\eta_2 > \tilde{a}(0)$ there exists a tax $\overline{\pi} \geq 0$ and a penalty $\pi \geq 0$ such that at least one equilibrium production schedule ξ^* of the corresponding hybrid scheme is a solution to*

$$\inf_{\xi \in \mathcal{U} \text{ s.t.} E^{\lambda}(\xi) \leq \eta_1 \ \& \ \mathbb{E}[\Delta + \Pi(\xi)] \leq \eta_2} \mathbb{E}[C(\xi)] \tag{25}$$

Again the proof of this result is straightforward though it requires an extension of Theorem 2 to the two constraints in (25). In this hybrid scheme the price of emissions remains in between $\bar{\pi}$ and $\bar{\pi} + \pi$. The existence of a floor is the main reason for the attraction of the idea of mixing the schemes.

Again this result can not be interpreted in the framework of the paper by Robert and Spence (1976) which is a generalization of Weitzman (1974).

4. Fixed Cap Generation Performance Standard

In this section we introduce the *Fixed Cap Generation Performace Standard (FCGPS)* which is a mixture of a cap-and-trade scheme with a generation performance standard. Our goal is to reduce electrity prices by relative allocation while keeping the amount of allowances in the market (the cap) fixed. To this end, part of the initial allocation is put into a pot from which the relative allocation is withdrawn. If at the end of the compliance period the relative allocation does not exceed the amount of allowances in the pot, the allowances that remain in the pot are auctioned, which fixes the number of allowances in the market. The exact allocation procedure is defined in Subsection 4.1, Subsection 4.2 is concerned with the corresponding trading bounds and profits while Subsection 4.3 gives the appropriate equilibrium definition. Finally Subsection 4.4 states the second main result of this paper, namely the equivalence between standard cap-and-trade schemes and the fixed cap generation performance standard.

4.1. Allocation Rule

Generation Performance Standard In a standard cap-and-trade scheme, each installation $i \in I$ is provided with a free initial allocation Λ_0^i at time $t=0$. This is in contrast to the *Generation Performance Standard* (or relative scheme) introduced and studied in Carmona et al. (2010), where each installation $i \in I$ is not only provided with the initial allocation of a specific number $\tilde{\Lambda}_0^i$ of free allowances at time $t=0$, but throughout the compliance period, it is also provided with free allowances on the basis of its production. To be more specific, for each good $k \in K$, the regulator chooses a relative allocation factor $y^k \geq 0$ and at each time t, firm $i \in I$ could claim $\sum_{k\in K}\sum_{j\in J^{i,k}} y^k \xi_t^{i,j,k}$ allowances if this amount were available. For each admissible strategy $\xi^i \in \mathcal{U}^i$ of agent $i \in I$, we denote by

$$Y^i(\xi^i) := \sum_{k\in K}\sum_{j\in J^{i,k}}\sum_{t=0}^{\mathfrak{T}-1} y^k \xi_t^{i,j,k} \tag{26}$$

the amount of allowances potentially earned by firm i in the name of the relative allocation up to time $\mathfrak{T}-1$, $\mathfrak{T} \in \{0,\dots,T-1\}$, and we denote by

$$Y(D) := \sum_{k\in K}\sum_{t=0}^{\mathfrak{T}-1} y^k D_t^k \tag{27}$$

the total allocation that the relative scheme would require before time $\mathfrak{T}$ for a production schedule satisfying the demand.

Fixed Cap Generation Performance Standard We now describe the allocation mechanism of the *Fixed Cap Generation Performence Standard* that we propose in this study. This scheme is characterized by a cap $\tilde{\Lambda} > 0$, an initial allocation of $\tilde{\Lambda}_0^i$ of free allowances given at time $t=0$ to each participating installation $i \in I$,r an initial period $[0,\mathfrak{T}]$ with $\mathfrak{T} \in \{0,\dots,T-1\}$, a relative distribution of free allowances during this initial period, and at time $\mathfrak{T}$, an auction of the remaining allowances if the number of allowances already given away by the regulator is still below the intended cap $\tilde{\Lambda} > 0$. We explain below what happen if the cap is reached before time $\mathfrak{T}$. We now give the specifics of this new scheme which we sometimes call hybrid.

Once the free initial allocation of the $\tilde{\Lambda}_0^i$ allowances to firm i are taken care of, the remaining

$$\Upsilon := \tilde{\Lambda} - \sum_{i\in I} \tilde{\Lambda}_0^i \tag{28}$$

allowances are set for relative distribution and auction. We use the suggestive terminology of a *pot* from which the allowances will be allocated relatively to production levels and possible auctioned off. If the size of the pot is large enough so that the relative allocation never exceeds the size of the pot, then the allocation mechanism is simple. In this case each agent following the strategy ξ^i is given $Y^i(\xi^i)$ allowances out of the pot, and the remaining allowances in the pot $\Upsilon - \sum_{i\in I} Y^i(\xi^i)$ are auctioned. For each agent, this mechanism reduces the marginal cost of production because, he/she obtains for each unit of good produced, a given number of allowances for free, relieving him/her from buying them later at the auction. This decrease in marginal production costs leads to reduced prices of goods, and to a tighter control of the windfall profits and fuel poverty.

If the size of the pot is not large enough to guaranty that the relative allocation stays smaller than the pot, then we have to deal with some technicalities. Given agents production strategies $(\xi^i)_{i\in I}$ we have to distinguish between scenarios for which $\{\sum_{i\in I} Y^i(\xi^i) < \Upsilon\}$ and $\{\sum_{i\in I} Y^i(\xi^i) \geq \Upsilon\}$. The first case was discussed above. If $\{\sum_{i\in I} Y^i(\xi^i) < \Upsilon\}$ on the other hand, then in order to prevent the free allocation to exceed the cap, each agent $i \in I$ is given the amount

$$\Gamma(\xi^{-i}) := \left(\Upsilon - \sum_{i'\in I\setminus\{i\}} Y^{i'}(\xi^{i'})\right)^+, \tag{29}$$

of free allowances where $\xi^{-i} := (\xi^{i'})_{i'\in I\setminus\{i\}}$ denotes the strategies of the other agents. Then as before, the $\Upsilon - \sum_{i\in I} \Gamma(\xi^{-i})$ remaining allowances in the pot are actioned.

Given these rules it is easy to see that the allowances allocated from the pot to each firm $i \in I$ is always given by $\Gamma(\xi^{-i}) \wedge Y^i(\xi^i)$. i.e. the amount of allowances given from the pot is bounded from above by $\Gamma(\xi^{-i})$ which depends on the strategies $\xi^{-i} := (\xi^{i'})_{i'\in I\setminus\{i\}}$ of the other agents.

Following this allocation rule the number of allowances that remain in the pot and are auctioned at time $\mathfrak{T}$ are given by

$$\Upsilon(\xi) = \begin{cases} \Upsilon - \sum_{i\in I} Y^i(\xi^i) & \text{if } \sum_{i\in I} Y^i(\xi^i) < \Upsilon \\ \Upsilon - \sum_{i\in I} \Gamma(\xi^{-i}) & \text{if } \sum_{i\in I} Y^i(\xi^i) \geq \Upsilon \end{cases}$$

In the sequel, we assume that the auction is an Open Ascending-Bid Auction (English Auction) and we denote by the $\mathcal{F}_{\mathfrak{T}}$-measurable random variable P the price of the auctioned allowances, and by φ^i the $\mathcal{F}_{\mathfrak{T}}$-measurable auction strategy, i.e. the number of allowances bought by firm i at the auction. For simplicity we assume that the allowances are paid and delivered at time T, even though they are auctioned at time $\mathfrak{T}$.

The following proposition is crucial for the hybrid scheme to work:

PROPOSITION 1. *The relative allocation in the hybrid allocation scheme does never exceed the size of the pot, i.e.*

$$\Upsilon(\xi) \geq 0 \tag{30}$$

holds almost surely for all $\xi = (\xi^i)_{i\in I}$ with $\xi^i \in \mathcal{U}^i$.

Proof of Proposition 1 It is sufficient to prove that $\sum_{i\in I} \Gamma(\xi^{-i}) \leq \Upsilon$ is true on $\{\sum_{i\in I} Y^i(\xi^i) \geq \Upsilon\}$. As $Y^i(\xi^i) \geq 0$ for all $i \in I$ it holds on $\{\sum_{i\in I} Y^i(\xi^i) \geq \Upsilon\}$ that

$$\Gamma(\xi^{-i}) \leq \left(\Upsilon - \sum_{i'\in I\setminus\{i\}} Y^{i'}(\xi^{i'}) \frac{\Upsilon}{\sum_{i\in I} Y^i(\xi^i)}\right)^+ = \left(\Upsilon - \sum_{i'\in I\setminus\{i\}} Y^{i'}(\xi^{i'}) \frac{\Upsilon}{\sum_{i\in I} Y^i(\xi^i)}\right)$$

where the last equality holds because $\sum_{i'\in I\setminus\{i\}} Y^{i'}(\xi^{i'}) \leq \sum_{i\in I} Y^i(\xi^{i'})$ implies that the last term is nonnegative. Moreover simple algebra yields

$$\begin{aligned} \Upsilon - \sum_{i'\in I\setminus\{i\}} Y^{i'}(\xi^{i'}) \frac{\Upsilon}{\sum_{i\in I} Y^i(\xi^i)} &= \Upsilon \frac{\sum_{i\in I} Y^i(\xi^i)}{\sum_{i\in I} Y^i(\xi^i)} - \sum_{i'\in I\setminus\{i\}} Y^{i'}(\xi^{i'}) \frac{\Upsilon}{\sum_{i\in I} Y^i(\xi^i)} \\ &= Y^i(\xi^i) \frac{\Upsilon}{\sum_{i\in I} Y^i(\xi^i)} \end{aligned}$$

and we conclude

$$\sum_{i\in I} \Gamma(\xi^{-i}) \leq \sum_{i\in I} \left(Y^i(\xi^i) \frac{\Upsilon}{\sum_{i\in I} Y^i(\xi^i)}\right) = \Upsilon$$

which concludes the proof. □

Since the rest of the allowances in the pot are auctioned, the total amount of allowances in the market is Λ as in the standard scheme. Notice that for $y^k = 0$ we recover the standard scheme where the whole Υ of the pot is auctioned. If on the other hand $\Upsilon = 0$, then we are in the realm of the standard cap-and-trade scheme without auction and without relative allocation.

4.2. Trading Constraints and Profits

If for each firm $i \in I$ we denote by φ^i the number of allowances purchased in the auction, by ξ^i its production strategy and by ξ^{-i} the other firms strategies, then the total number of allowances that an agent obtains through allocation and auctioning in the FCGPS hybrid scheme reads $\tilde{\Lambda}_0^i + \Gamma(\xi^{-i}) \wedge Y^i(\xi^i) + \varphi^i$. Hence this has to replace Λ_0^i in the trading constraints and profits from Section 2.

More precisely, since the number of (physical) allowance certificates $\tilde{\Lambda}_0^i + \Gamma(\xi^{-i}) \wedge Y^i(\xi^i) + \varphi^i + \theta_T^i$ that firm $i \in I$ surrenders for compliance must be nonnegative we obtain a lower bound for the trading at the last time point T of the compliance period, i.e.

$$\theta_T^i \geq -\left(\tilde{\Lambda}_0^i + \Gamma(\xi^{-i}) \wedge Y^i(\xi^i) + \varphi^i\right) \tag{31}$$

must hold almost surely. Moreover the penalty payment due by firm i at time T for using the strategy ξ^i while the other firms are using strategies ξ^{-i} is:

$$\pi\left(\Delta^i + \Pi^i(\xi^i) - \tilde{\Lambda}_0^i - \varphi^i - \theta_T^i - \Gamma(\xi^{-i}) \wedge Y^i(\xi^i)\right)^+. \tag{32}$$

Combining (4) – (5) together with (32), we obtain the following expression for the terminal wealth (profits and losses at time T) of firm i

$$\begin{aligned} H^{A,S,P,\xi^{-i},i}(\theta^i, \xi^i, \varphi^i) := &\sum_{t=0}^{T-1} \sum_{(j,k)\in M_i} (S_t^k - C_t^{i,j,k})\xi_t^{i,j,k} \\ &+ \sum_{t=0}^{T-1} \theta_t^i (A_{t+1} - A_t) - \theta_T^i A_T - \varphi^i P \\ &- \pi\left(\Delta^i + \Pi^i(\xi^i) - \tilde{\Lambda}_0^i - \varphi^i - \theta_T^i - \Gamma(\xi^{-i}) \wedge Y^i(\xi^i)\right)^+. \end{aligned}$$

4.3. Equilibrium Definition

In this subsection we adjust the equilibrium definition of Section 2 to the FCGPS. To this end we choose allowance and product price processes in the space $\mathcal{L}_T^1(\mathbb{R}) \times \mathcal{L}_{T-1}^1(\mathbb{R}^{|K|})$. Moreover, in line with the trading constraints spelled out above, we define the following spaces of admissible strategies:

$$\begin{aligned} \tilde{\mathcal{Q}}^{\xi^{-i},i}(A,P) := \Big\{ &(\theta^i, \xi^i, \varphi^i) \,\big|\, \xi^i \in \mathcal{U}^i,\ \varphi^i \geq 0,\ \varphi^i \in \mathcal{F}_{\mathfrak{T}}, \\ &\varphi^i P \text{ and } R_T^A(\theta^i) \text{ integrable, } \theta_T^i \geq -\left(\tilde{\Lambda}_0^i + \Gamma(\xi^{-i}) \wedge Y^i(\xi^i) + \varphi^i\right) \text{ a.s.}\Big\}. \end{aligned}$$

and

$$\tilde{\mathcal{Q}}(A,P) := \left\{(\theta^i, \xi^i, \varphi^i)_{i\in I} \,\big|\, (\theta^i, \xi^i, \varphi^i) \in \tilde{\mathcal{Q}}^{\xi^{-i},i}(A,P) \text{ for all } i \in I\right\}.$$

In addition to Assumptions 1 and 2, we shall also use the following technical property:

Assumption 3. *we assume that the potentially earned amount of allowances $Y(D)$ matches the size of the pot only on sets of zero probability, i.e. $\mathbb{P}[Y(D) = \Upsilon] = 0$.*

Following the intuition that given prices $A=\{A_t\}_{t=0}^T$, $S=\{(S_t^k)_{k\in K}\}_{t=0}^{T-1}$ and P, and the production strategies ξ^{-i} of the other firms, each firm aims at increasing its own wealth by maximizing the function

$$(\theta^i,\xi^i,\varphi^i)\hookrightarrow \mathbb{E}[H^{A,S,P,\xi^{-i},i}(\theta^i,\xi^i,\varphi^i)], \tag{33}$$

over its admissible investment and production strategies, we are led to define equilibrium in the following way:

Definition 3. The triple $(A^*,S^*,P^*)\in\mathcal{L}^1_T(\mathbb{R})\times\mathcal{L}^1_{T-1}(\mathbb{R}^{|K|})\times L^1(\mathcal{F}_{\mathfrak{T}})$ of integrable processes and random variable are an equilibrium for the hybrid scheme if there exists $(\theta^*,\xi^*,\varphi^*)\in\tilde{\mathcal{Q}}(A^*,P^*)$ such that:

(i) All financial positions are in zero net supply, i.e.

$$\sum_{i\in I}\theta_t^{*i}=0,\qquad t=0,\dots,T; \tag{34}$$

(ii) All allowances that are auctioned are bought, i.e.

$$\sum_{i\in I}\varphi^{*i}=\Upsilon_{\mathfrak{T}}(\xi^*); \tag{35}$$

(iii) Supply meets demand for each good

$$\sum_{i\in I}\sum_{j\in J^{i,k}}\xi_t^{*i,j,k}=D_t^k,\qquad k\in K,\ \ t=0,\dots,T-1; \tag{36}$$

(iv) Each firm $i\in I$ is satisfied by its own strategy given the strategies $\xi^{*-i}=(\xi^{*i'})_{i'\in I\setminus\{i\}}$ of the other firms, namely:

$$\mathbb{E}[H^{A^*,S^*,P^*,\xi^{*-i},i}(\theta^{*i},\xi^{*i},\varphi^{*i})]\geq \mathbb{E}[H^{A^*,S^*,P^*,\xi^{*-i},i}(\theta^i,\xi^i,\varphi^i)] \tag{37}$$

for all $(\theta^i,\xi^i,\varphi^i)\in\tilde{\mathcal{Q}}^{\xi^{*-i},i}(A^*,P^*)$.

From this definition it is obvious that $\varphi^{*i}(P^*-A^*_{\mathfrak{T}})=0$ almost surely for all $i\in I$. If it would hold that $\mathbb{P}[\{P^*<A^*_{\mathfrak{T}}\}]>0$ then the auction and trading strategies $\varphi^{*i}+\mathbf{1}_{\{P^*<A^*_{\mathfrak{T}}\}}$ and $\theta_T^{*i}-\mathbf{1}_{\{P^*<A^*_{\mathfrak{T}}\}}$ would outperform the original strategies φ^{*i} and θ^{*i}. However this does not lead to an equilibrium as on $\{P^*<A^*_{\mathfrak{T}}\}$ it holds that

$$\sum_{i\in I}\varphi^{*i}+\mathbf{1}_{\{P^*<A^*_{\mathfrak{T}}\}}>\sum_{i\in I}\varphi^{*i}=\Upsilon(\xi^*)$$

contradicting condition (ii) of an equilibrium. On the other hand on $\{P^*>A^*_{\mathfrak{T}}\}$ it holds that $\varphi^{*i}=0$ since it is cheaper buying allowances in the market than in the auction. Hence we conclude that $\varphi^{*i}(P^*-A^*_{\mathfrak{T}})=0$.

4.4. Equivalence of Standard Cap-and-Trade Schemes and Fixed Quantity Generation Performance Standards

In this section we prove a form of equivalence between standard cap-and-trade schemes and hybrid cap-and-trade schemes. To be more specific we prove that every equilibrium production strategy in a standard cap-and-trade scheme is also an equilibrium production strategy for a hybrid cap-and-trade scheme with the same penalty and cap, and vice versa. Therefore we assume that the caps in both schemes are identical, i.e. $\tilde{\Lambda}=\Upsilon+\sum_{i\in I}\tilde{\Lambda}_0^i=\sum_{i\in I}\Lambda_0^i=\Lambda$. The main result of this section is:

Theorem 3. *Under Assumptions 1 and 3 the following holds. If (A^*, S^*) is an equilibrium with production strategies ξ^* for the standard scheme then the prices $(A^*, S^{'*}, P^*)$ where*

$$S_t^{'*k} = S_t^{*k} - y^k \mathbb{E}[A^*_{\mathfrak{T}} \mathbf{1}_{\{Y(D)<\Upsilon\}} | \mathcal{F}_t] \quad k \in K, t = 0, \ldots, \mathfrak{T}-1 \tag{38}$$

$$S_t^{'*k} = S_t^{*k} \quad k \in K, \ t = \mathfrak{T}, \ldots, T-1 \tag{39}$$

$$P^* = A^*_{\mathfrak{T}} \tag{40}$$

define an equilibrium for the cap-and-trade scheme with the hybrid allocation having the same production strategies ξ^* and allowance price processes A^*. The converse statement is true as well.

A detailed proof of this result is given in the third appendix at the end of the paper. Theorem 3 implies that the results of Theorem 1 and Corollary 1 can be transferred to the hybrid scheme:

Corollary 4. *Under Assumptions 1-3 it holds that:*

(i) There exists a solution $\overline{\xi} \in \mathcal{U}$ of the global optimization problem (11) and the triple $(\overline{A}, \overline{S}', \overline{P})$ defined by

$$\overline{A}_t = \pi \mathbb{P}_t[\Gamma + \Pi(\overline{\xi}) \geq 0]$$

for all $t = 0, \ldots, T$, $\overline{P} = \overline{A}_{\mathfrak{T}}$ and

$$\overline{S}_t'^k = \max_{i \in I, \, j \in J^{i,k}} (C_t^{i,j,k} + e^{i,j,k} \overline{A}_t) 1_{\{\overline{\xi}_t^{i,j,k} > 0\}} - y^k \mathbb{E}[\overline{A}_{\mathfrak{T}} \mathbf{1}_{\{Y(D)<\Upsilon\}} | \mathcal{F}_t] \mathbf{1}_{\{t<\mathfrak{T}\}}$$

for all $t = 0, \ldots, T-1$, $k \in K$ form a market equilibrium of the hybrid cap-and-trade scheme. Moreover, the equilibrium allowance price process is almost surely unique, while the process $\overline{S}'$ is the smallest equilibrium price in the sense of Theorem 1.

(ii) The hybrid cap-and-trade scheme is socially optimal.

Proof of Corollary 4 (i) This follows directly from Theorem 1 and Theorem 3.

(ii) Corollary 1 implies that for every threshold $\lambda > 0$ and upper bound $\eta > \tilde{a}(\lambda)$ there exists an equilibrium of the standard scheme with corresponding equilibrium strategy $\overline{\xi}$ which is a solution of (19). Due to Theorem 3 there exists an equilibrium of the hybrid scheme with the same corresponding equilibrium strategy $\overline{\xi}$ proving the assertion. □

Notice that the conclusions of Theorem 3 are stronger than assertion (i) of Corollary 4. Namely Corollary 4 implicitly relates the lowest possible product prices $\overline{S}$ of the standard scheme with the lowest possible product prices $\overline{S}'$ of the hybrid scheme. On the other hand Theorem 3 relates *all* equilibrium prices S^* of the standard scheme to equilibrium prices S'^* of the hybrid scheme and vice versa. In the case study presented in Section 5, we will see that the hybrid scheme, in contrast to schemes in which all the allowances are auctioned, can reduce windfall profits to zero in average, while at the same time it preserves the social optimality of the standard scheme.

Remark 2. **Simplified Allocation Rule**. The complexity of the definition of a regulation is the main hindrance to its popularity and likelihood to be adopted. So in order to avoid the technicalities of handling scenarios for which a plain relative allocation would exceed the cap, one notice that if there is not much demand uncertainty and the size of the pot is chosen large enough for the probability of an empty pot to be small, and we can define a simpler allowance allocation scheme by

$$\sum_{t=0}^{\mathfrak{T}-1} \sum_{(j,k) \in J^{i,k}} y^k \xi_t^{i,j,k} \mathbf{1}_{\{\sum_{s=0}^t \sum_{(i,j,k)} y^k \xi_s^{i,j,k} < \Upsilon\}}. \tag{41}$$

In this scheme, the relative allocation is simply stopped when it starts to exceed the size of the pot. The equilibrium equivalence with a standard cap-and-trade scheme does not hold any longer, but the equilibrium production strategies are in most applications very similar to those of the hybrid scheme introduced in this paper.

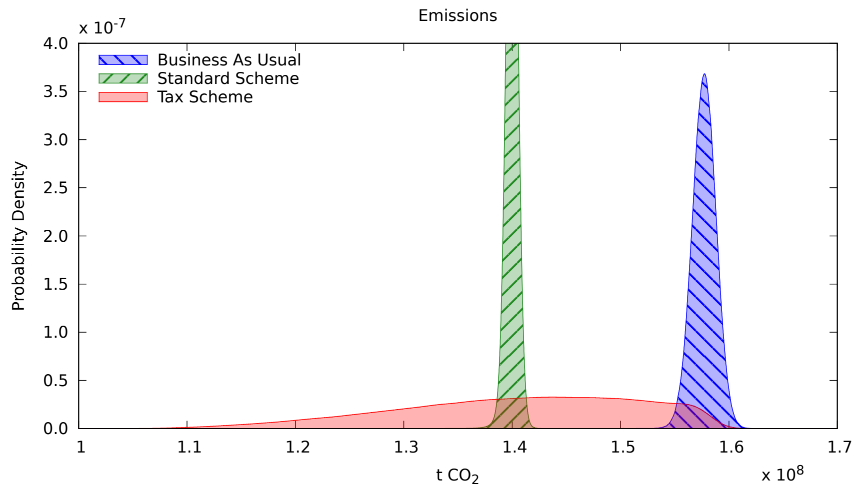

Figure 1 Histograms of emission distributions in the cases of BAU, a pure tax of 28$/$t_{CO_2}$, and a cap-and-trade scheme with penalty 100$/$t_{CO_2}$ and cap of 140 Mega-ton.

5. Case Study

We compare the properties of the three schemes discussed in this paper when implemented on the Korean electricity market with data calibrated to the period 2006 - 2008.

5.1. Case Study: the Korean Electricity Market

At the core of our analysis is the main abatement mechanism in electricity production, the fuel switch from coal to gas (i.e. when for some period of time, coal plants are turned off in favour of cleaner gas plants). This was the main abatement mechanism in EU ETS Reinaud (2007). It has large abatement potential in the Korean electricity market. We assume that the Korean electricity market is totally deregulated and use the following production capacities.

Production Capacity in GW	
Nuclear	18
Coal	19
LNG	23

5.2. Emissions

Corollaries 1 and 2 shed new light on the difference between standard cap-and-trade schemes and taxes. If a regulator fixes a cap and does not want to overshoot the cap by too much in average, he should use a standard cap-and-trade scheme. If on the other hand he is only interested in the average emission reduction he should use a tax. In a deterministic setting both schemes are equivalent. Hence the tax might perform very well if we want to regulate markets with not so volatile reduction costs like e.g. end consumers. However the results do not tell us how well or how badly a tax performs if it is used to reach a fixed cap under stochastic reduction costs. This questions can only be answered numerically. Indeed, we show at the example of the Korean electricity sector how difficult it can be to control emissions with a tax. To this end we consider a tax of 28$/$t_{CO_2}$ and compare it to an emission trading scheme with cap 140 Mega-ton and penalty of 100$/$t_{CO_2}$. The resulting plots are shown in Figure 1. As expected the cap-and-trade scheme leads to an emission distribution which is centered narrowly around the cap. At the same time the tax gives an emission distribution which in average corresponds to the cap but at the same time

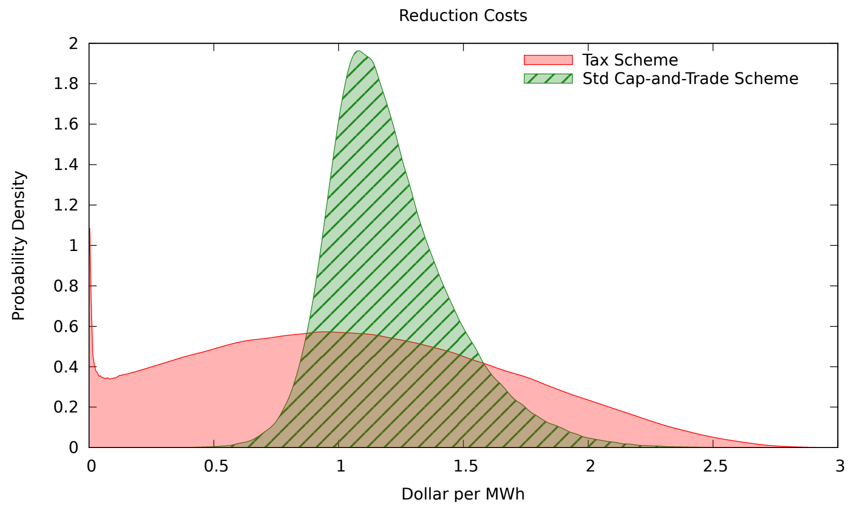

Figure 2 Histograms of reduction costs for the pure tax and the cap and trade scheme. The average reduction cost of the tax and the cap-and-trade scheme are $1.05\$/MWh$ and $1.2\$/MWh$ respectively.

is extremely broad - even much broader than the BAU distribution. In view of Corollary 2 this is not surprising because the tax reaches average emission targets, with no constraint on the width of the distribution.

Because both the tax and the cap-and-trade scheme give the same average reduction Corollary 2 tells us that the tax should be cheaper in average. Indeed the average reduction cost of the tax is $1.05\$/MWh$ while the average reduction cost of the cap-and-trade scheme are $1.2\$/MWh$. So in terms of average cost the tax performs better but not much better. And the question is whether the small difference in reduction costs justifies the broad emission distribution. Figure 2 compares the histograms of reduction costs. We saw that the tax had a broader emission distribution than the cap-and-trade scheme. We see now that it also has a broader reduction cost distribution. This should not come as a surprise since the cap-and-trade scheme is more selective in triggering emission reductions.

In many countries the power sector is the biggest emitter, so this broad emission distribution can become problematic. This is a serious shortcoming for a policy based on a straight tax, but in all fairness, it also has advantages. First and foremost, it gives a stable long term price signal that triggers investment in low carbon technologies. Therefore it makes sense to think about combining it with a cap-and-trade scheme as we will discuss in Section 5.4.

5.3. Costs, Windfall Profits and Fuel Poverty

Based on the results of Section 4 we compare different allocation mechanisms of cap-and-trade schemes. The mechanisms we consider are the standard allocation, 100% auctioning and the fixed cap generation performance standard (FCGPS) hybrid scheme. Notice that all three schemes have the same production policies, reduction costs and emission processes. Hence we are particularly interested in how consumer costs and windfall profits relate to reduction costs and penalty payments under the different allocation mechanisms. In our case study, the standard scheme has a cap of 140 Mega-ton of carbon dioxide and a penalty of 100 USD.

5.3.1. Free upfront allocation Obviously, costs of production are higher in the presence of a cap-and-trade scheme. This is because, due to emission constraints, producers switch to cleaner and more expensive technologies to avoid penalty payments. However from Figure 3 we see that for a 10% reduction target, average abatement costs are only 1.21$ per MWh of produced electricity!

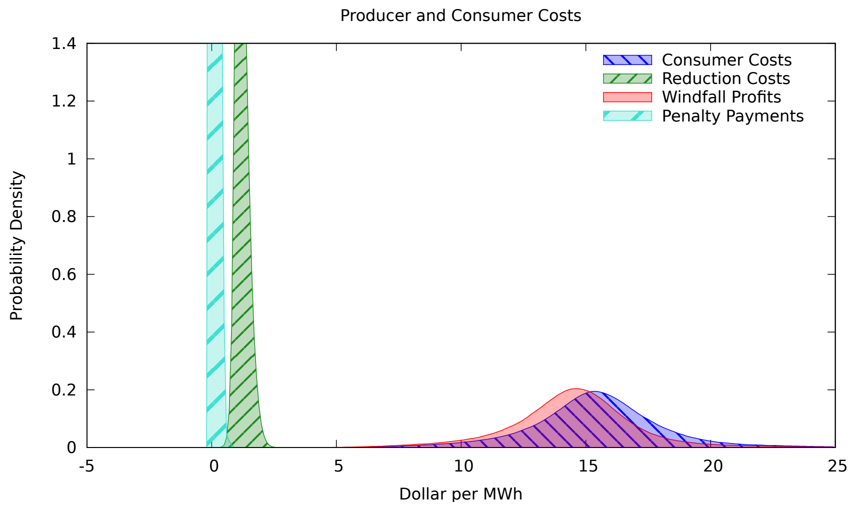

Figure 3 Histograms of consumer cost, reduction cost, windfall profits and penalty payments of a standard cap-and-trade scheme calibrated to reach the emissions target with 95% probability.

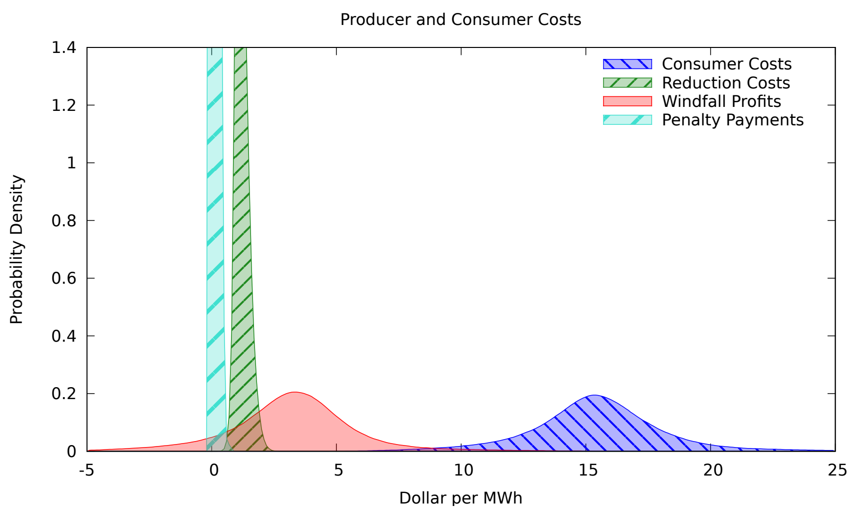

Figure 4 Histograms of consumer cost, reduction cost, windfall profits and penalty payments of a standard cap-and-trade where all allowances are auctioned at the beginning of the trading period.

Though as observed in EU ETS consumers costs (in average $15.45\$/MWh$) exceed the overall reduction costs by far ($\times 8$ in the present case). This has two related side effects first this can lead to fuel poverty, secondly this gives rise to windfall profits which were observed in the first phase of EU ETS, and have been the core of the main criticism of cap-and-trade schemes by consumer advocates.

5.3.2. Auction The most frequently proposed approach to the reduction of windfall profits is to replace the free allocation of allowances by an auction. This is a tempting proposition because the auction proceeds constitute extra income for the auctioneers. These proceeds could be returned to consumers, invested in cleaner technology Research & Development (R&D) or new emission reduction projects, or even used as subsidies for the most vulnerable households. However, not only is there no guarantee that any of these options will actually be exercised, but what is commonly overseen, is that the consumer costs can exceed the auction revenue by far.

As shown in Figure 4 in the case of our Korea case study, auctioning does neither reduce consumer costs (in average 15.45$/$MWh$) nor does it reduce windfall profits to a satisfactory level. These are in average still 3.26$/$MWh$. Because the auction revenue corresponds to 11.26$/$MWh$ consumer costs exceed significantly the revenue from the auction. This amount can only cover about three forth of the consumer costs. Hence the commonly believed argument that auction revenues can be used to cover costs of endconsumers is wrong and there is still significant wealth transfer from consumers to producers.

The reason why auctioning of allowances does not cover consumer costs is the following: by selling allowances, one is able to collect an amount which is essentially equal to the total number of allowances times the allowance price. This money will, in general not match the overall consumer burden, since the latter is related not to the number of allowances, but instead to the number of product units consumed within one compliance period.

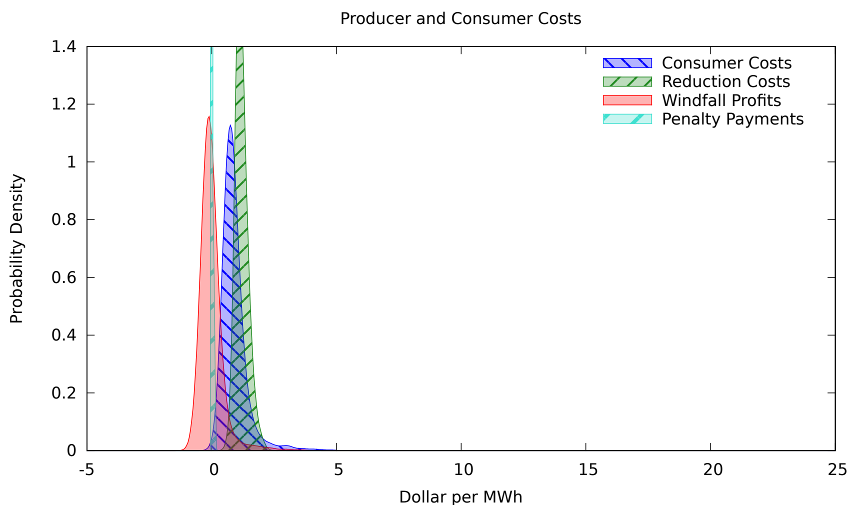

Figure 5 Histograms of consumer cost, reduction cost, windfall profits and penalty payments in the case of a hybrid scheme.

5.3.3. Hybrid Scheme Because the equilibrium electricity spot price is given by

$$\overline{S}_t^k = \max_{i\in I,\, j\in J^{i,k}} (C_t^{i,j,k} + e^{i,j,k}\overline{A}_t)1_{\{\xi_t^{i,j,k}>0\}} - \mathbf{1}_{\{t\le\mathfrak{T}-1\}} y^k \mathbb{E}[\overline{A}_{\mathfrak{T}} \mathbf{1}_{\{\sum_{t=0}^{\mathfrak{T}-1} D_t \ge \Upsilon\}} | \mathcal{F}_t] \tag{42}$$

for all $t = 0,\dots,T-1$ and $k\in K$, the hybrid scheme reduces windfall profits at its origin: at the formation of the electricity price. Hence the reduction of windfall profits is related to the product units consumed within 0 and $\mathfrak{T}-1$ rather than the number of allowances in the market, as in the case of an auctioning of allowances. This results in a much more efficient reduction of windfall profits, even if a significant amount of allowances is left over for free allocation and can be used to set incentives to develop a sustainable production portfolio.

In this case study we used parameters $y = 0.52$ for electricity and a *pot size* $\Upsilon = 1.2\times 10^8$, which leaves 15% of the allowances for a free upfront allocation. As in Carmona et al. (2010) the relative allocation is only given for electricity that was not produced by nuclear power, as nuclear power is never the marginal technology.

Figure 5 shows that in contrast to the auction scheme the reduction in windfall profits comes with a reduction of consumer costs which are 0.91$/$MWh$ in average which implies that reduction costs are split on endconsumers and producers. Also it reduces windfall profits to zero in average.

Notice also that consumer costs are much less volatile than under an upfront allocation or an auction.

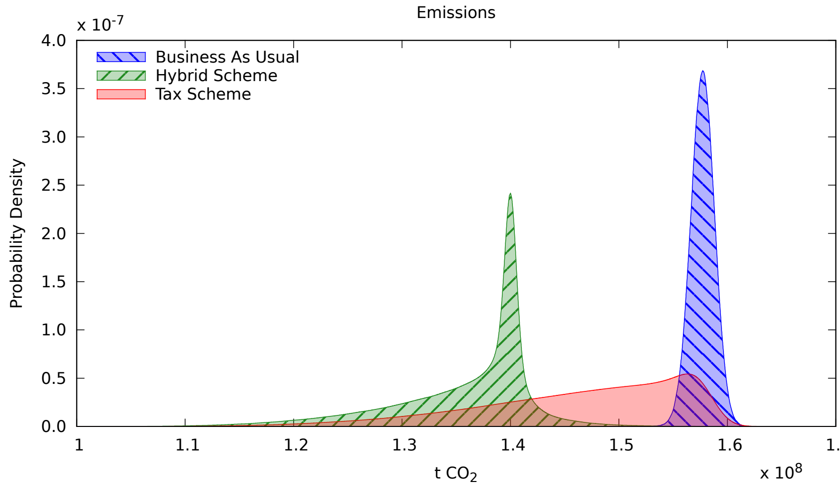

Figure 6 Histograms of emission distributions in BAU, a pure tax of $25\$/t_{CO_2}$ and the same tax mixed with a cap and trade scheme with penalty $20\$/t_{CO_2}$ and a cap of 140 Mega-ton. .

5.4. Mixing Tax with Cap-and-Trade Schemes and Generation Perforance Standards

We saw in the previous section that cap-and-trade schemes increase prices of goods and showed that FCGPSs can help to reduce this price increase. In the following example we show that this can be useful when we combine an emission tax with a cap-and-trade scheme. To this end we consider again the example of Korea and assume that the government introduces a carbon tax of $25\$/t_{CO2}$ (Now slightly lower than in Section 5.2) over the whole economy and discuss what happens if the tax is extended either by a standard cap-and-trade scheme or a FCGPS. Mixing tax with cap-and-trade is pritty much the same as introducing a carbon price floor as was recently done by the United Kingdom (see e.g. Ares (2011)).

Figure 6 shows the effect of different policy alternatives on the emissions in the Korean electricity sector. As already seen in Section 5.2 the tax leads to a broad emission distribution, that is even broader than in BAU.

Because the power sector is one of the biggest emitters this broad emission distribution introduces a large emission uncertainty for the country. In case that policy makers want more certainty about emission reductions from sectors with volatile reduction costs they could extend the economy wide tax by a cap-and-trade scheme covering only those sectors. In the following we discuss the effect of mixing the tax with a cap-and-trade scheme at our example of the Korean power sector. We choose a penalty of $20\$/t_{CO2}$ and in order to trigger even more emission reductions from the power sector we use a relatively strict cap 140 Mega-ton.

While this leads to a good control of the emissions it has one important drawback. While for the tax the carbon price is fixed at $25\$/t_{CO2}$ the carbon price for the power market takes now values in a range from $25\$/t_{CO2}$ to $45\$/t_{CO2}$. So not only does it increase the carbon price it also increases its volatility. Power producers have the flexibility and are used to fully exploit this flexibility in order to react on varying production costs and varying carbon price. However as seen in the previous section the full carbon price is passed trough to the electricity price and power consumers are faced with the same volatile carbon price that takes values in the $25\$ - 45\$$ range instead of solely the tax of 25\$. This is totally unnecessary, in particular because endconsumers do not have the flexibility to react quickly on varying carbon price. To illustrate the effect of this path through on the electricity price we show in 7 the difference between average electricity price in BAU compared to the different regulations. The average price increase in case of a pure tax is $13\$/MWh$. (With average emission factor $0.52t_{CO_2}/MWh$ this corresponds to a carbon price of 25\$, matching of

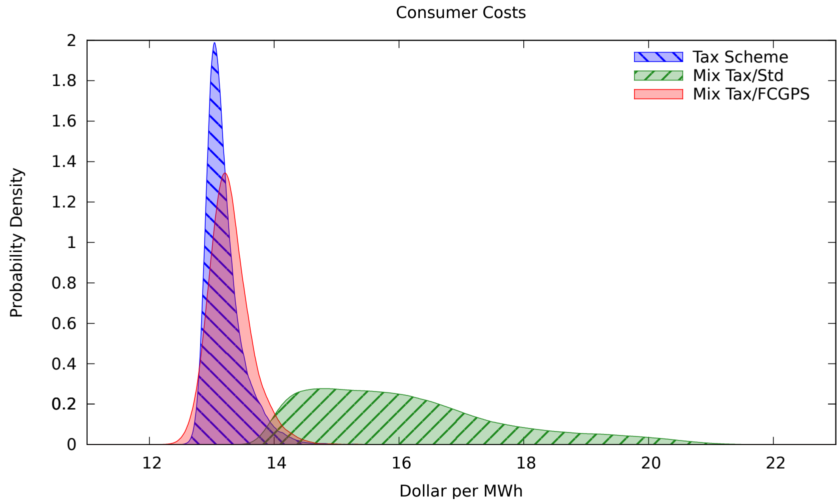

Figure 7 Histograms of the difference between consumer costs in BAU and consumer costs under the pure tax, the tax mixed with a standard scheme and the tax mixed with FCGPS. This indicates that the tax mixed with FCCPS gives a clear carbon price signal embedded in the electricity price, which is at the same level as the pure tax. If we mix a tax with a standard cap-and-trade scheme the carbon price embedded in power is higher and more volatile.

course the tax). On the other hand the average price increase with a mixed scheme compared to BAU is 16$/$MWh$. Which corresponds to a carbon price of 31$. It is important to notice that while the carbon price is higher in the mixed scheme it is also much more volatile.

If a policy maker wants to omit this pass through of the allowance price on the electricity price and instead wants to fix the carbon price embedded in the power price at the level of the tax there is a simple solution. With the FCGPS allocation mechanism we have a simple mechanism to reduce the influence of the allowance price on power consumers carbon price. This is because the relative allocation prevents the allowances from being priced into electricity. This is shown in Figure 7 where for the FCGPS the carbon price embedded in power is very close to the case of a pure tax. The discrepancy comes from the constant relative allocation factor. If this would be chosen as the instantaneous emission factor, the influence of the allowance price on the carbon price could be eliminated. While a pure tax leads to widely fluctuating emissions and a standard cap-and-trade scheme leads to widely fluctuating carbon prices, the mix of tax with FCGPS allows both to reach a tight emission target and fixing endconsumers carbon price.

6. Conclusion

The main thrust of this paper is to design cap-and-trade schemes capable of reaching reasonable pollution targets at low reduction costs while controlling end consumer prices. We prove that emission trading schemes which allow for excess emissions modulo the payment of a penalty, such as the EU ETS, are socially optimal in a multi-period stochastic setting, social optimality meaning that a given a regulatory emission target (upper bound on the expected excess emissions) is reached at lowest expected costs. As a special case of this result, we show that if the emission target is given as a bound on the expected emissions, an emission tax reaches the target at lowest expected cost.

Next, we build on the theory and numerical implementations of Carmona et al. (2010) to compare standard cap-and-trade schemes – with and without auctioning of the initial allocation – with a new allocation scheme which mimics a generation performance standard while preserving a fixed cap and social optimality. The main results of this comparison are:

Standard cap-and-trade schemes are socially optimal in a stochastic setting if they are given the right emission targets and penalties. However, the numerical case study shows that the consumers' burden far exceeds the overall reduction costs (8 times the reduction cost for our case study), giving rise for huge windfall profits.

Auctioning, despite its popularity among the supporters of cap-and-trade schemes puzzled by the magnitude of the windfall profits of the first phase of EU ETS, cannot lower windfall profits to a reasonable level. They merely help the re-distribution of these costs. Indeed, the revenues of the auctions (expected to be in the range of $9,5$ \$ per MWh in our case study) can remain significantly smaller than the consumer costs, covering only approximately two thirds of the latter. Hence the common belief that auction revenues can be used to cover consumer costs needs to be substantiated, as there is still significant wealth transfer from consumers to producers.

Hybrid cap-and-trade schemes incorporate the best of the standard and relative schemes: they are socially optimal, they respect a cap fixed ex ante, and provide a tight control of the level at which allowances prices enter the prices of products at the source of the emissions. This has two interesting applications: Firstly they can be used to reduce windfall profits and fuel poverty. In contrast to auctioning, it gives good control of both quantities because these are reduced at their source, by reducing the factor by which emissions are priced into goods whose production causes pollution. Secondly they can incorporate a tax together with a cap-and-trade scheme, only embedding the emission tax (and not the allowance price) in prices of goods whose production causes the externality.

Appendix A: Costs of Cap-and-Trade Schemes

For the sake of completeness, we review the definitions of the various costs resulting from a cap-and-trade regulation.

A.1. Reduction Costs

For any equilibrium production schedule $\xi^* \in \mathcal{U}$, we define the reduction costs RC as the random variable given by the difference between the production costs under this production schedule and the production costs incurred in the same random scenarios had we used the Business As Usual (BAU) equilibrium production schedule $\xi^\dagger \in \mathcal{U}$ instead:

$$RC = \sum_{i,j,k} \sum_{t=0}^{T-1} (\xi_t^{*i,j,k} - \xi_t^{\dagger i,j,k}) C_t^{i,j,k}. \tag{43}$$

Notice that as defined, the reduction costs do not depend upon the trading strategies of the individual firms in the emissions market.

A.2. Consumer Costs

As for the reduction costs, in order to define the consumer costs we compare the results of the cap-and-trade scheme to a BAU equilibrium. If $\xi^\dagger$ is a BAU equilibrium production strategy, the lowest BAU price $\hat{S}_t^k$ for good k is given by:

$$S_t^{\dagger k} := \max_{i\in I, j\in J^{i,k}} C_t^{i,j,k} \mathbf{1}_{\{\xi_t^{\dagger i,j,k} > 0\}}. \tag{44}$$

So if we denote by S^* the lowest possible equilibrium prices in our cap-and-trade scheme, the markets overall consumer costs are defined as

$$CC = \sum_{t=0}^{T-1} \sum_{k\in K} (S_t^{*k} - S_t^{\dagger k}) D_t^k. \tag{45}$$

A.3. Windfall Profits

If ξ^* is an optimal production strategy associated with the equilibrium (A^*, S^*), we define the target price $\hat{S}_t^k$ of good k as:

$$\hat{S}_t^k := \max_{i \in I, j \in J^{i,k}} C_t^{i,j,k} \mathbf{1}_{\{\xi_t^{*i,j,k} > 0\}}. \tag{46}$$

This price is the marginal cost under the optimal production schedule without taking into account the cost of pollution. We then define the windfall profits of firm i as:

$$\sum_{t=0}^{T-1} \sum_{(j,k) \in M_i} (S_t^{*k} - \hat{S}_t^k) \xi_t^{*i,j,k},$$

and the overall windfall profits as

$$WP = \sum_{t=0}^{T-1} \sum_{k \in K} (S_t^{*k} - \hat{S}_t^k) D_t^k. \tag{47}$$

These windfall profits measure the profits from the production of goods in excess over what the profits would have been, had the same production schedule been used and the prices did not include the costs of pollution.

Appendix B: Case Study: the Korean Electricity Market

We ran numerical experiments with data from the Korean electricity market which is based on an isolated grid, so that we can study the impact of the introduction of an emission regulation without having to take into account emission leakage across boarders. For the sake of convenience, we model electricity demand and fuel switch cost as continuous processes $(D(t))_{t \in [0,T]}$ and $(F(t))_{t \in [0,T]}$ with a time unit equal to one year. By sampling at a daily rate, we obtain discrete-time versions of the processes on which the numerical computations are performed. We explain below how the parameters for $(D(t))_{t \in [0,T]}$ and $(F(t))_{t \in [0,T]}$ are estimated by linear regression.

B.1. Demand Process

The continuous–time demand process is modeled by

$$D(t) = \min\{(P_D(t) + X_D(t))^+, \kappa^n + \kappa^c + \kappa^g\} \qquad t \in [0,T]$$

where κ^n, κ^c and κ^g are the production capacities from nuclear, coal and natural gas power plants respectively, and where the deterministic part

$$P_D(t) = a_D + b_D t + \sum_{j=0}^{6} c_j \cos(2\pi\varphi_j t + l_j) \qquad t \in [0,T] \tag{48}$$

accounts for a linear trend superimposed onto seasonal and weekly demand fluctuations. The stochastic part $(X(t))_{t \in [0,T]}$ is modeled by an Ornstein-Uhlenbeck process satisfying

$$dX_D(t) = \gamma_D(\alpha_D - X_D(t))dt + \sigma_D dW(t) \tag{49}$$

with parameters $\gamma_D, \alpha_D, \sigma_D \in \mathbb{R}$. The estimation of the parameters is based on historical load data for the time period April 2005 - April 2008 available on the website of the Korean Power Exchange. The parameters of the process were identified in two steps. First the deterministic harmonics in (48) are identified from peaks in the Fourier transform. Secondly, after removing the deterministic part $(P_D(t))_{t \in [0,T]}$ (red line in Figure B.3) the residual component $(X_D(t))_{t \in [0,T]}$ is estimated by the standard linear regression described in Subsection B.3. The resulting parameters for the stochastic part are

Stochastic Part $(X_D(t))_{t\in[0,T]}$		
γ_D	α_D	σ_D
258	0	1.434×10^6

those for the affine part of $(P_D(t))_{t\in[0,T]}$ read

Affine Part of $(P_D(t))_{t\in[0,T]}$	
a_D	b_D
971112	56165

and the 7 main oscillations of $(P_D(t))_{t\in[0,T]}$ are given by

Periodic Part of $(P_D(t))_{t\in[0,T]}$							
	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
c_i	46919	51179	24336	8134	63094	29422	9704
φ_i	1	2	3	4	52.14	104.29	208.57
l_i	1.17	2.29	−0.31	2.31	1.71	0.27	1.74

Long term periodicities were computed with a yearly periodic Fourier transform while short term periodicities were computed with a weekly periodic Fourier transform.

B.2. Fuel Switch Price Process

The continuous–time fuel switch process is modeled by

$$F(t) = a_F + b_F t + X_F(t) \qquad t \in [0,T]$$

where the stochastic part $(X_F(t))_{t\in[0,T]}$ is again modeled by an Ornstein-Uhlenbeck process whose evolution follows the stochastic differential equation

$$dX_F(t) = \gamma_F(\alpha_F - X_F(t))dt + \sigma_F dW(t) \tag{50}$$

with parameters $\gamma_F, \alpha_F, \sigma_F \in \mathbb{R}$. Here again, $(W(t))_{t\in[0,T]}$ is a Wiener process.

For the estimation of the parameters of the fuel spread appearing in the fuel switch cost, we used Asian LNG import prices from Bloomberg, and Asian coal prices from Argus Media Group, from January 2006 to July 2008. Taking into account expected long term gas and coal prices we fixed $a = 60\$$ neglecting the recent fuel switch price increase. As for the electricity demand process, the parameters of the stochastic component $(X_F(t))_{t\in[0,T]}$ were calibrated using the procedure described in Subsection B.3. We found the parameters:

Fuel Switch Price Process $(F(t))_{t\in[0,T]}$				
a_F	b_F	γ_F	α_F	σ_F
33.68	4.81	5.18	0	28.21

B.3. Linear Regression

The parameters $\gamma_i, \alpha_i, \sigma_i$ for $i \in \{D, F\}$ of the Ornstein-Uhlenbeck processes (49) and (50) are estimated by a standard linear regression method applied as follows. From the formulas for conditional mean and variance

$$\begin{aligned} \mathrm{E}(X(t)|\mathcal{F}_s) &= X(s)e^{-\gamma(t-s)} + \alpha(1-e^{-\gamma(t-s)}) \quad s \le t \\ \mathrm{Var}(X(t)|\mathcal{F}_s) &= \frac{\sigma^2}{2\gamma}(1-e^{-2\gamma(t-s)}) \quad s \le t \end{aligned}$$

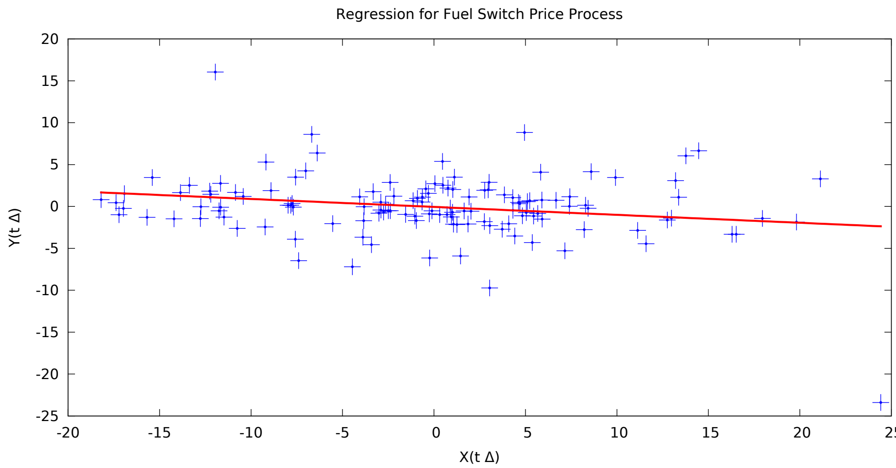

Figure 8 Scatterplot of $(X(t\Delta), Y(t\Delta))$ calculated by (51) based on historical fuel switch prices for the Korean electricity market. The straight line depicts the linear regression estimated by least squares.

we obtain the regression

$$Y(t\Delta) := X((t+1)\Delta) - X(t\Delta) = \beta_0 + \beta_1 X(t\Delta) + \beta_2 \epsilon_t \qquad t = 1, \dots, n-1 \tag{51}$$

where $(\epsilon_t)_{t=1}^{n-1}$ are independent, standard Gaussian random variables and $\beta_0, \beta_1, \beta_2$ are connected to α, γ, σ by

$$\alpha = -\frac{\beta_0}{\beta_1}, \qquad \gamma = -\frac{1}{\Delta}\ln(1+\beta_1), \qquad \sigma = \sqrt{\frac{2\gamma\beta_2}{1-e^{-2\gamma\Delta}}}.$$

Appendix C: Proof of Theorem 3

In this Section we prove Theorem 3. To this end we show that the existence of equilibrium strategies (θ^*, ξ^*) for the standard scheme with prices (A^*, S^*) guarantee existence of equilibrium strategies $(\theta'^*, \xi^*, \varphi^*)$ for the hybrid scheme with prices (A^*, S'^*, P^*) and vice versa. In fact we can prove this without knowing the specific form of the equilibrium strategies.

The rationale is as follows: If there exist strategies (θ^*, ξ^*) that are feasible and fulfill conditions (i)-(iii) of Definition 1 under (A^*, S^*) then the strategies $(\theta'^*, \xi^*, \varphi^*)$ related to (θ^*, ξ^*) by

$$\begin{aligned} \varphi^{*i} &= \Upsilon(\xi^*)/|I| \\ \theta'^{*i}_T &= \theta^{*i}_T + \Lambda^i_0 - \tilde{\Lambda}^i_0 - \Gamma(\xi^{*-i}) \wedge Y^i(\xi^{*i}) - \varphi^{*i} \\ \theta'^*_t &= \theta^*_t \text{ for all } t = 0, \dots, T-1 \end{aligned}$$

are feasible and fulfill conditions (i)-(iv) of Definition 3 under (A^*, S'^*, P^*) and vice versa. Notice that for the strategies so defined, the number of allowances surrendered for compliance at the end of the compliance period is the same in both the standard and the hybrid scheme.

Feasibility of the above strategies is based on the following Lemma:

Lemma 1. *Let A an integrable allowance price process and consider an agent $i \in I$ with auction strategy φ^i such that $\varphi^i A_{\mathfrak{T}}$ is integrable. Moreover fix $\xi \in \prod_{i\in I} \mathcal{U}^i$ and assume that the trading strategies θ^i and θ'^i are related by*

$$\theta^i_t = \theta'^i_t \text{ for all } t = 0, \dots, T-1 \tag{52}$$

and

$$\Lambda^i_0 + \theta^i_T = \tilde{\Lambda}^i_0 + \varphi^i + \theta'^i_T + \Gamma(\xi^{-i}) \wedge Y^i(\xi^i) \tag{53}$$

almost surely. Then, $(\theta^i, \xi^i) \in \mathcal{Q}^i(A)$ if and only if $(\theta'^i, \xi^i, \varphi^i) \in \tilde{\mathcal{Q}}^{\xi^{-i},i}(A, A_{\mathfrak{T}})$.

Proof of Lemma 1 Since θ^i and θ'^i differ only at time $t=T$, the equivalence between the integrability of $R_T^A(\theta^i)$ and $R_T^A(\theta'^i)$ relies on the equivalence between the integrability of $\theta_T^i A_T$ and $\theta_T'^i A_T$. But this is clear from (53) because $\varphi^i A_{\mathfrak{T}}$ is integrable and $\Gamma(\xi^{-i}) \wedge Y^i(\xi^i)$ is bounded. Considering now the inequality constraint, if we assume that

$$\theta_T^i \geq -\Lambda_0^i,$$

then both the left and right hand sides of (53) are positive which implies that

$$\theta_T'^i \geq -\left(\tilde{\Lambda}_0^i + \Gamma(\xi^{-i}) \wedge Y^i(\xi^i) + \varphi^i\right).$$

The converse statement is proven analogously. □

The following lemma shows that if (θ^*, ξ^*) fulfills conditions (i) and (ii) of Definition 1, then it follows that $(\theta'^*, \xi^*, \varphi^*)$ fulfills conditions (i), (ii) and (iii) of Definition 3 and vice versa.

Lemma 2. *Let (A, S) be integrable allowance and product price processes. Moreover, suppose that for all agents $i \in I$ the strategies $(\theta^i, \xi^i)_{i\in I} \in \mathcal{Q}(A)$ and $(\theta'^i, \xi^i, \varphi^i)_{i\in I} \in \tilde{\mathcal{Q}}(A, A_{\mathfrak{T}})$ satisfy conditions (52), (53) and*

$$\sum_{i\in I} \varphi^i = \Upsilon(\xi) \tag{54}$$

almost surely. Then it holds that the couple $(\theta^i, \xi^i)_{i\in I}$ fulfills conditions (i) and (ii) of Definition 1 if and only if $(\theta'^i, \xi^i, \varphi^i)_{i\in I}$ fulfill conditions (i) and (iii) of Definition 3.

Proof of Lemma 2 Condition (54) implies that

$$\sum_{i\in I} \varphi^i = \Upsilon - \sum_{i\in I}(\Gamma(\xi^{-i}) \wedge Y^i(\xi^i)).$$

Summing the left and right hand side of (53) this yields

$$\sum_{i\in I}\Lambda_0^i + \sum_{i\in I}\theta_T^i = \sum_{i\in I}\tilde{\Lambda}_0^i + \sum_{i\in I}\Gamma(\xi^{-i}) \wedge Y^i(\xi^i) + \Upsilon - \sum_{i\in I}\Gamma(\xi^{-i}) \wedge Y^i(\xi^i) + \sum_{i\in I}\theta_T'^i,$$

while (28) implies that

$$\sum_{i\in I}\theta_T^i = \sum_{i\in I}\theta_T'^i.$$

Hence if $(\theta^i, \xi^i)_{i\in I}$ satisfies condition (i) of Definition 1 then $(\theta'^i, \xi^i, \varphi^i)_{i\in I}$ fulfills conditions (i) of Definition 3 and vice versa. If $(\theta^i, \xi^i)_{i\in I}$ satisfy condition (ii) of Definition 1 then $(\theta'^i, \xi^i, \varphi^i)_{i\in I}$ fulfill conditions (iii) of Definition 3 and vice versa. □

Until now we have shown that all the equilibrium conditions but the individual optimality conditions (condition (iii) of Definition 1 and condition (iv) of Definition 3) are fulfilled. It remains to prove the following assertion: if (ξ^*, θ^*) fulfills

$$\mathbb{E}[L^{A^*,S^*,i}(\theta^{*i}, \xi^{*i})] - \mathbb{E}[L^{A^*,S^*,i}(\theta^i, \xi^i)] \geq 0 \tag{55}$$

for all $(\theta^i, \xi^i) \in \mathcal{Q}^i(A^*)$ then $(\theta'^*, \xi^*, \varphi^*)$ fulfills

$$\mathbb{E}[H^{A^*,S'^*,P^*,\xi^{*-i},i}(\theta'^{*i}, \xi^{*i}, \varphi^{*i})] - \mathbb{E}[H^{A^*,S'^*,P^*,\xi^{*-i},i}(\theta^i, \xi^i, \varphi^i)] \geq 0 \tag{56}$$

for all $(\theta^i, \xi^i, \varphi^i) \in \tilde{\mathcal{Q}}^i(A^*, A^*)$ and vice versa.

A simple concavity argument gives us lower bounds for which it is easier to prove non-negativity than for (55) and (56). A family of lower bounds is given in the following lemma:

LEMMA 3. *Let (A,S) be integrable price processes and P an auction price. Moreover, fix an agent $i \in I$ and suppose that the strategies $\xi^{*i}, \xi^i \in \mathcal{U}^i$ and θ^{*i}, θ^i, φ^{*i}, φ^i are such that $\varphi^{*i}P$, $\varphi^i P$, $R_T^A(\theta^{*i})$ and $R_T^A(\theta^i)$ are integrable and condiser the following linear combinations*

$$\xi^i(\mu) := \xi^{*i} + \mu(\xi^i - \xi^{*i}) \qquad \theta^i(\mu) := \theta^{*i} + \mu(\theta^i - \theta^{*i}) \qquad \varphi^i(\mu) := \varphi^{*i} + \mu(\varphi^i - \varphi^{*i})$$

for $0 < \mu \leq 1$. If in addition Assumption 1 is satisfied, then it holds that

$$\mathbb{E}\Big[L^{A,S,i}(\theta^{*i}, \xi^{*i})\Big] - \mathbb{E}\Big[L^{A,S,i}(\theta^i, \xi^i)\Big] \geq \frac{1}{\mu}\mathbb{E}\Big[L^{A,S,i}(\theta^{*i}, \xi^{*i}) - L^{A,S,i}(\theta^i(\mu), \xi^i(\mu))\Big]$$

in case of the standard scheme as well as

$$\begin{aligned}\mathbb{E}\Big[H^{A,S,P,\xi^{*-i},i}(\theta^{*i}, \xi^{*i}, \varphi^{*i})\Big] - \mathbb{E}\Big[H^{A,S,P,\xi^{*-i},i}(\theta^i, \xi^i, \varphi^i)\Big] \\ \geq \frac{1}{\mu}\mathbb{E}\Big[H^{A,S,P,\xi^{*-i},i}(\theta^{*i}, \xi^{*i}, \varphi^{*i}) - H^{A,S,P,\xi^{*-i},i}(\theta^i(\mu), \xi^i(\mu), \varphi^i(\mu))\Big]\end{aligned}$$

in case of the hybrid scheme.

Proof of Lemma 3 From the concavity of L we conclude for all $0 < \mu \leq 1$ that

$$\mathbb{E}[L^{A,S,i}(\theta^i(\mu), \xi^i(\mu))] \geq (1-\mu)\mathbb{E}[L^{A,S,i}(\theta^{*i}, \xi^{*i})] + \mu\mathbb{E}[L^{A,S,i}(\theta^i, \xi^i)]$$

which translates into

$$\mathbb{E}[L^{A,S,i}(\theta^i(\mu), \xi^i(\mu))] - \mathbb{E}[L^{A,S,i}(\theta^{*i}, \xi^{*i})] \geq \mu\big(\mathbb{E}[L^{A,S,i}(\theta^i, \xi^i)] - \mathbb{E}[L^{A,S,i}(\theta^{*i}, \xi^{*i})]\big)$$

and proves the assertion. A similar argument holds for the hybrid scheme. □

For each agent $i \in I$, the following lemma will help us to express the lower bounds from Lemma 3 for the standard/hybrid scheme in terms of the objective function of the hybrid/standard scheme and the function $\mathcal{U}^i \ni \xi^i \hookrightarrow K^{A,\xi^{-i},i}(\xi^i)$ defined as

$$K^{A,\xi^{-i},i}(\xi^i) = -Y^i(\xi^i)A_{\mathfrak{T}}\mathbf{1}_{\{Y(D)<\Upsilon\}} + \Big(Y^i(\xi^i) \wedge \Gamma(\xi^{-i})\Big)A_{\mathfrak{T}}$$

for all processes A and ξ^{-i}. For small perturbations of the equilibrium strategies, this gives a simple relation between the objective functions in the standard and the hybrid schemes since Lemma 5 shows that the terms originating from K vanish as μ converges to zero.

LEMMA 4. *Let A, S, S' be integrable price processes such that*

$$S_t^{'k} = S_t^k - y^k\mathbb{E}[A_{\mathfrak{T}}\mathbf{1}_{\{Y(D)<\Upsilon\}}|\mathcal{F}_t]\mathbf{1}_{\{t\leq\mathfrak{T}-1\}} \tag{57}$$

holds almost surely for all $k \in K$, $t = 0, \ldots, T-1$ and suppose that P is an integrable auction price. Moreover fix an agent $i \in I$ as well as other agents strategies $\xi^{-i} \in \prod_{i' \in I\setminus\{i\}} \mathcal{U}^i$ and suppose that the strategies $(\theta^i, \xi^i) \in \mathcal{Q}^i(A)$ and $(\theta'^i, \xi^i, \varphi^i) \in \tilde{\mathcal{Q}}^{\xi^{-i},i}(A,P)$ satisfy the conditions (52), (53) as well as $\varphi^i(P - A_{\mathfrak{T}}) = 0$ almost surely. Then it holds under Assumption 1 that

$$\mathbb{E}\Big[H^{A,S',P,\xi^{-i},i}(\theta'^i, \xi^i, \varphi^i)\Big] = \mathbb{E}\Big[L^{A,S,i}(\theta^i, \xi^i) - (\Lambda_0^i - \tilde{\Lambda}_0^i)A_T + K^{A,\xi^{-i},i}(\xi^i)\Big].$$

Proof of Lemma 4 Due to (52) and (53) it holds that

$$\begin{aligned}
&\mathbb{E}\Big[H^{A,S',P,\xi^{-i},i}(\theta'^i,\xi^i,\varphi^i)\Big]\\
&=\mathbb{E}\Big[\sum_{t=0}^{T-1}\sum_{(j,k)\in M_i}(S'^k_t-C^{i,j,k}_t)\xi^{i,j,k}_t+\sum_{t=0}^{T-1}\theta'^i_t(A_{t+1}-A_t)-\theta'^i_T A_T-\varphi^i P\\
&\quad-\pi\Big(\Delta^i+\Pi^i(\xi^i)-\tilde\Lambda^i_0-\Gamma(\xi^{-i})\wedge Y^i(\xi^i)-\varphi^i-\theta'^i_T\Big)^+\Big]\\
&=\mathbb{E}\Big[\sum_{t=0}^{T-1}\sum_{(j,k)\in M_i}(S^k_t-C^{i,j,k}_t-1_{\{t\le\mathfrak{T}-1\}}y^k\mathbb{E}\left[A_{\mathfrak{T}}\mathbf{1}_{\{Y(D)<\Upsilon\}}|\mathcal{F}_t\right])\xi^{i,j,k}_t\\
&\quad+\sum_{t=0}^{T-1}\theta^i_t(A_{t+1}-A_t)-\theta^i_T A_T-(\Lambda^i_0-\tilde\Lambda^i_0)A_T-\varphi^i(P-A_{\mathfrak{T}})\\
&\quad+\big(\Gamma(\xi^{-i})\wedge Y^i(\xi^i)\big)A_{\mathfrak{T}}-\pi\left(\Delta^i+\Pi^i(\xi^i)-\Lambda^i_0-\theta^i_T\right)^+\Big].
\end{aligned}$$

With (26) this reads

$$\begin{aligned}
&\mathbb{E}\Big[\sum_{t=0}^{T-1}\sum_{(j,k)\in M_i}(S^k_t-C^{i,j,k}_t)\xi^{i,j,k}_t+\sum_{t=0}^{T-1}\theta^i_t(A_{t+1}-A_t)-\theta^i_T A_T-(\Lambda^i_0-\tilde\Lambda^i_0)A_T\\
&\quad-\varphi^i(P-A_{\mathfrak{T}})-Y^i(\xi^i)A_{\mathfrak{T}}\mathbf{1}_{\{Y(D)<\Upsilon\}}+\big(\Gamma(\xi^{-i})\wedge Y^i(\xi^i)\big)A_{\mathfrak{T}}\\
&\quad-\pi\left(\Delta^i+\Pi^i(\xi^i)-\Lambda^i_0-\theta^i_T\right)^+\Big]\\
&=\mathbb{E}\Big[L^{A,S,i}(\theta^i,\xi^i)-(\Lambda^i_0-\tilde\Lambda^i_0)A_T+K^{A,\xi^{-i},i}(\xi^i)\Big]
\end{aligned}$$

which concludes the proof. □

Using the results from Lemma 4 to rewrite the lower bounds from Lemma 3 the terms originating from K vanish as μ converges to zero. This is shown in the following Lemma.

LEMMA 5. *Let $\xi^*\in\mathcal{U}$ and $\xi^i\in\mathcal{U}^i$ for some $i\in I$. Moreover assume that Assumption 3 is fulfilled and define $\xi^i(\mu):=\xi^{*i}+\mu(\xi^i-\xi^{*i})$ for all $0<\mu\le 1$. Then taking the limit $\mu\searrow 0$ along a countable set $(0,1]\cap\mathbb{Q}$ it holds that*

$$\lim_{\mu\to 0}\frac{1}{\mu}\mathbb{E}\big[K^{A,\xi^{*-i},i}(\xi^{*i})-K^{A,\xi^{*-i},i}(\xi^i(\mu))\big]=0$$

for all integrable allowance price processes A.

Proof of Lemma 5 Since $\xi^*\in\mathcal{U}$ it holds on $\{Y(D)<\Upsilon\}$ that $Y^i(\xi^{*i})<\Gamma(\xi^{*-i})$ and hence

$$\lim_{\mu\to 0}\frac{1}{\mu}\Big(Y^i(\xi^{*i})\wedge\Gamma(\xi^{*-i})-Y^i(\xi^i(\mu))\wedge\Gamma(\xi^{*-i})\Big)=Y^i(\xi^{*i})-Y^i(\xi^i).$$

On the other hand it holds on $\{Y(D)>\Upsilon\}$ that $Y^i(\xi^{*i})>\Gamma(\xi^{*-i})$ and therefore

$$\lim_{\mu\to 0}\frac{1}{\mu}\Big(Y^i(\xi^{*i})\wedge\Gamma(\xi^{*-i})-Y^i(\xi^i(\mu))\wedge\Gamma(\xi^{*-i})\Big)=0.$$

Since moreover Assumption 3 implies $\mathbb{P}[\{Y(D)=\Upsilon\}]=0$ we conclude that

$$\begin{aligned}\lim_{\mu\to 0}\ &\frac{1}{\mu}\Big(Y^i(\xi^{*i})\wedge\Gamma(\xi^{*-i})-Y^i(\xi^i(\mu))\wedge\Gamma(\xi^{*-i})\Big)\\ &=\big(Y^i(\xi^{*i})-Y^i(\xi^i)\big)\mathbf{1}_{\{Y(D)<\Upsilon\}}=\frac{1}{\mu}\Big(Y^i(\xi^{*i})-Y^i(\xi^i(\mu))\Big)\mathbf{1}_{\{Y(D)<\Upsilon\}}\end{aligned}$$

holds almost surely. Hence it follows that

$$\lim_{\mu\to 0}\frac{1}{\mu}\Big(K^{A,\xi^{*-i},i}(\xi^{*i})-K^{A,\xi^{*-i},i}(\xi^i(\mu))\Big)=0$$

almost surely. Moreover it holds by the triangle inequality for all $0<\mu\leq 1$ that

$$\begin{aligned}&\frac{1}{\mu}\Big|K^{A,\xi^{*-i},i}(\xi^{*i})-K^{A,\xi^{*-i},i}(\xi^i(\mu))\Big|\\ &\leq\frac{1}{\mu}\Big|\Big(Y^i(\xi^{*i})-Y^i(\xi^i(\mu))\Big)A_{\mathfrak{T}}\mathbf{1}_{\{Y(D)<\Upsilon\}}\Big|+\frac{1}{\mu}\Big|\Big(Y^i(\xi^{*i})\wedge\Gamma(\xi^{*-i})-Y^i(\xi^i(\mu))\wedge\Gamma(\xi^{*-i})\Big)A_{\mathfrak{T}}\Big|\\ &\leq\frac{2}{\mu}\Big|\Big(Y^i(\xi^{*i})-Y^i(\xi^i(\mu))\Big)A_{\mathfrak{T}}\Big|=\frac{2}{\mu}\Big|\sum_{k\in K}\sum_{j\in J^{i,k}}\sum_{t=0}^{\mathfrak{T}-1}y^k\mu(\xi_t^{*i,j,k}-\xi_t^{i,j,k})A_{\mathfrak{T}}\Big|\\ &\leq 2\Big|\sum_{k\in K}\sum_{j\in J^{i,k}}\sum_{t=0}^{\mathfrak{T}-1}y^k\kappa^{i,j,k}A_{\mathfrak{T}}\Big|\end{aligned}$$

while $A_{\mathfrak{T}}$ is integrable by assumption. Hence the assertion follows from dominated convergence. □

We can now turn to the proof of Theorem 3.

Proof of Theorem 3 Assume first that (A^*,S^*) is an equilibrium of the standard scheme with strategies $(\theta^*,\xi^*)\in\mathcal{Q}(A^*)$.

Based on these strategies we define optimal strategies $(\theta'^*,\xi^*,\varphi^*)$ for the hybrid scheme, where for all $i\in I$ the strategies φ^{*i} and $\theta^{'*i}$ are given by

$$\varphi^{*i}=\Upsilon(\xi^*)/|I| \tag{58}$$
$$\theta_T^{'*i}=\theta_T^{*i}+\Lambda_0^i-\tilde{\Lambda}_0^i-\Gamma(\xi^{*-i})\wedge Y^i(\xi^{*i})-\varphi^{*i} \tag{59}$$

and $\theta_t'^*=\theta_t^*$ for all $t=0,\dots,T-1$. Since $\varphi^{*i}A_{\mathfrak{T}}$ is integrable and φ^{*i} is almost surely non negative for all $i\in I$ we deduce from Lemma 1 that $(\theta'^{*i},\xi^{*i},\varphi^{*i})\in\tilde{\mathcal{Q}}^{\xi^{*-i},i}(A^*,A^*_{\mathfrak{T}})$. Further $(\varphi^{*i})_{i\in I}$ so defined fulfills condition (ii) of Definition 3. This together with Lemma 2 implies that $(\theta^{'*i},\xi^{*i},\varphi^{*i})_{i\in I}$ fulfill conditions (i) and (iii) of Definition 3. To show that $(A^*,S^{'*},P^*)$ is an equilibrium of the hybrid scheme it remains to prove that condition (iv) of Definition 3 is satisfied for any $(\theta^{'i},\xi^i,\varphi^i)\in\tilde{\mathcal{Q}}^{\xi^{*-i},i}(A^*,A^*_{\mathfrak{T}})$. To this end we define linear combinations

$$\xi^i(\mu):=\xi^{*i}+\mu(\xi^i-\xi^{*i})\qquad\theta'^i(\mu):=\theta^{*i}+\mu(\theta'^i-\theta^{*i})\qquad\varphi^i(\mu):=\varphi^{*i}+\mu(\varphi^i-\varphi^{*i}).$$

Based on these strategies we define also a trading strategy $\theta^i(\mu)$ for the standard scheme given by

$$\theta_T^i(\mu)\ :=\ -\Lambda_0^i+\tilde{\Lambda}_0^i+\Gamma(\xi^{*-i})\wedge Y^i(\xi^i(\mu))+\varphi^i(\mu)+\theta_T^{'i}(\mu)$$

and $\theta_t^i(\mu):=\theta_t^{'i}(\mu)$ for all $t=0,\dots,T-1$.

Note that since θ_T^{*i} and $\theta_T'^i$ are feasible they satisfy

$$\begin{aligned}\theta_T'^i(\mu)\ &=\ (1-\mu)\theta_T^{*i}+\mu\theta_T'^i\\ &\geq\ -\tilde{\Lambda}_0^i-(1-\mu)(\Gamma(\xi^{*-i})\wedge Y^i(\xi^{*i})+\varphi^{*i})-\mu(\Gamma(\xi^{*-i})\wedge Y^i(\xi^i)+\varphi^i).\end{aligned}$$

Since moreover $\xi^i \hookrightarrow \Gamma(\xi^{*-i}) \wedge Y^i(\xi^i)$ is concave this implies for all $0 \le \mu \le 1$ that

$$\theta'^i_T(\mu) \ge -\tilde{\Lambda}^i_0 - \Gamma(\xi^{*-i}) \wedge Y^i(\xi^i(\mu)) - \varphi^i(\mu)$$

meaning that $(\theta'^i(\mu), \xi^i(\mu), \varphi^i(\mu)) \in \tilde{\mathcal{Q}}^{\xi^{*-i},i}(A^*, A^*_{\mathfrak{T}})$. This together with Lemma 1 implies that $(\theta^i(\mu), \xi^i(\mu)) \in \mathcal{Q}^i(A^*)$. Knowing this we are ready to prove condition (iv) of Definition 3. From Lemmas 3 and 4 we derive that

$$\begin{aligned}
&\mathbb{E}\Big[H^{A^*,S'^*,P^*,\xi^{*-i},i}(\theta'^{*i},\xi^{*i},\varphi^{*i})\Big] - \mathbb{E}\Big[H^{A^*,S'^*,P^*,\xi^{*-i},i}(\theta'^i,\xi^i,\varphi^i)\Big] \\
&\ge \frac{1}{\mu}\mathbb{E}\Big[H^{A^*,S'^*,P^*,\xi^{*-i},i}(\theta'^{*i},\xi^{*i},\varphi^{*i}) - H^{A^*,S'^*,P^*,\xi^{*-i},i}(\theta'^i(\mu),\xi^i(\mu),\varphi^i(\mu))\Big] \\
&= \frac{1}{\mu}\mathbb{E}\Big[L^{A^*,S^*,i}(\theta^{*i},\xi^{*i}) + K^{A^*,\xi^{*-i},i}(\xi^{*i}) - L^{A^*,S^*,i}(\theta^i(\mu),\xi^i(\mu)) - K^{A^*,\xi^{*-i},i}(\xi^i(\mu))\Big] \\
&\ge \frac{1}{\mu}\mathbb{E}\Big[K^{A^*,\xi^{*-i},i}(\xi^{*i}) - K^{A^*,\xi^{*-i},i}(\xi^i(\mu))\Big] \qquad (60)
\end{aligned}$$

for all $0 < \mu \le 1$, where the last inequality holds due to the optimality of (θ^{*i}, ξ^{*i}) for the standard scheme. Hence we conclude from Lemma 5 that

$$\begin{aligned}
&\mathbb{E}\Big[H^{A^*,S'^*,P^*,\xi^{*-i},i}(\theta'^{*i},\xi^{*i},\varphi^{*i})\Big] - \mathbb{E}\Big[H^{A^*,S'^*,P^*,\xi^{*-i},i}(\theta'^i,\xi^i,\varphi^i)\Big] \\
&\ge \lim_{\mu\to 0}\frac{1}{\mu}\mathbb{E}\Big[K^{A^*,\xi^{*-i},i}(\xi^{*i}) - K^{A^*,\xi^{*-i},i}(\xi^i(\mu))\Big] = 0 \qquad (61)
\end{aligned}$$

where we take the limit $\mu \searrow 0$ along a countable set $(0,1] \cap \mathbb{Q}$. This argumentation holds for any $(\theta'^i, \xi^i, \varphi^i) \in \tilde{\mathcal{Q}}^{\xi^{*-i},i}(A^*, A^*_{\mathfrak{T}})$ proving condition (iv) of Definition 3.

To prove the converse statement assume that (A^*, S'^*, P^*) is an equilibrium of the hybrid scheme with corresponding strategies $(\theta'^{*i}, \xi^{*i}, \varphi^{*i}) \in \tilde{\mathcal{Q}}^{\xi^{*-i},i}(A^*, P^*)$. For condition (iii) of Definition 3 to be fulfilled it is necessary that $\varphi^{*i}(P^* - A^*_{\mathfrak{T}}) = 0$ almost surely for all agents $i \in I$.

Based on the optimal strategies in the hybrid scheme we define optimal strategies (θ^*, ξ^*) for the standard scheme where for all $i \in I$ the trading strategy θ^{*i} is given by

$$\theta^{*i}_T := \theta'^{*i}_T - \Lambda^i_0 + \tilde{\Lambda}^i_0 + \Gamma(\xi^{*-i}) \wedge Y^i(\xi^{*i}) + \varphi^{*i}$$

and $\theta^{*i}_t := \theta'^{*i}_t$ for all $t = 0, \ldots, T-1$. Since $\varphi^{*i}(P^* - A^*_{\mathfrak{T}}) = 0$ and $(\theta'^{*i}, \xi^{*i}, \varphi^{*i}) \in \tilde{\mathcal{Q}}^{\xi^{*-i},i}(A^*, P^*)$ it holds that for all that $\varphi^{*i}A^*_{\mathfrak{T}}$ is integrable, hence Lemma 1 implies that $(\theta^*, \xi^*) \in \mathcal{Q}(A^*)$. Moreover Lemma 2 implies that $(\theta^{*i}, \xi^{*i})_{i\in I}$ fulfill conditions (i) and (ii) of Definition 1.

It remains to prove that condition (iii) of the equilibrium definition for standard schemes is fulfilled. Therefore, we compare the strategy (θ^*, ξ^*) with all other strategies $(\theta^i, \xi^i) \in \mathcal{Q}^i(A^*)$ and $i \in I$ and define linear combinations by

$$\begin{aligned}
\xi^i(\mu) &:= \xi^{*i} + \mu(\xi^i - \xi^{*i}) \\
\theta^i(\mu) &:= \theta^{*i} + \mu(\theta^i - \theta^{*i}) \\
\theta'^i_T(\mu) &:= \Lambda^i_0 - \tilde{\Lambda}^i_0 - \Gamma(\xi^{*-i}) \wedge Y^i(\xi^i(\mu)) - \varphi^{*i} + \theta^i_T(\mu). \\
\theta'^i_t(\mu) &:= \theta^i_t(\mu) \quad \text{ for all } t = 0, \ldots, T-1.
\end{aligned}$$

Again it holds that $(\theta^i(\mu), \xi^i(\mu)) \in \mathcal{Q}^i(A^*)$ and hence $(\theta'^i(\mu), \xi^i(\mu), \varphi^{*i}) \in \tilde{\mathcal{Q}}^{\xi^{*-i},i}(A^*, A^*_{\mathfrak{T}})$ by Lemma 1. Moreover it holds for any $(\theta^i, \xi^i) \in \mathcal{Q}^i(A^*)$ and $0 < \mu$ that

$$\mathbb{E}[L^{A^*,S^*,i}(\theta^{*i},\xi^{*i})] - \mathbb{E}[L^{A^*,S^*,i}(\theta^i,\xi^i)]$$

$$
\begin{aligned}
&\geq \frac{1}{\mu}\mathbb{E}[L^{A^*,S^*,i}(\theta^{*i},\xi^{*i}) - L^{A^*,S^*,i}(\theta^i(\mu),\xi^i(\mu))] \\
&= \frac{1}{\mu}\mathbb{E}[(H^{A^*,S'^*,P^*,\xi^{*-i},i}(\theta'^{*i},\xi^{*i},\varphi^{*i}) - K^{A^*,\xi^{*-i},i}(\xi^{*i})) \\
&\qquad - (H^{A^*,S'^*,P^*,\xi^{*-i},i}(\theta'^{i}(\mu),\xi^i(\mu),\varphi^{*i}) - K^{A^*,\xi^{*-i},i}(\xi^i(\mu)))] \\
&\geq \frac{1}{\mu}\mathbb{E}[K^{A^*,\xi^{*-i},i}(\xi^i(\mu)) - K^{A^*,\xi^{*-i},i}(\xi^{*i}))]
\end{aligned}
$$

where the first and second equality are consequences of Lemma 3 and 4 respectively. The last inequality holds due to the optimality of $(\theta^{*i},\xi^{*i},\varphi^{*i})$ for the hybrid scheme. Hence taking the limit $\mu \searrow 0$ along a countable set $(0,1]\cap\mathbb{Q}$ we derive from Lemma 5 that

$$
\begin{aligned}
&\mathbb{E}[L^{A^*,S^*,i}(\theta^{*i},\xi^{*i})] - \mathbb{E}[L^{A^*,S^*,i}(\theta^i,\xi^i)] \\
&\geq \lim_{\mu\to 0}\frac{1}{\mu}\mathbb{E}[K^{A^*,\xi^{*-i},i}(\xi^i(\mu)) - K^{A^*,\xi^{*-i},i}(\xi^{*i}))] = 0
\end{aligned}
$$

holds for any $(\theta^i,\xi^i)\in\mathcal{Q}^i(A^*)$. This proves condition (iii) of Definition 1 and concludes the proof. □

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