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Predatory Trading: a Game on Volatility and Liquidity

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We propose and analyze a continuous time stochastic differential game as a mathematical model for the liquidation of a financial position by a distressed trader facing a predator trading with looser time constraints. Temporary and permanent price impacts are used to model liquidity effects. We construct Nash equilibria by solving a system of coupled degenerate HJB equations and analyze the results numerically. The main thrust of the paper is to exhibit the impact of aggregate noise traders in the market, showing among other things that higher price volatility results in an increase of the predator's trading activity. We also identify market conditions under which the presence of the predator is beneficial to the distressed trader.

1. Introduction and Motivation

Liquidity risk has recently drawn increasing attention in the finance and financial mathematics literature. Characterizing the liquidity of a financial market is a complex task, and so far no clear consensus has emerged as how to define it. Liquidity should characterize the ease of trading a security (5), however quite often, a quantitative characterization of liquidity, or lack thereof, features the degree to which the buying or selling of an asset may affect the market price of the security. In classical asset pricing theory, e.g. (10), (20), or (14), market models are assumed to have perfect liquidity in the sense that the seller (resp. buyer) can sell (resp.buy) an asset whenever she wants, and the market price of the asset is not affected by the size or the direction of the transaction. Clearly, this assumption is highly unrealistic.

Financial crises, e.g. the Black Monday in 1987, the Long Term Capital Management (LTCM) collapse following the Russian government's default, and most recently the credit derivatives crunch during the summer of 2007 and the global crisis following Lehman Brothers bankruptcy, have highlighted the importance of studying and understanding liquidity in all of its forms, both for individual institutions and for central banks and regulators. Several academic and industry research reports (25, 29, 12, 17) pointed out that both the sudden dry-up of market-liquidity

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and the instability of the liquidity structure of a market, exacerbate the impact of the nitial crisis triggers.

In the finance literature, four sources of illiquidity have been investigated. The first and simplest one is a *friction-like* illiquidity, which includes brokerage fees, order-processing costs and transaction taxes. Important works in this area include (5, 4), as wel as (18, 6). Another track in liquidity research, represented by (12), is on the funding problem of the traders and the mutual interaction between the margin settings by the financiers and the funding of traders' collateralized positions. Thirdly, we mention the literature on financial market microstructure defined as the study of the process of exchanging assets under explicit trading rules, aiming at deciphering how specific trading mechanisms affect the price formation process, as highlighted in the classic overview (26), or (15) which surveys most NBER papers on liquidity and asset pricing, and (21). The market impact models we adopt in this paper enjoy support from the market microstructure research community. The fourth, and to us the most engaging direction of research, emphasizes the strategic interplay between an oligopoly of traders with large positions or trading capacities during an adverse period of illiquidity. When an institutional trader under financial distress is compelled to liquidate an unusually large position, other strategic traders may take advantage of the liquidation constraints and reap extra profits from the situation. Such a phenomenon is called *predatory trading* (11). A famous example of financial crisis led by predatory trading is the 1998 LTCM debacle in the foreign government bonds trading business. In fact, the loss induced by the default of Russian government was only a relatively small proportion to LTCM's whole business, but it was the explicit compulsion for LTCM to liquidate its positions on European and Asian bonds that elicited the gory predatory trading by its peers. Many trading partners of LTCM immediately recognized the intention of the troubled giant in foreign government bonds, and relentlessly carried on predatory strategies (11, 24). A more recent instance of predatory trading is the collapse of the hedge fund Amaranth after his star trader accumulated large positions in natural gas futures which the fund was unable to liquidate at reasonable price because of the awareness of the size of the positions and the need for the liquidation by predators.

The economic model of predatory trading is as follows. Due to some exogenous reasons, a certain 'distressed' trader is forced to generate cash from selling off another risky asset within a limited amount of time. Other solvent traders, who observe the liquidation need of the troubled victim, can tactically design trading strategies and make a profit from the price movement caused by the compulsory liquidation of the former. Essentially, this can be modeled as a game between these strategic players, each one optimizes his own payoff, yet also takes into account the optimal response of his opponent. To closely capture the reality of the trading industry, we wish to allow the agents to be able to trade in continuous-time. Hence, the optimal response of each agent, in terms of his instantaneous trading intensity, is a process in continuous-time. Each agent's decision-making problem, taking the strategy of his opponents as given, is then an optimal control problem and the overall evolution of the economy appears as a stochastic differential game, see (8) and (19). The main goal of the present study is to identify a Nash equilibrium in which each side optimizes his payoff within his admissible strategies and has no extra incentive to deviate, and compute comparative statics which reveal how the various market parameters influence the outcome of the predation game.

The first systematic studies of predatory trading were initiated in (11) and (7), and further developed in (13) and (27). Except possibly for an informal discussion in (13), all the previous studies focused on the search for open-loop equilibria whereby the participants are not allowed to get feedback during the game, nor update their responses. In these studies, only the mean price process matters, and the volatility component in the price dynamics disappears from the analysis after expectations are taken: the strategic players do not need to care about the influence of the noise traders in the market, and predation becomes a deterministic game between the strategic players. In the model studied in this paper, we allow each player to use closed-loop strategies and continuously observe the market price. As pointed out in (30), a closed-loop equilibrium structure has advantages over an open-loop equilibrium structure: it is always consistent across diffeomorphic state-space representations besides being subgame-perfect, whereas an open-loop equilibrium structure is not guaranteed to have either of these properties. Moreover, for modeling predatory trading per se, adopting a closed-loop structure incorporates feedbacks of the market fluctuations, and lets the noise trader aggregate play a non-trivial role in equilibrium. In addition to market plasticity and market elasticity that quantify the two main forms of market price impact, and are readily recognized in the previous studies (13, 27), our model acknowledges volatility, a measure closely associated with the trading activities of the noise traders at large, as the third market parameter relevant to the interplay of predatory trading.

The paper is organized as follows. In Section 2, we introduce the standard model for temporary and permanent price impacts which we use to capture illiquidity of the market. The two linearity assumptions made throughout the paper are justified by the results of prior empirical studies and *folk* theorems on the form of the price impact functions. We provide rigorous arguments justifying these assumptions in an appendix at the end of the paper. The non-zero sum stochastic differential game used as a model for our analysis of predatory trading is described in Section 3, while numerical results and discussions of the implications of the properties of the Nash equilibrium are given in Section 4.

2. Market Impact Modeling

In order to analyze strategic interactions between rational market participants, we first introduce a quantitative model for the various forms of market impact induced by trading activity. We argue that in continuous-time, if the market impact is time-homogeneous, then its permanent component should be linear in the speed of trading in order to avoid diversifiable statistical arbitrage opportunities. We give precise definitions and rigorous statements in what follows. These results are essentially known, even if their proofs are often given under different conditions. But since they are part of the folklore, we only give proofs in appendix for the sake of completeness.

Consider a security with imperfect liquidity, an exchange-market where this security is traded, and a rational trader who actively participates in this market. Without loss of generality, we normalize the interest rate to 0. Let $X(t)$ denote the position in this security of the trader at time $t \in \mathbb{R}_+$, and $\xi(t)$ denote the speed of trading

$$X(t) = X(0) + \int_0^t \xi(s)ds.$$

Although the position of the trader in the security is a piecewise constant function of time, an absolutely-continuous function can offer a reasonable approximation as Figure 1 shows. Hence we assume that the process $X(t)$ is differentiable at almost every t in $\mathbb{R}_+$, and $\xi(t)$ is its derivative.

We denote by $Z(t)$ the market quote price of this security at time $t \in \mathbb{R}_+$. When there is a bid-ask gap, $Z(t)$ indicates the mid-quote. The cumulative cost to the trader is $\int_0^t \tilde{Z}(s)\xi(s)ds$, where $\tilde{Z}(t)$ is the actual transaction price. Note that since the market is not perfectly liquid, $\tilde{Z}(t)$ is not always the same as $Z(t)$. The difference $\tilde{Z} - Z$ is denoted by g, and is called the *temporary price impact*. We suppose that g depends upon the speed of trading ξ at that instant in the sense that $\tilde{Z}(t) - Z(t) = g(\xi(t))$. Intuitively, the function g captures qualitatively how the limit order books available in the market are eaten up at different levels of trading intensities.

The price distortion g is the *temporary component* of market impact as it disappears imme-

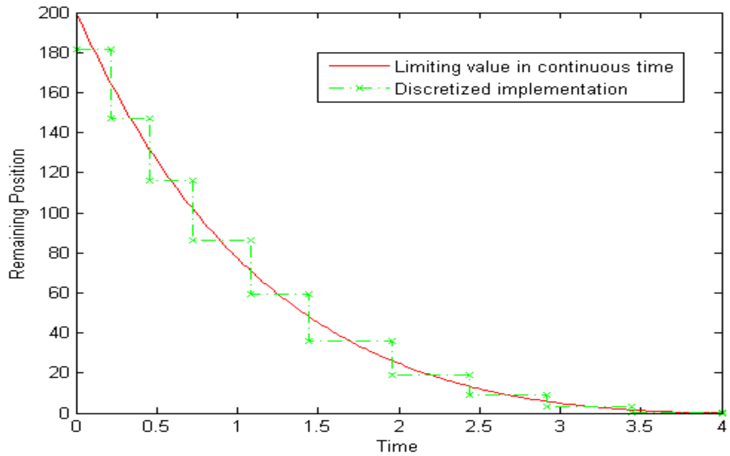

Figure 1. Absolutely continuous approximation of a real-time piecewise constant holding. We determine the optimal trading trajectory in continuous-time as a guide for executing the trades in tiny slices.

diately when trading activity ceases. The *permanent component* of the market impact is given by a function f of the trading speed $\xi(t)$ of the agent, and appears in the time evolution of the quote in the form:

$$dZ(t) = f\left(\xi(t)\right)dt + \mu(t)dt + \sigma\left(t, Z(t)\right)dW(t) \tag{1}$$

in addition to the usual drift term $\mu(t)dt$. The specific form $\sigma(t,z)$ of the volatility is irrelevant at this stage. Notice that the permanent impact is also time homogeneous in the sense that f is independent of t. We will also assume that $f(0) = 0$, i.e. the impact ceases with trading activity. Moreover, for the sake of simplicity we assume that there is no fundamental drift in the asset's price diffusion (1) besides the permanent impact $f(\xi(t))dt$, namely $\mu \equiv 0$. Hence any potential trading strategy of the agent is solely based on the nature of the market impact terms $f(\cdot)$ and $g(\cdot)$ and the volatility pattern $\sigma(\cdot,\cdot)$.

We call a trading scheme *hands-clean* if the path $(\xi(t))_{t\in[0,T]}$ satisfies

$$X(T) - X(0) = \int_0^T \xi(t)dt = 0,$$

and we say that a hands-clean scheme is admissible if the holdings process $X(t) = X(0) + \int_0^t \xi(s)ds$ is bounded. This technical condition is merely added to avoid doubling strategies in continuous time. Traders explore the expected return of hands-clean trades

$$\Pi = \mathbb{E}\left[-\int_0^T \tilde{Z}(t)\xi(t)dt\right],$$

and try to design and use schemes that yield strictly positive expected returns. Therefore, a reasonable model for the market impact functions $f(\cdot)$ and $g(\cdot)$ should preclude the existence of such forms of arbitrage. Formally, the functions $f(\cdot)$ and $g(\cdot)$ should meet the following

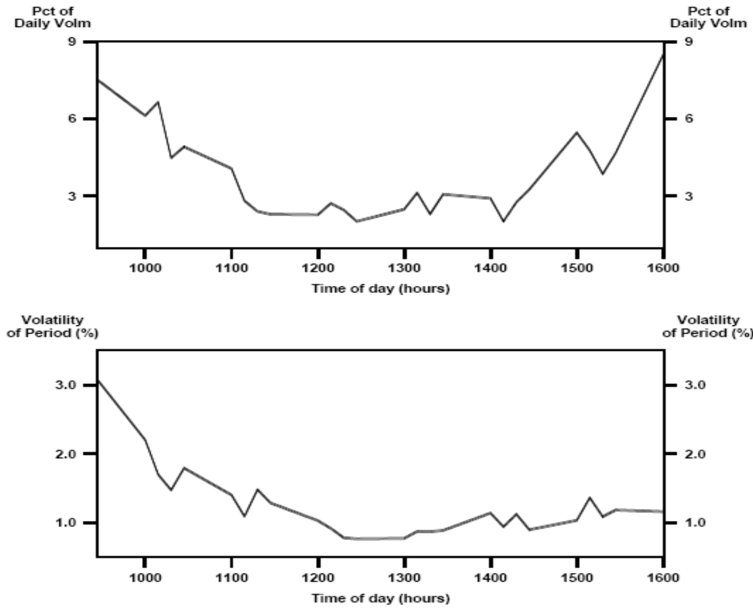

Figure 2. The upper panel displays a ten-day average intra-day volume profile of a large-cap stock on NYSE, and the lower panel displays a ten-day average intra-day variance profile of this stock, both on 15-minute intervals. Our model setup of constant 'volatility' in volume time implicitly expects the variance profile to have the same curve as that of the volume profile, which is approximately valid, as shown in this figure. Courtesy of Almgren et al. at Citigroup Global Quantitative Research.

requirement:

$$\Pi = \mathbb{E}\left[-\int_0^T \tilde{Z}(t)\xi(t)dt\right] \leq 0 \tag{2}$$

for all admissible hands-clean scheme $(\xi(t))_t$.

Under these assumptions we are able to prove that the temporary impact $g(\xi)$ needs to have the same sign as ξ, and that the permanent impact $f(\xi)$ must be linear in ξ as there exists a positive constant γ such that $f(\xi) = \gamma\,\xi$. To be more specific, we have:

Theorem 2.1 Linearity of the Permanent Component of Market Impact *If the market precludes deterministic hands-clean schemes that yield a strictly positive expected return, then the permanent impact function f must be odd and linear in ξ, namely there exists a positive constant γ, such that*

$$f(\xi) = \gamma\,\xi, \qquad \xi \in \mathbb{R} \tag{3}$$

The proof is given in appendix. It begins with a proof that the function f is odd. We are not able to establish the oddness of the temporary impact function g. We can only show that it must satisfy $g(\xi) \geq g(-\xi)$ for all positive ξ's and the opposite relation for negative ξ's.

The above argument shows that $g(\xi_1) \geq g(-\xi_2)$, for any $\xi_1,\ \xi_2 > 0$, thus positivity of the function g on the positive half line. However, unlike f, there are no obvious reason for the linearity of function g. In fact, even the monotonicity of g is an open question.

Remark 1: Notice that both permanent and temporary impacts are assumed to be time-homogeneous. This should not be regarded as a restrictive assumption since it can be obtained by scaling the physical clock time τ to the so-called *volume time* t, which represents the fraction of an average day's volume that has been executed up to the corresponding clock time. In this way, a constant-rate trajectory in *volume time* would be tuned into a VWAP execution in *clock time*, which is a common practice on the algorithmic trading desks. A *Volume-Weighted Average Price* (VWAP) execution essentially trades the asset at a uniform speed with respect to the volume time t. The relationship between τ and t is independent of the total daily volume. We normalize it so that $t = 0$ at market open and $t = 1$ at market close of the same trading day. Keep in mind that while we scale the clock time to volume time, the intra-day volatility, or more precisely the intra-day variance, needs to be rescaled too. Fortunately as pointed out in the empirical studies by Almgren et al. (3), the (population mean of) intra-day variance grows on a curve very similar to that of the (population mean of) intra-day volume. See for instance, Figure (2). This justifies the use of constant volatility models (as long as one is willing to work in volume-time scale).

3. Setup of the Predatory Trading Game

We consider a continuous-time economy with two assets, a risk-free asset and a risky asset. For simplicity we normalize the risk-free interest rate to 0, and call the risk-free asset *cash*. We suppose that there is a finite set of large strategic risk neutral traders seeking to maximize their expected profits. They form a tight oligopoly. They observe the order flow and have first-hand information regarding transient liquidity needs in the market. They attempt to maximize their objectives through their ability to forecast price moves and affect asset prices. The second category of traders is a cohort of noise traders. They are less informed, trade randomly based on exogenous reasons, and help clear the market.

Within the first category, we further assume that there is a distressed trader (*victim*), who has x_0 amount of the risky asset on his book, and needs to convert it to as much cash as possible by the end of a time horizon T. The other strategic traders (*predators*) are assumed to be completely solvent, they can go either long or short in this risky asset during the horizon $[0, T]$, and have a longer trading horizon $\tilde{T} > T$ to become hands-clean. Although the dynamics of the market price movement (as we shall see right away in equation (4)) and the observable state variables in our model setup are essentially the same as the previous studies (13) and (27), the main novelty of our study is to allow strategic traders to use closed-loop control strategies, rather than confining them to employing (essentially deterministic) open-loop strategies. Closed-loop Nash equilibriums are stable under diffeomorphisms of the state-space representations of the game and they are subgame-perfect, while open-loop equilibriums are not guaranteed to have either of these properties. Moreover, in our approach, the price volatility plays a non-trivial role, and the analysis cannot be reduced to a deterministic game by considering the mean process of the future price movements as in the previous open-loop analyses.

Dynamics of the State of the System

For tractability reasons, we focus on the duopoly case of one distressed trader and one predatory trader. Wee dnote by $\xi(t)$ (resp. $\eta(t)$) the trading speed of the distressed trader (resp. predator) at volume time t. For the sake of brevity, we shall simply refer to the time $t \in [0, T]$ or $t \in [T, \tilde{T}]$, instead of referring explicitly to volume time. $X(t) = \int_0^t \xi(s)ds$ (resp. $Y(t) = \int_0^t \eta(s)ds$) is the position in the risky asset of the distressed trader (resp. predator) at time t. The last state variable $Z(t)$ is the market quote price. As explained earlier, the permanent component of market impact should be linear in both $\xi(t)$ and $\eta(t)$, so we choose the following model

$$dZ(t) = \gamma\,[\xi(t) + \eta(t)]dt + \sigma\, dW(t) \tag{4}$$

where the value of the coefficient γ, the so-called *plasticity* of the market, is in the common knowledge of both players: from a market-microstructure point of view, the higher the level of information asymmetry the risky asset is associated with, the higher the value of γ. The diffusion part $\sigma dW(t)$ characterizes the aggregate random impact of demand and supply from all of the noise traders in the market, for which a random walk in discrete-time, or a martingale in continuous-time, is a reasonable model. Although we could use the log-normal model of the Black-Scholes theory, or even more sophisticated local or stochastic-volatility models, given its simplicity and the relatively short time scale of the transactions under consideration, we choose to work in the Bachelier framework. Our goal is to compare qualitatively how the predatory trading game evolves under low and high volatility environments, and the Bachelier model is good enough to serve this purpose.

In addition to the permanent component of market impact, when the two players trade with intensities $\xi(t)$ and $\eta(t)$ respectively, the transaction price suffers a slippage $g(\xi(t)+\eta(t))$ from the market quote $Z(t)$ averaged through the infinitesimal period around t. This is the temporary market impact discussed in the previous section. For example, if the selling intensity of the distressed trader is exactly offset by the buying intensity of the predator, the microstructure interpretation is that the limit/market orders they send to the exchange instantaneously pair with each other, thus the overall effect is that the transaction price both of them get averaged through a small time interval around t is exactly equal to the market mid-quote $Z(t)$, i.e. $g(\xi(t) + \eta(t)) = g(0) = 0$. Actually, no permanent impact will be incurred in this situation either. In another example, if the predator is selling the asset at the same intensity as the distressed trader, i.e. $\xi(t) = \eta(t) < 0$, then they eat up the bid limit-order book faster than the distressed trader would do on his own, and the market mid-quote shifts after each transaction. The average slippage of the transaction price they get over a small time interval around t is larger than the average slippage in the case only one player trades, i.e. $g(\xi(t)+\eta(t)) = g(2\xi(t)) < g(\xi(t)) = g(\eta(t)) < 0$, and the market mid-quote price shifts at the rate of $\gamma 2\xi(t)$ due to the permanent market impact. As a first order approximation, we shall just take a linear model for the temporary component of market impact, namely $g(\xi(t) + \eta(t)) = \lambda[\xi(t) + \eta(t)]$, where λ is a positive constant that is known to both players. Sometimes, we also refer to λ as the *elasticity* of the market, which inversely characterizes how deep the limit-order book is. The thinner the limit-order book, the higher λ. We note that this permanent/temporary market impact setup is backed by most of the optimal execution literature (9, 1, 2), and is consistent with the models used in previous studies on predatory trading (13, 27).

In addition to the trading activities of the two strategic players, the noise traders also submit limit and/or market orders randomly, and shift the market mid-quote $Z(t)$ over time. This random-walk type of movement is captured by the martingale diffusion term $\sigma dW(t)$ in (4), and the parameter σ, referred to as the *volatility* of the market, characterizes qualitatively the influence of these smaller, less informed outsiders, to the evolution of the market price process: the more fickle and tumultuous the limit-order books fluctuate (apart from the duopolistic players' trading impacts), the higher the value of σ. Although it is true that the random submissions of limit and market orders by noise traders also alters the transaction prices the two strategic players get, and add (or subtract) to the price slippage from the mid-quote, the net effect from noise traders' order submission to the two players' temporary price slippage is null on average. Incorporating a diffusion term into the price dynamics, we have acknowledged three types of players: the distressed trader, the predator(s), and the rest of the market for which we use the umbrella terminology *noise traders*. From a game theory viewpoint, the third participant plays the role of *nature* in the extensive-form perspective of the game (8). In this spirit, we allow both strategic players to use closed-loop strategies, and to keep observing the market fluctuations which are not in exact control of their combined actions. So in addition to the state variables x and y (positions of the two strategic players respectively), which are readily used in the previous studies, we incorporate z, the market quote price, into the system's state as well. We shall use the notation $\mathbf{x} = (x, y, z)$ and $\mathbf{X}(t) = (X(t), Y(t), Z(t))$ for the state of the system.

As in Schöneborn and Schied (27) we give the predator a longer trading horizon. Our predatory trading model is comprised of two stages. During the first stage $[0, T]$, both players can trade in the market; and during the second stage $[T, \tilde{T}]$, only the predator has the freedom to trade in order to unwind her position back to 0. The distressed trader's objective is to generate as much cash as possible by time T, with up to $x_0 > 0$ units of the risky asset to sell. Since he is *distressed*, it is sensible to require that he should be monotonically selling, without buying back any of the risky asset during the period $[0, T]$. On the other hand, the predator is assumed to be completely solvent. She can hold any long or short position in the risky asset during the first stage. While she is free to use any admissible strategy to unwind her position by time $\tilde{T}$, for the sake of simplicity, we shall assume that she is using a plain VWAP to unload her position $Y(T)$ by time $\tilde{T}$. The main effect of this simplifying assumption is to shorten the computing time by giving an easily computed expression to the scrap value at time T (see details below). There is no agreement about the relative values of $\tilde{T}$ and T. Since the sole purpose of examining the second stage is to assign a scrap value $S(\mathbf{x}) = \hat{S}(y, z)$ to the value function of the predator at the end of $[0, T]$, we ran a large number of numeric experiments and we found that the qualitative features of the game's equilibrium is pretty stable with respect to the ratio of $\tilde{T}$ to T or the choice of the scrap function $\hat{S}(\cdot, \cdot)$. For the sake of definiteness we choose $\tilde{T} = 5\,T$ (i.e., a trading week vs. a trading day) and we use the value of a VWAP liquidation of $Y(Y)$ as scrap value for $\hat{S}$.

System of Coupled HJB Equations

We first concentrate on the first stage $[0, T]$. We denote by $U(t, \mathbf{x})$ (resp. $V(t, \mathbf{x})$) the value function in equilibrium of the distressed trader (resp. predator) when the system is in state $\mathbf{X}(t) = \mathbf{x}$ at time t. We also consider two measurable functions ϕ and $\psi : \mathbb{T} \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{R}$ that the two players will adopt as their closed-loop strategies.

For any given closed-loop strategy $\psi : (t, \mathbf{x}) \hookrightarrow \psi(t, \mathbf{x})$ of the predator, the value function $U^\psi(t, \mathbf{x})$ of the stochastic optimal control problem of the distressed trader is:

$$U^\psi(t, \mathbf{x}) = \max_{\xi \in \mathcal{A}^{[1]}} \mathbb{E}\left[-\int_t^T \xi(s)[Z(s) + \lambda[\xi(s) + \psi(s, \mathbf{X}(s))]] ds \;| \mathbf{X}(t) = \mathbf{x} \right] \tag{5}$$

subject to

$$\begin{cases} dX(t) &= \xi(t) dt \\ dY(t) &= \psi(t, \mathbf{X}(t)) dt \\ dZ(t) &= (\gamma\, \xi(t) + \gamma\, \psi(t, \mathbf{X}(t))) dt + \sigma dW(t) \end{cases} \tag{6}$$

where the requirement for the admissible set $\mathcal{A}^{[1]}$ is that $\xi(t) \le 0$, and $X(t) = x_0 + \int_0^t \xi(s) ds \in [0, x_0]$, for all $t \in [0, T]$. The Hamilton-Jacobi-Bellman (HJB) equation (31, 22) of the distressed trader writes:

$$U_t^{\psi}+\max_{\xi\leq 0}\{-\lambda\,\xi^2-\lambda\,\psi(t,\mathbf{x})\,\xi-z\,\xi+\psi(t,\mathbf{x})\,U^{\psi}{}_y+\xi\,U^{\psi}{}_x+(\gamma\,\xi+\gamma\,\psi(t,\mathbf{x}))U^{\psi}{}_z+\frac{\sigma^2}{2}\,U^{\psi}{}_{zz}\}=0. \quad (7)$$

where we used subscripts to denote partial derivatives. The maximum is attained for

$$\xi^*=-\frac{1}{2\lambda}\left(z+\lambda\,\psi(t,\mathbf{x})-U^{\psi}{}_x-\gamma\,U^{\psi}{}_z\right)^+, \quad (8)$$

and plugging this value back into (7), we obtain:

$$U^{\psi}{}_t=-\,\psi(t,\mathbf{x})(U^{\psi}{}_y+\gamma\,U^{\psi}{}_z)-\frac{1}{2}\sigma^2\,U^{\psi}{}_{zz}-\frac{1}{4\lambda}\left[\left(z+\lambda\,\psi(t,\mathbf{x})-U^{\psi}{}_x-\gamma\,U^{\psi}{}_z\right)_+\right]^2. \quad (9)$$

Similarly, given any closed-loop strategy $\phi:(t,\mathbf{x})\hookrightarrow\phi(t,\mathbf{x})$ for the distressed trader, the value function of the stochastic optimal control problem of the predator is:

$$V^{\phi}(t,\mathbf{x})=\max_{\eta\in\mathcal{A}^{[2]}}\mathbb{E}\left[-\int_t^T(Z(s)+\lambda[\eta(s)+\phi(s,\mathbf{X}(s))])\eta(s)ds+S(\mathbf{X}(T))\,|\mathbf{X}(t)=\mathbf{x}\right] \quad (10)$$

subject to

$$\begin{cases} dX(t) &= \phi\,(t,\mathbf{X}(t))\,dt \\ dY(t) &= \eta(t)dt \\ dZ(t) &= (\gamma\,\phi(t,\mathbf{X}(t))+\gamma\,\eta(t))dt+\sigma dW(t)\end{cases} \quad (11)$$

where the only requirement for the admissible set $\mathcal{A}^{[2]}$ is that $Y(t)=\int_0^t\eta(s)ds$ exists and is bounded on $[0,T]$. As explained earlier, the scrap function S will be given by the result of VWAP trading by the predator over the second period $T,\tilde{T}]$, but for the purpose of the present discussion, its value is irrelevant so we do not specify it yet. The HJB equation of the predator reads:

$$V_t^{\phi}+\max_{\eta\in\mathbb{R}}\{-\lambda\,\eta^2-\lambda\,\phi(t,\mathbf{x})\,\eta-z\,\eta+\phi(t,\mathbf{x})\,V^{\phi}{}_x+\,\eta\,V^{\phi}{}_y+(\gamma\,\eta+\gamma\,\phi(t,\mathbf{x}))\,V_z+\frac{1}{2}\sigma^2\,V^{\phi}{}_{zz}\}=0. \quad (12)$$

Note that since there is no sign restriction on the predator's optimal response, the maximum is attained for:

$$\eta^*=-\frac{1}{2\lambda}\left(z+\lambda\,\phi(t,\mathbf{x})-V^{\phi}{}_y-\gamma\,V^{\phi}{}_z\right), \quad (13)$$

and plugging this back into (12), we get:

$$V^{\phi}{}_t+\frac{1}{4\lambda}\left(z+\lambda\,\phi(t,\mathbf{x})-V^{\phi}{}_y-\gamma\,V^{\phi}{}_z\right)^2+\phi(t,\mathbf{x})(V^{\phi}{}_x+\gamma\,V^{\phi}{}_z)+\frac{1}{2}\sigma^2\,V^{\phi}{}_{zz}=0\,. \quad (14)$$

We look for a Nash-equilibrium of the this predatory trading game by looking for a couple (ϕ,ψ) of closed loop strategies which solve the problems (5) -(6) and (10) - (11) simultaneously or equivalently (9) and (14). Putting (8) and (13) together, we can express the optimal responses of the two players in terms of the partial derivatives of their value functions:

$$\begin{aligned} \phi(t,\mathbf{x}) &= -\frac{1}{3\lambda}\left(z - 2\,U^{\psi}{}_x + V^{\phi}{}_y - 2\gamma\,U^{\psi}{}_z + \gamma\,V^{\phi}{}_z\right)_+ \\ &= -\frac{1}{3\lambda}\delta(t,\mathbf{x})^+ \\ \psi(t,x,y,z) &= \begin{cases} -\frac{1}{3\lambda}(z + U_x - 2\,V_y + \gamma\,U_z - 2\gamma\,V_z) & \text{if} \quad \delta(t,\mathbf{x}) > 0 \\ -\frac{1}{2\lambda}(z - V_y - \gamma\,V_z) & \text{if} \quad \delta(t,\mathbf{x}) \le 0 \end{cases} \end{aligned} \tag{15}$$

where we set

$$\delta(t,\mathbf{x}) := z - 2\,U^{\psi}{}_x + V^{\phi}{}_y - 2\gamma\,U^{\psi}{}_z + \gamma\,V^{\phi}{}_z \tag{16}$$

for the *determinant function* which dichotomizes the time-state domain $[0,T] \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$ into two regimes according to its sign. Furthermore, if we define the conjugate $\hat{\delta}$ by

$$\hat{\delta}(t,\mathbf{x}) := z - 2\,V^{\phi}{}_y + U\psi_x - 2\gamma\,V^{\phi}{}_z + \gamma\,U^{\psi}{}_z \tag{17}$$

we see that the search for a Nash-equilibrium of the predatory trading game can be summarized in the following free boundary system of nonlinear PDEs:

$$\begin{aligned} &\text{(a) where} \quad \delta(t,\mathbf{x}) > 0 \\ &-\frac{\partial}{\partial t}\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} -\frac{1}{3\lambda}\hat{\delta}(t,x,y,z)(U_y + \gamma\,U_z) + \frac{1}{9\lambda}\left(\delta(t,x,y,z)\right)^2 + \frac{1}{2}\sigma^2\,U_{zz} \\ -\frac{1}{3\lambda}\delta(t,x,y,z)(V_x + \gamma\,V_z) + \frac{1}{9\lambda}\left(\hat{\delta}(t,x,y,z)\right)^2 + \frac{1}{2}\sigma^2\,V_{zz} \end{bmatrix} \\ &\text{(b) where} \quad \delta(t,\mathbf{x}) \le 0 \\ &-\frac{\partial}{\partial t}\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} -\frac{1}{2\lambda}(z - V_y - \gamma\,V_z)(U_y + \gamma\,U_z) + \frac{1}{2}\sigma^2\,U_{zz} \\ +\frac{1}{4\lambda}(z - V_y - \gamma\,V_z)^2 + \frac{1}{2}\sigma^2\,V_{zz} \end{bmatrix} \end{aligned} \tag{18}$$

where the functions δ and $\hat{\delta}$ are defined as in (16) and (17) from U and V. If we can uncover a solution to this system, then not only do we know the value functions of the players in equilibrium, but through (15), we also have the optimal closed-loop strategies that neither player has an incentive to deviate from.

Terminal and Boundary Conditions

For the distressed trader's value function, the terminal condition is known to be $U(T,\mathbf{x}) = 0$, i.e. he cannot continue to trade after his deadline. One obvious boundary condition is $U(t,0,y,z) = 0$, for all $t \in [0,T]$, meaning that the distressed trader can sell up to x_0 units of the risky asset that is already on his book, but will not go any further with short sales or leverage. Another boundary condition is that $U(t,x,y,0) = 0$, meaning the risky asset can go defunct, and if the price process ever hits 0, it will be delisted from the exchange for the entire future (no revival). Thus even if there were some risky asset on the distressed trader's book at that time, he could not get any cash from the delisted security. The initial price quote z_0 is chosen away from 0. Similarly, for the predator, it is also true that $V(t,x,y,0) \equiv 0$, whether her position y is positive or negative at that moment. In case she holds a short position in the risky asset when it gets delisted, the cash she needs to pay to cover her short position is, not surprisingly, 0. The remaining terminal condition is the scrap value the predator assigns to her terminal condition.

We choose:

$$
\begin{aligned}
V(T,\mathbf{x}) &= S(\mathbf{x}) \qquad (19)\\
&= y\int_0^1 \Bigg[(z-\lambda\,\frac{y}{(\tilde{T}-T)}-\gamma\,y\,\theta)\,\Phi\left(\frac{z-\gamma\,y\,\theta}{\sigma\sqrt{(\tilde{T}-T)\,\theta}}\right)+\\
&\qquad (z+\lambda\,\frac{y}{(\tilde{T}-T)}+\gamma\,y\,\theta)\,e^{\frac{2\gamma\,z\,y}{\sigma^2(\tilde{T}-T)}}\,\Phi\left(-\frac{z+\gamma\,y\,\theta}{\sigma\sqrt{(\tilde{T}-T)\,\theta}}\right)\Bigg]\,d\theta\\
&=: \hat{S}(y,z)
\end{aligned}
$$

where we use the notation Φ for the cummulative distribution function (cdf) of the standard Gaussian distribution. A detailed derivation of this scrap value is given in Appendix B. Although there can be other ways for the predator to unload the remnant position $Y(T)$ during $[T,\tilde{T}]$, numerical experiments show that the qualitative features of the game's equilibrium are pretty insensitive to the specific form of the scrap value $S(\mathbf{x})$ used by the predator, as long as it is differentiable and monotonically increasing in both y and z (monotonically decreasing in z when y is negative).

4. Equilibrium of the Predatory Trading Game and Comparative Statics

We now solve for the closed-loop Nash-equilibrium of the predatory trading game taking place during the first period $[0,T]$ by solving numerically the PDE system (18). Recall that once a solution to (18) is computed, we obtain the optimal (closed-loop) trading strategy of each player via (15). We solve the PDE system by a backward explicit finite difference scheme with a sufficiently small time step Δt (see for example (28)) from the terminal conditions $U(T,\mathbf{x})$ and $V(T,\mathbf{x})$. Our system of nonlinear parabolic equations is highly degenerate since second order terms exist only in the second partial derivatives with respect to the variable z. So standard elliptic estimates are not available to help with existence proofs. Moreover, the nonlinearities appearing in the equations are too strong to use any scheme based on a forward-backward stochastic differential equation formulation of the problem. Finally, the vector nature of the system prevents us from using monotonicity arguments to prove existence of viscosity solutions via the convergence of *monotone numerical schemes.* So we rely on the time-honored method of vanishing viscosity in the spirit of the numerical scheme in (16): we add some *artificial viscosity* to the degenerate dimensions x and y, and eventually let the added viscosity terms tend to 0. So for each $\epsilon>0$, we first consider the solution $(U^{(\epsilon)},V^{(\epsilon)})$ of the following system:

$$
\begin{aligned}
&\text{(a) where}\quad \delta^{(\epsilon)}(t,\mathbf{x})>0\\
&-\frac{\partial}{\partial t}\begin{bmatrix}U^{(\epsilon)}\\ V^{(\epsilon)}\end{bmatrix}=\begin{bmatrix}-\frac{1}{3\lambda}\hat{\delta}^{(\epsilon)}(t,\mathbf{x})(U_y^{(\epsilon)}+\gamma U_z^{(\epsilon)})+\frac{1}{9\lambda}\left(\delta^{(\epsilon)}(t,\mathbf{x})\right)^2\frac{1}{2}\sigma^2(U_{zz}^{(\epsilon)}+\epsilon\,U_{xx}^{(\epsilon)}+\epsilon\,U_{yy}^{(\epsilon)})\\ -\frac{1}{3\lambda}\delta^{(\epsilon)}(t,\mathbf{x})(V_x^{(\epsilon)}+\gamma V_z^{(\epsilon)})+\frac{1}{9\lambda}\left(\hat{\delta}^{(\epsilon)}(t,\mathbf{x})\right)^2+\frac{1}{2}\sigma^2(V_{zz}^{(\epsilon)}+\epsilon\,V_{xx}^{(\epsilon)}+\epsilon\,V_{yy}^{(\epsilon)})\end{bmatrix}\\
&\text{(b) where}\quad \delta^{(\epsilon)}(t,\mathbf{x})\le 0\\
&-\frac{\partial}{\partial t}\begin{bmatrix}U^{(\epsilon)}\\ V^{(\epsilon)}\end{bmatrix}=\begin{bmatrix}-\frac{1}{2\lambda}(z-V_y^{(\epsilon)}-\gamma V_z^{(\epsilon)})(U_y^{(\epsilon)}+\gamma U_z^{(\epsilon)})+\frac{1}{2}\sigma^2(U_{zz}^{(\epsilon)}+\epsilon\,U_{xx}^{(\epsilon)}+\epsilon\,U_{yy}^{(\epsilon)})\\ \frac{1}{4\lambda}(z-V_y^{(\epsilon)}-\gamma V_z^{(\epsilon)})^2+\frac{1}{2}\sigma^2(V_{zz}^{(\epsilon)}+\epsilon\,V_{xx}^{(\epsilon)}+\epsilon\,V_{yy}^{(\epsilon)})\end{bmatrix}
\end{aligned}
\qquad (20)
$$

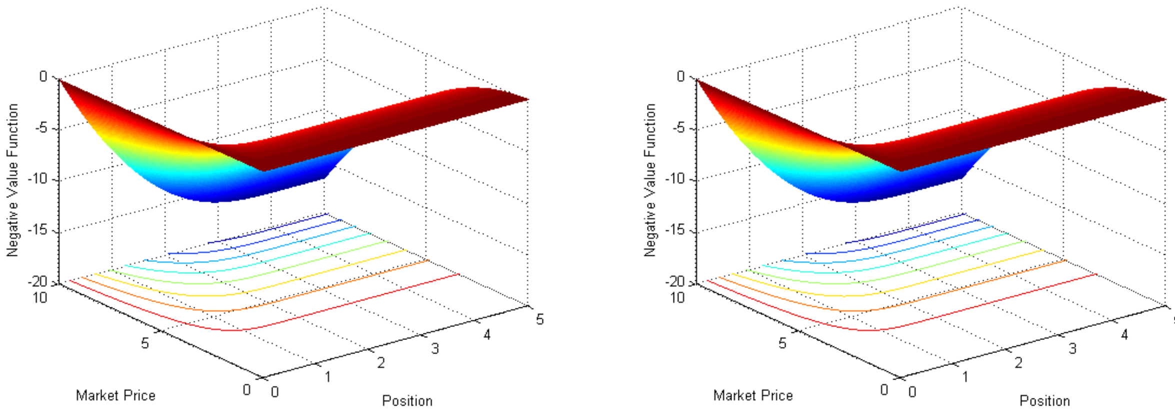

Figure 3. Numerical evidence that the ϵ-regularized solutions numerically *converge* as we tune ϵ down to 0. As we can observe by comparing the two panels, the value function surfaces (plotted upside-down) stay essentially the same as we shrink ϵ from $\epsilon = 2.5 \times 10^{-3}$ (left panel) to $\epsilon = 1.0 \times 10^{-4}$ (right panel).

where $\delta^{(\epsilon)}(t, \mathbf{x})$ and $\hat{\delta}^{(\epsilon)}(t, \mathbf{x})$ are defined from $U^{(\epsilon)}$ and $V^{(\epsilon)}$ as in (16) and (17). While we employ the *vanishing viscosity* technique to pursue a numerical solution to the Nash-equilibrium of this predatory trading game, the first-order of business is to check, at least numerically, that the regularized solutions $(U^{(\epsilon)}, V^{(\epsilon)})$ do converge as ϵ tends to 0. Namely, for the same set of market parameters γ, λ, and σ and same time horizons T and $\tilde{T}$, we need to check that the regularized solutions stay essentially the same as we take ϵ to 0. Figure 3 shows the 3-d surface plot of the value function in equilibrium of the distressed trader, at time $t = 0$ and the position of the predator (predator) being 0, the added viscosity levels being $\epsilon = 2.5 \times 10^{-3}$ (left pane) and $\epsilon = 1.0 \times 10^{-4}$ (right pane). We intentionally plot the negative of the value function, i.e. put the tent-like surface upside-down, so that we have a better viewing angle for the curvature of the surface while still producing visible contour plots underneath. We observe that the value functions are numerically indistinguishable as further illustrated on the 2-d plots of cross sections over the *market price* dimension and the *distressed trader initial position* dimension displayed in Figure 4. We computed that the largest difference in the regularized solutions is on the order of magnitude of 10^{-3}, even if we shrink ϵ by a factor of 25. This is a strong numerical evidence that the regularized solutions converge to a limit, a viscosity solution of the system. In the following numerical investigation of the Nash-equilibrium, we use $\epsilon = 1.0 \times 10^{-4}$, since tuning ϵ further down does not contribute numerically to the solution, but definitely make the computation time dreadful.

Once we obtain a stable numerical approximation for (U, V), it is still not trivial to visualize the value functions, since there are actually three relevant dimensions x, y and z. Recall that the predator (predator) always starts from a 0 position when $t = 0$. One possibility is to plot the value functions over the (x, z) plane to view how each player's value function looks like in equilibrium for each level of initial market quote z and initial inventory x of the distressed trader. This is done in Figure 5. Once again, we intentionally plot the value function surfaces upside-down to better visualize the curvature. Figure 6 and Figure 7 further illustrate the properties of the value functions by intercepting them with five equally spaced initial market quote levels and with five equally spaced distressed trader's initial inventory levels respectively.

We note from Figure 6 that the value function of the distressed trader is always positive. This

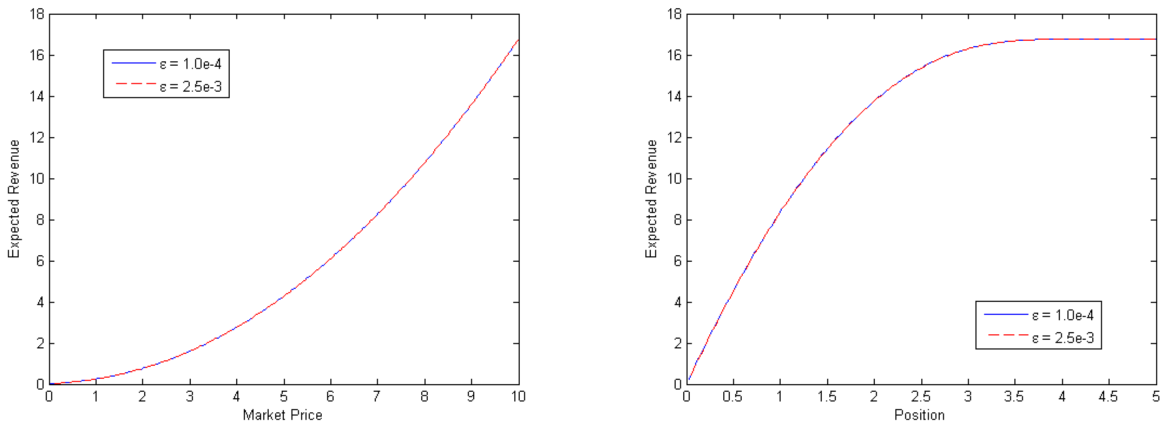

Figure 4. Cross sections of the value function for two different values of ϵ, over the market price for a fixed initial position level (left panel), and over the position position for a fixed market price level (right panel).

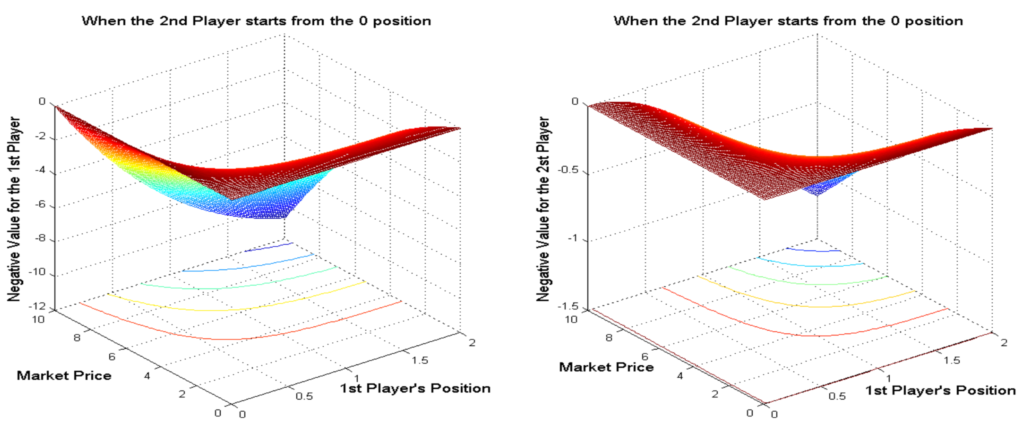

Figure 5. Value function surfaces (plotted upside-down) of each player at the beginning of the game. Recall that the predator starts from a zero position $Y(0) = y = 0$.

is because the distressed trader is required to be monotonically selling, always generating a non-negative amount of cash from his trading trajectory. Moreover, we note from Figure 7 that the value function of the predator is always positive at $y = 0$. Indeed, whatever closed-loop strategy $\phi(t, \mathbf{x})$ the distressed trader chooses, the "do-nothing" strategy $\psi(t, \mathbf{x}) \equiv 0$ is admissible and yields zero profit in the end. As her equilibrium strategy $\psi^*(t, \mathbf{x})$ must yield a payoff at least as favorable as the "do-nothing" strategy, her value function must be non-negative at $y = 0$ for every $z > 0$, $x > 0$, and $t < T$.

In order to get a sense of the deformations of the value functions over time, we can repeat the graphical visualizations described above at intermediate times. For the sake of definiteness, we generated the previous three plots again in Figures 8, (9) and (10) at volume time $t = T/2$.

In the same spirit, Figure 11 shows the properties of the value function of the predator over

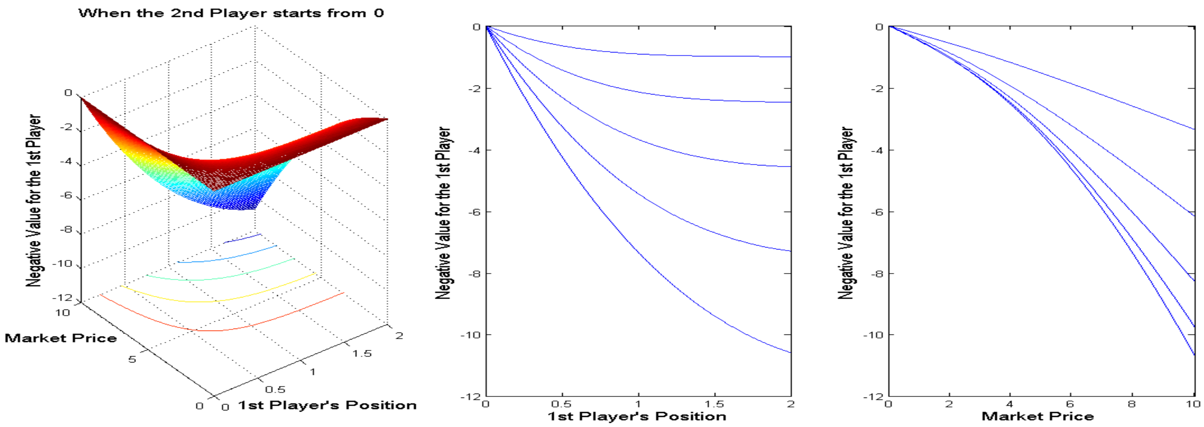

Figure 6. Distressed trader's expected revenue at the beginning of the game as a function of his initial inventory x and the initial market quote z.

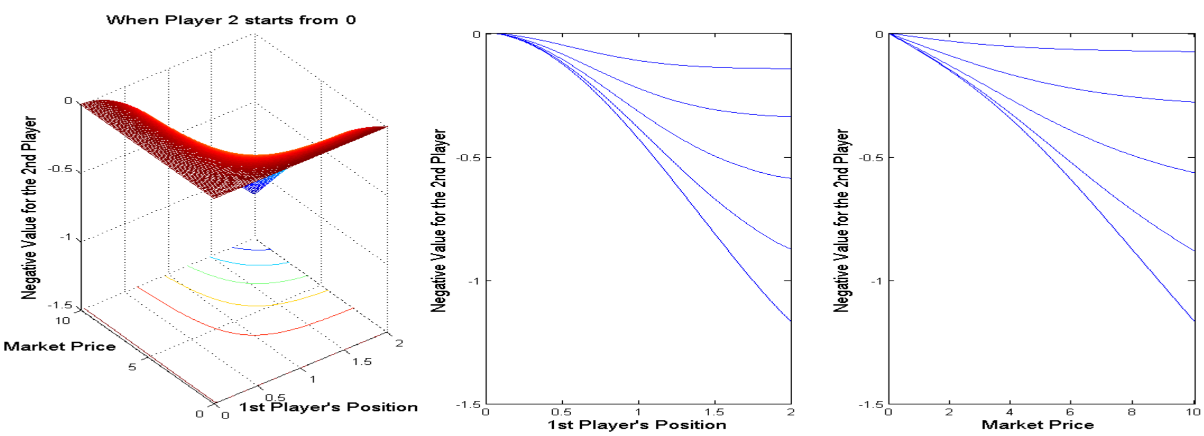

Figure 7. Predator's expected profit at the beginning of the game as a function of the initial inventory x of the distressed trader and the initial market quote z.

the (y, z) plane for a fixed level x of the distressed trader inventory.

Figure 12 shows the value function of the distressed trader over the (y, z) plane, for a fixed level of his own inventory x. It appears from this plot that the value function of the distressed trader is not very sensitive to the initial position y of the predator, but primarily depends upon the current market quote z. This fact supports our previous claim that the equilibrium value of the distressed trader will not be very sensitive to the precise curvature of the scrap function of the predator $S(x, y, z) = \hat{S}(y, z)$.

Impact of the Volatility. As explained earlier, the main thrust of our model is to allow each player to use closed-loop control strategies. Before each infinitesimal time step, both players observe how the market integrates the moves of the noise traders to decide what optimal responses they each should choose for the next infinitesimal time interval $[t, t + dt)$. Henceforth, the de-

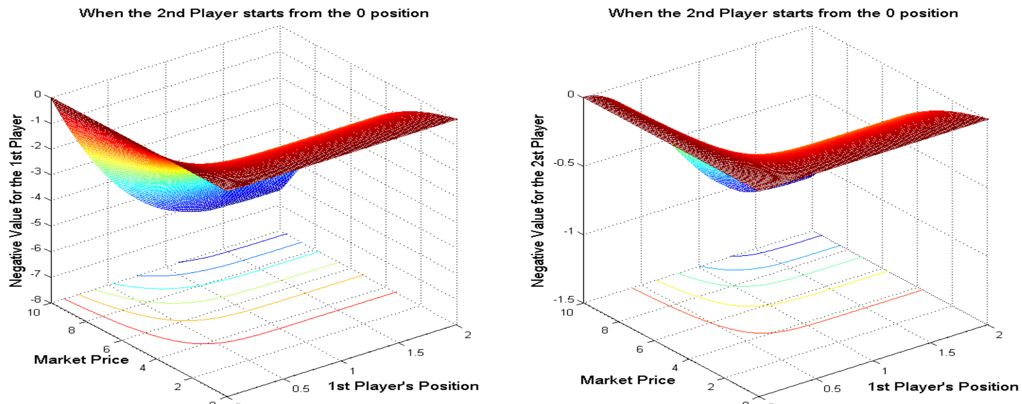

Figure 8. Same plots as in Figure (5) at time $t = T/2$.

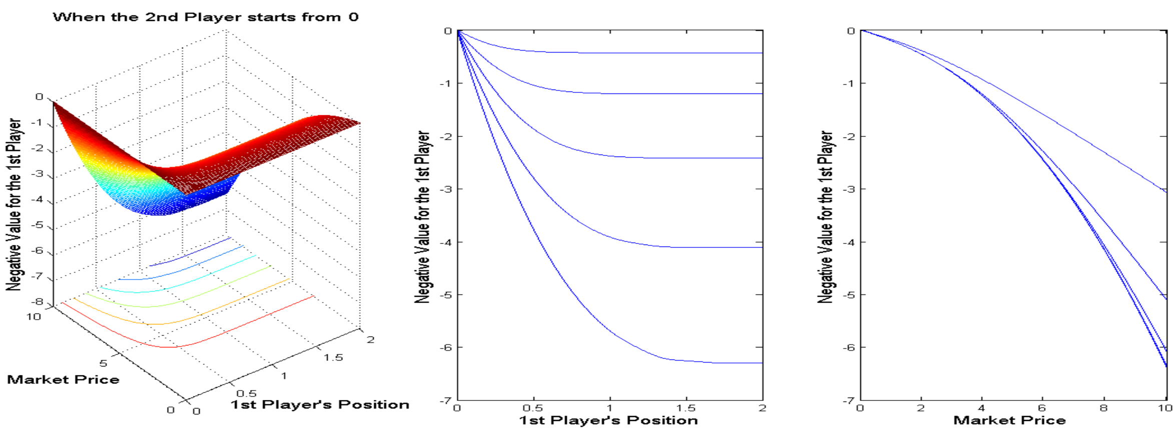

Figure 9. Same plots as in Figure (6) at time $t = T/2$.

gree of fickleness of the market as captured by the volatility parameter σ, affects the dynamic equilibrium of the game. As a first illustration of this impact, Figures 13 and 14 show that the predator enjoys a higher expected profit in most parts of the $\mathbf{x} = (x, y, z)$ domain in a higher volatility environment.

Monte Carlo Simulations. Once the equilibrium closed loop strategies ϕ and ψ are determined, we can simulate Monte Carlo samples of the time evolution of the 3-dimensional state $\mathbf{X}(t) = (X(t), Y(t), Z(t))$, under different volatility settings, but driven by the same innovation

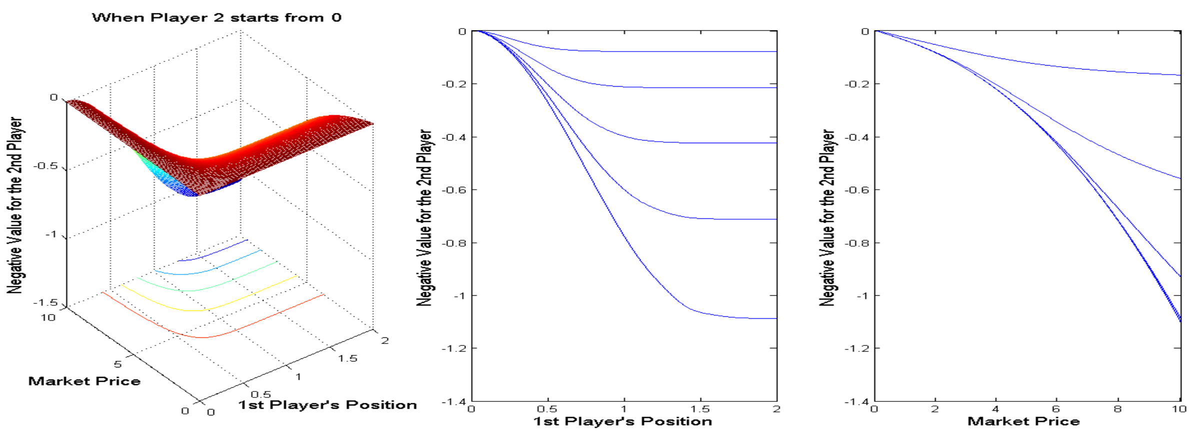

Figure 10. Same plots as in Figure (7) at time $t = T/2$.

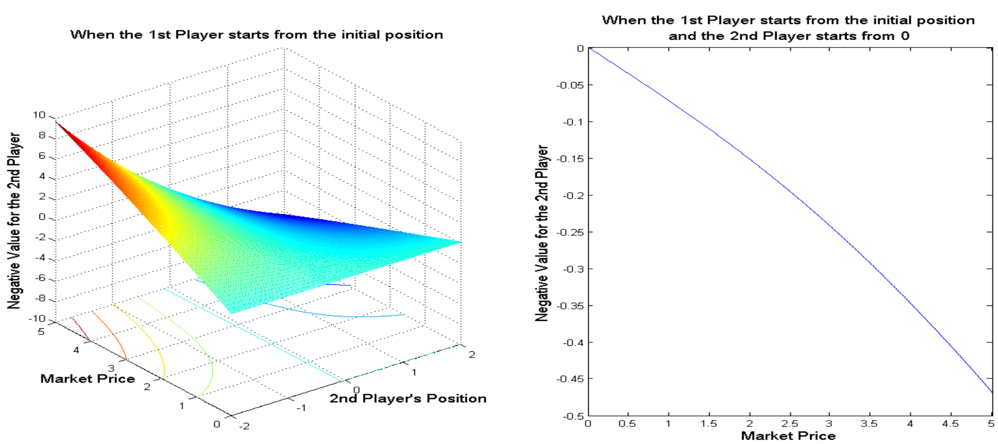

Figure 11. Value function (still plotted upside-down) of the predator over the (y, z) plane at an intermediate time for a given level x of the distressed trader's inventory. The right panel gives the cross section for $y = 0$.

process $(W(t))_{t\in[0,T]}$ using the expressions

$$X(t) = x_0 + \int_0^t \xi(s)ds = x_0 + \int_0^t \phi\left(s, X(s), Y(s), Z(s)\right) ds$$

$$Y(t) = 0 + \int_0^t \eta(s)ds = 0 + \int_0^t \psi\left(s, X(s), Y(s), Z(s)\right) ds$$

$$Z(t) = = z_0 + \int_0^t \gamma\left(\phi\left(s, X(s), Y(s), Z(s)\right) + \psi\left(s, X(s), Y(s), Z(s)\right)\right) ds + \sigma W(t).$$

Two typical samples are shown in Figure (15). The plots on the top row are identical: they

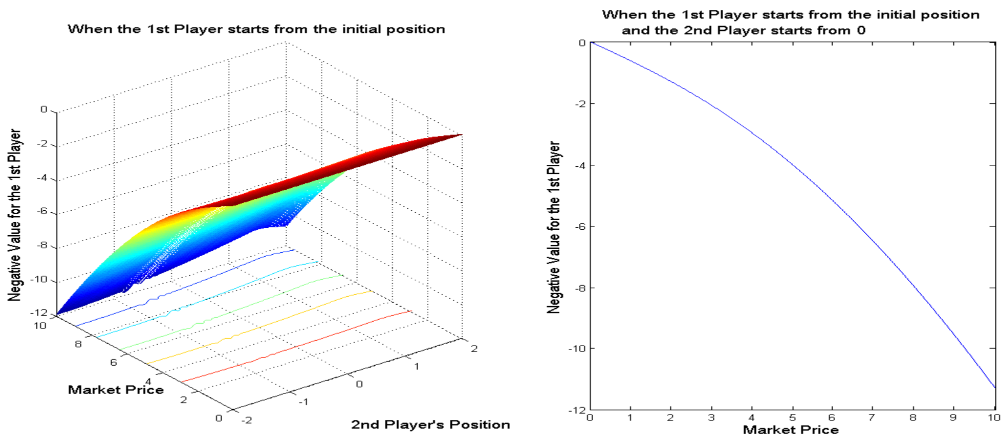

Figure 12. The value function surface (plotted upside-down to reveal the underneath contour plot) of the distressed trader against z and y at an interim moment of the game, for a given level of his own inventory x. The right panel further illustrate the value function curve intercepted at $y = 0$.

give sample paths of the quote price in a low (resp. high) volatility scenario. The corresponding trading intensity samples $(\xi(t))_{t\in[0,T]}$ and $(\eta(t))_{t\in[0,T]}$, are given in the second and third row of the first column in Figure (15), facing the position samples (i.e. their integral over time) in the second column. We observe that both the distressed trader and the predator trade more intensely when the volatility of the market increases. For example, between the 'volume time' 0.2 and 0.3, when the market price of the risky asset is hovering relatively low, in equilibrium the distressed trader slows down his selling activity, more so in the higher volatility environment. At a later stage of the game, say around 'volume time' 0.8, the distressed trader accelerates his trading intensity, prominently more in the higher volatility environment. The position process of the distressed trader, as a result, lags behind in the higher volatility case starting from around 'volume time' 0.2, speeds up around 'volume time' 0.7 and finally catches up and slightly surpasses the lower volatility scenario around 'volume time' 0.95. For the predator's equilibrium behavior on the other hand, both her trading intensity and cumulative position in the higher volatility case exceed those in the slightly lower volatility case, for almost the entire second half of the game. The impact of higher volatility on the trading activity of the predator can be deduced from the statistical distribution of $Y(T)$, her holdings at the end of the first period of the game. Using a large number of Monte Carlo samples, the changes in this distribution due to increased volatility can easily be visusalized. Figure 16 gives the plots of Gaussian-kernel density estimates for the position of the predator at the end of the first stage. We observe that the distribution of $Y(T)$ is clearly tilted to the right in the higher volatility case. This is consistent with our discussion of the results of Figures 13 and 14 showing that the predator will prey more aggressively and make more profit in higher volatility situations.

Impact of the Predator Presence. Lastly, our model also shows that in some cases, the presence of the predator can be beneficial to the distressed trader. In our model, the absence of predator corresponds to setting $\psi(t, x, y, z) \equiv 0$. From (9), we see that the HJB equation for the value function $\hat{U}$ of the distressed trader in the absence of predation is given by:

$$-\hat{U}_t = \frac{1}{4\lambda}\left[\left(z - \hat{U}_x - \gamma\hat{U}_z\right)_+\right]^2 + \frac{1}{2}\sigma^2\hat{U}_{zz}, \tag{21}$$

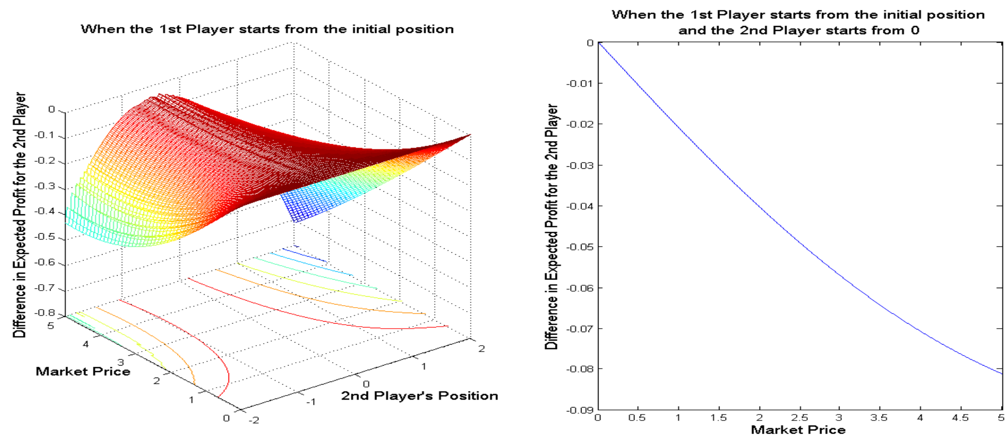

Figure 13. Difference between the value functions of of the predator over the (y, z) plane for a given level of the distressed trader's inventory x computed under a low volatility environment and a high volatility environment. The right panel gives the cross section of this difference over $y = 0$.

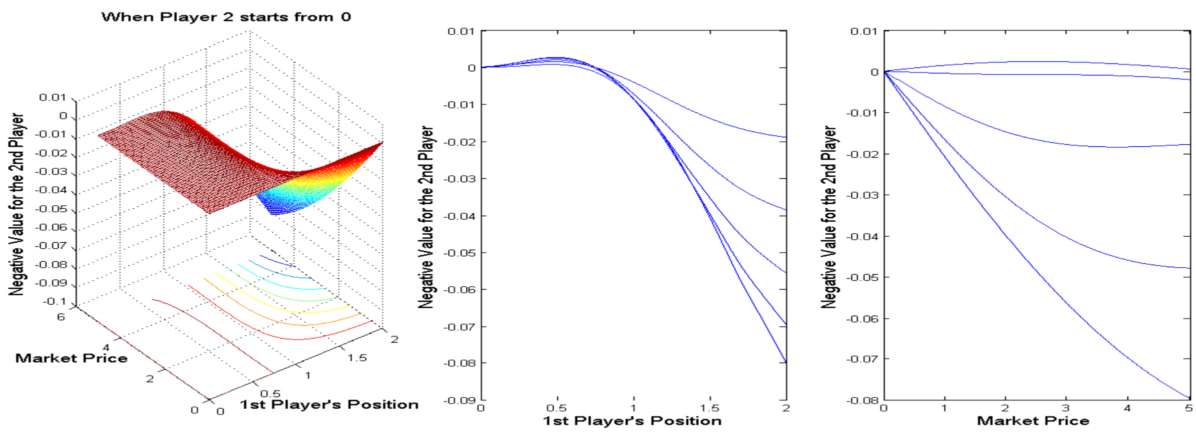

Figure 14. Difference between the value functions of the predator over the (x, z) plane for $y = 0$, at low volatility and high volatility. The middle and right panels show cross sections of this difference over five equally spaced values of the respective variables.

and its solution can be used to compare the distressed trader's welfare change due to the presence of the predator. Contrary to popular belief, under some particular market liquidity conditions, e.g. low permanent impact, high temporary impact and medium market volatility – conditions usually called *elastic environment* – the expected revenue of the distressed trader actually increases due to the 'predation' of the solvent trader whose role is more a "white knight" than a predator. Figure 17 gives an example of such a welfare increase over the (x, z) plane. The intuitive reason for a possible welfare increase is that the solvent player has a longer time horizon than the distressed trader, so that she can spread her trading activity over the entire horizon, especially so during the second stage $[T, \tilde{T}]$, thus significantly reducing their combined welfare loss due to the temporary component of market impact on fire-sales of the risky-asset. If in equilibrium the solvent trader delicately buys over the period $[0, T]$ a sizable proportion of the risky-asset while the distressed trader sells off to the market, and unloads her long position slowly over the longer period $[T, \tilde{T}]$, it can be perceived as if the solvent trader takes over the entire inventory

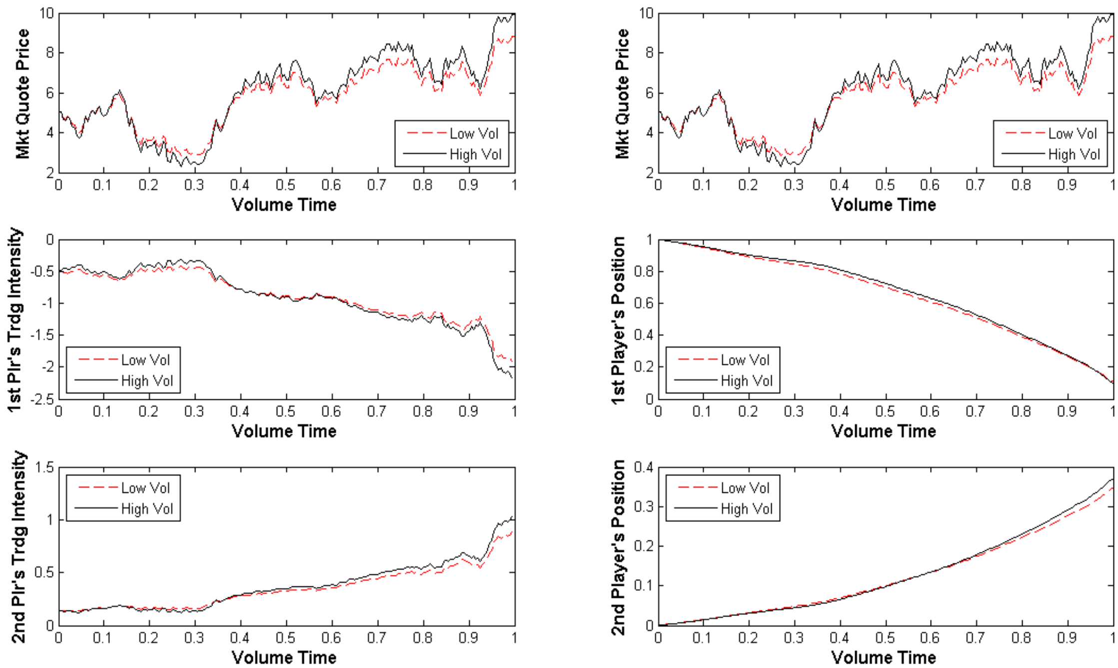

Figure 15. For a common innovation process $(W(t))_{t\in[0,T]}$, the simulated sample paths of how the market quote price and positions of the two players interact during the predatory trading game, under different volatility settings. The sample paths of the players' controls, i.e. the trading intensity processes, are illustrated as well in the lower two panels on the left hand side. The position processes (the lower two panels on the right hand side) are the mere integrals of the trading intensity processes.

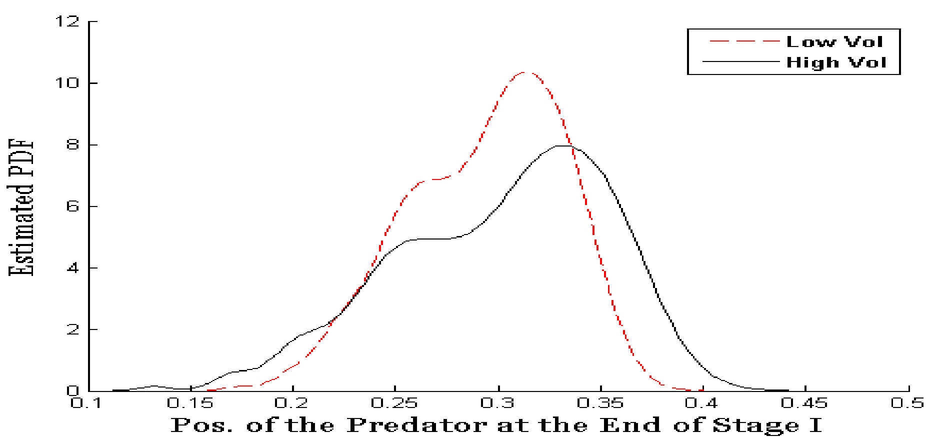

Figure 16. Gaussian kernel density estimate for $Y(T)$, position of the predator at the end of the first stage, for two different volatility levels. The distribution of $Y(T)$ is clearly more tilted to the right in the higher volatility case than in the lower volatility case.

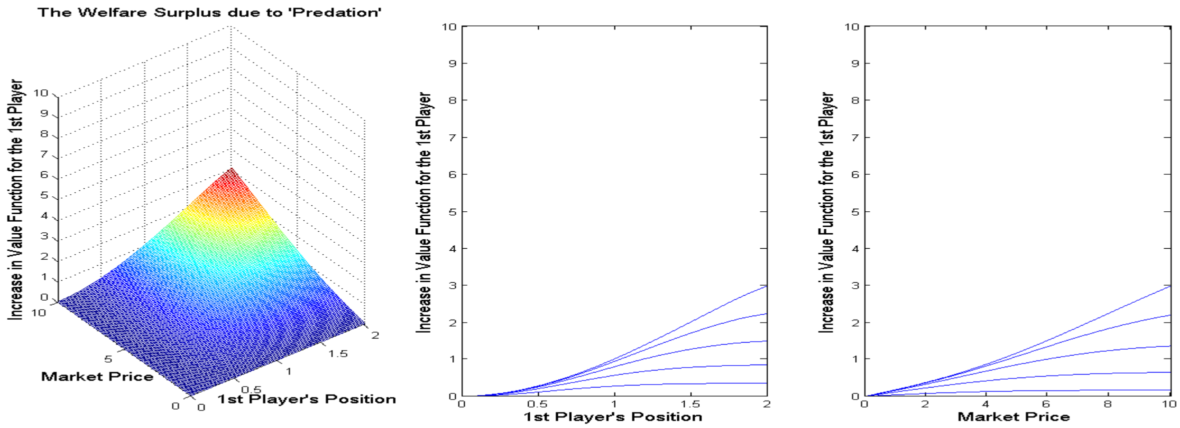

Figure 17. Increase over the (x, z)-plane in the value function for the distressed trader due to the presence of the predator. The middle and right panels further display cross sections of the value function increases for five equally spaced levels of the alternative variables.

of the distressed trader, sells on behalf of him over her longer trading horizon, and charges the distressed trader a premium for the liquidation service. Henceforth, it is not surprising that the welfare of the distressed trader also increases due to the presence of the solvent trader, although each player optimizes his or her own interest only, and this Pareto welfare increase happens solely as a result of a completely noncooperative game. This feature of our model is consistent with the discussion of (27).

5. Summary

We focused on a specific market-liquidity research application, and presented a stochastic differential game model of predatory trading allowing participants to use closed-loop strategies. In our model, the noise traders aggregate plays a non-trivial role in the evolution of the predatory trading phenomenon. This is in sharp contrast to the previous studies (e.g. (11), (13), or (27)) which limited their analyses to open-loop strategies, reducing the problem to a deterministic game between oligopolistic players, while the rest of the market participants did not matter. Our model recognizes that in addition to market plasticity and market elasticity, the volatility of the market is an important parameter influencing the outcome of predatory trading. Among other things, we show that the predator is likely to enjoy a higher expected profit under a more volatile market environment, and that the distressed trader may benefit from the presence of the predator under some market conditions.

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We would like to thank Markus Brunnermeier for engaging discussions on predatory trading, and Robert Almgren for insights concerning market impact models and optimal execution in continuous time.

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Appendix A: Proof of Theorem (2.1)

We first prove that the function f is odd. Without loss of generality we assume that the starting position $X(0)$ of a hands-clean trade is zero. For a deterministic hands-clean trading path $(\xi(t))_{t\in[0,T]}$, we have

$$\begin{aligned}\Pi &= \mathbb{E}\left[-\int_0^T (Z(t)+g(\xi(t)))\xi(t)dt\right]\\ &= \mathbb{E}\left[-\int_0^T Z(t)dX(t) - \int_0^T g(\xi(t))\xi(t)dt\right]\\ &= \mathbb{E}\left[-\,Z(t)X(t)|_0^T + \int_0^T X(t)\left(f(\xi(t))\mathrm{d}t + \sigma(t,Z(t))\mathrm{d}W(t)\right) - \int_0^T g(\xi(t))\xi(t)dt\right]\\ &= \int_0^T X(t)f(\xi(t))dt - \int_0^T g(\xi(t))\xi(t)dt + \mathbb{E}\left[\int_0^T X(t)\sigma(t,Z(t))\mathrm{d}W(t)\right], \qquad \text{(A1)}\end{aligned}$$

where the integral $\int_0^T X(t)\sigma(t,Z(t))\,dW(t)$ is known to be a local martingale. In addition, since we require that the process $(X(t))_t$ be bounded and virtually all major volatility models satisfy the technical condition in (23) that $(\sigma(t,Z(t)))_{t\in[0,T]} \in \mathcal{H}^2_{[0,T]}$, we can conclude that the integrand is in $\mathcal{H}^2_{[0,T]}$ as well. Therefore, $\int_0^T X(t)\sigma(t,Z(t))\,dW(t)$ is indeed a true martingale and its expectation is 0, and the expected return of a deterministic hands-clean scheme can be re-expressed as

$$\Pi = \mathbb{E}\left[-\int_0^T \tilde{Z}(t)\xi(t)dt\right] = \int_0^T X(t)f(\xi(t))dt - \int_0^T g(\xi(t))\xi(t)dt. \qquad \text{(A2)}$$

Let us now consider the following trading strategy: over the first half of the time interval, let the trader increase his position at the constant rate $+\xi$, and decrease it back to 0 at the constant rate $-\xi$ for the second half. Using (A2), the expected return reads:

$$\begin{aligned}\Pi &= \int_0^{T/2} \xi\, t\, f(\xi)dt + \int_{T/2}^T \xi(T-t)f(-\xi)dt - \int_0^{T/2} g(\xi)\xi dt - \int_{T/2}^T g(-\xi)(-\xi)dt\\ &= \frac{T^2}{8}\xi\left(f(\xi)+f(-\xi)\right) + \frac{T}{2}\xi\left(g(-\xi)-g(\xi)\right). \qquad \text{(A3)}\end{aligned}$$

Note that if the value of ξ is negative, meaning selling short in the first half and buying back to become flat in the end, the expression for Π takes exactly the same form.

If there exists a $\hat{\xi}$ such that $f(\hat{\xi})+f(-\hat{\xi})>0$, then we can let $\xi=|\hat{\xi}|$ and the coefficient for the T^2 term becomes $\frac{1}{8}\xi\left(f(\xi)+f(-\xi)\right)>0$, and there always exists a positive T such that $\Pi>0$, contradictory to the no-arbitrage condition (2). If there exists a $\hat{\xi}$ such that $f(\hat{\xi})+f(-\hat{\xi})<0$, then we can let $\xi=-|\hat{\xi}|$ and the coefficient for the T^2 term becomes again $\xi\left(f(\xi)+f(-\xi)\right)/8>0$, leading to contradictions of the no-arbitrage condition (2). Therefore we have verified that for all values of ξ, one has $f(-\xi)=-f(\xi)$.

Now, in order to prove that f is linear, we make use of (A2) once more, and consider the following deterministic hands-clean trading strategy: the trader increases his position at the constant rate $+\xi_1$ over the period $[0,\xi_2 T/(\xi_1+\xi_2)]$ and decrease his position back to 0 at the constant rate $-\xi_2$ over the period $[\xi_2 T/(\xi_1+\xi_2),T]$. His expected return is:

$$\begin{aligned}
\Pi &= \int_0^{\frac{\xi_2}{\xi_1+\xi_2}T}\xi_1\,t\,f(\xi_1)dt+\int_{\frac{\xi_2}{\xi_1+\xi_2}T}^{T}\xi_2(T-t)f(-\xi_2)dt\\
&\quad-\int_0^{\frac{\xi_2}{\xi_1+\xi_2}T}g(\xi_1)\xi_1dt-\int_{\frac{\xi_2}{\xi_1+\xi_2}T}^{T}g(-\xi_2)(-\xi_2)dt\\
&=\xi_1 f(\xi_1)\frac{1}{2}\cdot\frac{\xi_2^2T^2}{(\xi_1+\xi_2)^2}+\xi_2 f(-\xi_2)\frac{1}{2}\cdot\frac{\xi_1^2T^2}{(\xi_1+\xi_2)^2}-g(\xi_1)\xi_1\frac{\xi_2T}{\xi_1+\xi_2}+g(-\xi_2)\xi_2\frac{\xi_1T}{\xi_1+\xi_2}\\
&=\frac{T^2}{2}\frac{\xi_1\xi_2}{(\xi_1+\xi_2)^2}\left(\xi_2 f(\xi_1)-\xi_1 f(\xi_2)\right)+T\frac{\xi_1\xi_2}{\xi_1+\xi_2}\left(g(-\xi_2)-g(\xi_1)\right),
\end{aligned}\tag{A4}$$

where we used the oddness of f in the third equality.

We first restrict ourselves to positive ξ_1 and ξ_2. If there exist $\hat{\xi}_1$ and $\hat{\xi}_2$ such that $\hat{\xi}_2 f(\hat{\xi}_1)-\hat{\xi}_1 f(\hat{\xi}_2)>0$, then we simply let $\xi_1=\hat{\xi}_1$ and $\xi_2=\hat{\xi}_2$ and the coefficient for the T^2 term becomes $\frac{1}{2}\frac{\xi_1\xi_2}{(\xi_1+\xi_2)^2}\left(\xi_2 f(\xi_1)-\xi_1 f(\xi_2)\right)>0$ contradicting the no-arbitrage condition (2). If there exist $\hat{\xi}_1$ and $\hat{\xi}_2$ such that $\hat{\xi}_2 f(\hat{\xi}_1)-\hat{\xi}_1 f(\hat{\xi}_2)<0$, then we pick $\xi_1=\hat{\xi}_2$ and $\xi_2=\hat{\xi}_1$, and again have $\frac{1}{2}\frac{\xi_1\xi_2}{(\xi_1+\xi_2)^2}\left(\xi_2 f(\xi_1)-\xi_1 f(\xi_2)\right)>0$. Therefore, for all positive values of ξ_1 and ξ_2, we always have $\xi_2 f(\xi_1)-\xi_1 f(\xi_2)=0$, and there exists a positive constant γ such that $f(\xi)=\gamma\,\xi$ for all $\xi>0$. We conclude the proof using the oddness of f.

Appendix B: Derivation of the Scrap Value for the Predator via a VWAP Execution

Suppose that the predator enters the extra period $[T,\tilde{T}]$ with a remnant position of $Y(T)$ in the risky asset. As suggested in the optimal execution literature (9) (2), a VWAP execution is a reasonable choice for a risk-neutral agent like the predator. Thus, she just trades the risky asset at the constant speed of $\overline{\eta}=-Y(T)/(\tilde{T}-T)$ with respect to the volume timer t. By (4), the market quote price satisfies:

$$Z(t)=Z(T)+\gamma\,\overline{\eta}\cdot(t-T)+\sigma\left(W(t)-W(T)\right),\qquad t\in[T,\tilde{T}].\tag{B1}$$

Over every infinitesimal time interval $[t,t+\mathrm{d}t]$, because of the temporary market impact, she has to pay the amount $g(\overline{\eta})\,\overline{\eta}dt$ due to execution slippage. To be consistent with the model setup in Section 3, we still take $g(\overline{\eta}) = \lambda \cdot \overline{\eta}$. Also, recall that we do not discard the possibility that the risky asset may become defunct, and once its market price hits 0, it gets delisted from the exchange. In such a case, our protagonist settles whatever short or long position left on her book at no cost. Thus, if we denote $\tau = \inf\{t \geq T \,|\, Z(t) = 0\}$ the first hitting time of 0 by the market price process, the expected cash revenue (a negative value stands for a payment) is:

$$\begin{aligned}
\hat{S}(y,z) &= -\,\mathbb{E}\left[\int_T^{\tilde{T}\wedge\tau} Z(t)\,\overline{\eta}dt + \int_T^{\tilde{T}\wedge\tau} g(\overline{\eta})\,\overline{\eta}dt \mid \begin{bmatrix} Y(T) \\ Z(T) \end{bmatrix} = \begin{bmatrix} y \\ z \end{bmatrix}\right] \\
&= -\,\mathbb{E}\left[\int_0^{\hat{T}\wedge\hat{\tau}} \left(\overline{\eta}\,z + \lambda\,\overline{\eta}^2 + \gamma\,\overline{\eta}^2\,\hat{t} + \overline{\eta}\,\sigma\,\hat{W}(\hat{t})\right) d\hat{t}\right] \\
&= -\,\mathbb{E}\left[\int_0^{\hat{T}} \overline{\eta}\left(z + \lambda\,\overline{\eta} + \gamma\,\overline{\eta}\,\hat{t} + \sigma\,\hat{W}(\hat{t})\right) \mathbf{1}_{\{\hat{Z}^*(\hat{t})\geq 0\}} d\hat{t}\right] \\
&= -\,\overline{\eta}\int_0^{\hat{T}} \mathbb{E}\left[\left(z + \lambda\,\overline{\eta} + \gamma\,\overline{\eta}\,\hat{t} + \sigma\,\hat{W}(\hat{t})\right) \mathbf{1}_{\{\hat{Z}^*(\hat{t})\geq 0\}}\right] d\hat{t}
\end{aligned} \tag{B2}$$

where $\hat{t} = t - T$, $\hat{T} = \tilde{T} - T$, $\hat{W}(\hat{t}) = W(T + \hat{t}) - W(T)$, $\overline{\eta} = -y/\hat{T}$, $\hat{Z}^*(\hat{t}) = \min_{\hat{s}\in[0,\hat{t}]} \hat{Z}(\hat{s})$ is the *running minimum* of the truncated process $\hat{Z}(\hat{t}) = Z(T + \hat{t})$ for $\hat{t} \in [0, \hat{T}]$, and $\hat{\tau} = \tau - T = \inf\{\hat{t} \geq 0 \,|\, z + \gamma\,\overline{\eta}\,\hat{t} + \sigma\,\hat{W}(\hat{t}) = 0\}$. Note that we obtained the second equality using the Markov property of the process $Z(t)$.

To calculate the expected value inside the above integral, we use the following standard result on Brownian motion with drift: if $a > 0$, $\mu \in \mathbb{R}$, and $\sigma > 0$ and $\tau = \inf\{t \geq 0; a + \mu\,t + \sigma\,W(t) < 0\}$,

$$\mathbb{E}\left[(a + b + \mu\,t + \sigma\,W(t))\,\mathbf{1}_{\{\tau\geq t\}}\right] = (a+b+\mu\,t)\,\Phi\left(\frac{a+\mu\,t}{\sigma\sqrt{t}}\right) + (a-b-\mu\,t)\,e^{-\frac{2\mu\,a}{\sigma^2}}\,\Phi\left(\frac{-\,a+\mu\,t}{\sigma\sqrt{t}}\right).$$

Indeed, applying this result to (B2), we get a simple expression for the scrap value of the predator:

$$\begin{aligned}
\hat{S}(y,z) &= -\,\mathbb{E}\left[\int_T^{\tilde{T}\wedge\tau} Z(t)\,\overline{\eta}dt + \int_T^{\tilde{T}\wedge\tau} g(\overline{\eta})\,\overline{\eta}dt \mid \begin{bmatrix} Y(T) \\ Z(T) \end{bmatrix} = \begin{bmatrix} y \\ z \end{bmatrix}\right] \\
&= -\,\overline{\eta}\int_0^{\hat{T}}\left[(z + \lambda\,\overline{\eta} + \gamma\,\overline{\eta}\,\hat{t})\,\Phi\left(\frac{z + \gamma\,\overline{\eta}\,\hat{t}}{\sigma\sqrt{\hat{t}}}\right) + (z - \lambda\,\overline{\eta} - \gamma\,\overline{\eta}\,\hat{t})\,e^{-\frac{2\gamma\,\overline{\eta}\,z}{\sigma^2}}\,\Phi\left(\frac{-\,z + \gamma\,\overline{\eta}\,\hat{t}}{\sigma\sqrt{\hat{t}}}\right)\right] d\hat{t} \\
&= y\int_0^1\left[(z - \lambda\,\frac{y}{\hat{T}} - \gamma\,y\,\theta)\,\Phi\left(\frac{z - \gamma\,y\,\theta}{\sigma\sqrt{\hat{T}\,\theta}}\right) + (z + \lambda\,\frac{y}{\hat{T}} + \gamma\,y\,\theta)\,e^{\frac{2\gamma\,z\,y}{\sigma^2\hat{T}}}\,\Phi\left(-\,\frac{z + \gamma\,y\,\theta}{\sigma\sqrt{\hat{T}\,\theta}}\right)\right] d\theta
\end{aligned}$$

which is the analytic form we used in (19) of Section (3).