A STRUCTURAL MODEL FOR ELECTRICITY PRICES

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Abstract. In this paper we propose a new and highly tractable structural approach to spot price modeling and derivative pricing in electricity markets, thus extending the growing branch of literature which describes power price dynamics via its primary supply and demand factors. Using a bid stack approach, our model translates the demand for power and the prices of fuels, used in the power generation process, into spot prices for electricity. We capture both the heavy-tailed nature of spot prices and the complex dependence structure between power and its underlying factors (fuel prices and demand), while retaining simple and commonly used assumptions on the distributions of these factors. Moreover, the derived spot price process then leads to closed form formulae for forward contracts on electricity and for dark and spark spread options, which are widely used for the valuation of power plants. As the stack structure and merit order dynamics are embedded into the model and fuel forward prices are inputs into the formulae, we capture a much richer and more realistic dependence structure than can be achieved through classical reduced-form price models. We illustrate this advantage through several comparisons with other common models for spread option pricing such as Margrabe's formula and a simple cointegration approach to power and fuels.

1. Introduction

Since the onset of electricity market deregulation in the 1990s, the modeling of prices in these markets has become an important topic of research both in academia and industry. The valuation of both physical assets and financial contracts requires a sophisticated model to capture the unusual features of the market. Key challenges include prominent periodicities and mean-reversion at various time scales, the sudden and erratic spikes that are most striking in historic spot price data, and the strong relationship between the prices of electricity and the commodities that are used for its production (see Figure ?? for sample daily historical spot prices). While most of these features stem from the non-storability of electricity and the resulting matching of supply and demand at all times, much literature in the area has either ignored or oversimplified these links with underlying supply and demand considerations, primarily in the interest of mathematical simplicity and the availability of convenient closed-form expressions derivative prices. In this work, we aim to exploit the well-known relationships between power price and its primary drivers and yet maintain the advantage of closed-form expressions for spot, forward and option prices.

The existing literature on electricity price modeling can be approximately divided into three categories. The first one models the dynamics of the electricity forward curve directly (c.f. [?] and [?]). In this setting, spot prices are treated as forwards with instantaneous delivery but are not a focus of the approach. Furthermore, forward curve models typical ignore fuel prices, or introduce them as exogenous correlated processes, and are hence not successful at capturing the important afore mentioned dependence structure between fuels and electricity. The second category, which we refer to as reduced-form spot price models, is closest to the paradigm of thought in equity markets, the geometric Brownian motion model for the evolution of stock prices. In this case, the starting point is an exogenously specified stochastic process for the electricity spot price for which many variations have been proposed (c.f. [?], [?], [?], [?], [?], [?], [?], [?]). Derivatives are then valued under an equivalent martingale measure, the pricing measure. Like the direct modeling of the forward curve, this approach also suffers from the inability to capture the intricate relationships between fuel and electricity prices described in this paper. Further, spikes are usually only obtained through the inclusion of jump processes or regime switches, which provide little insight into the causes that underly these sudden price swings.

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The structural approach to electricity price modeling stems from the seminal work by Barlow (c.f. [?]). We use the adjective structural to describe models, which — to varying degrees of detail and complexity — explicitly construct the supply curve in electricity markets (commonly known as the bid stack due to the price-setting auction)[1]. The market price is then obtained under the equilibrium assumption that demand and supply have to match. In Barlow's work the bid stack is simply an exogenously specified parametric function, which is evaluated at a random demand level. Later works have refined the modeling of the bid curve and taken into account its dependency generating capacity available (c.f. [?], [?], [?], [?], [?]), as well as fuel prices (c.f. [?], [?], [?], [?], [?]) and / or the cost of carbon emissions (c.f. [?], [?]). The raison d'être of all structural models is very clear. If the bid curve is chosen appropriately, then observed stylized facts of historic data — including the occurrence of sudden spikes — can be well matched. Moreover, because price formation is explained using fundamental variables and costs of production, these models offer insight into the causal relationships in the market; for example, price spikes are observed to coincide with states of high very demand or low capacity; similarly, at times of low demand, power prices are correlated more closely with fuel prices of cheaper technologies, while at times of high demand, more expensive fuels tend to set the power price. As a direct consequence, this class of models also performs best at capturing the varied dependencies between electricity, fuel prices, demand and capacity.

The model we propose falls into the category of structural models. For the purpose of parametrization of the bid stack, several authors have chosen an exponential function of demand (c.f. [?],[?]), while others have stressed the need for a 'heat rate function' multiplicative in the fuel price (c.f. [?]). We build on both of these concepts. We extend the analysis to two or more fuels, allowing for switch in the merit order. Capturing the delicate interplay between demand, capacity, and multiple fuel prices is a challenging undertaking, which is addressed by very few authors, particularly in a derivative pricing framework. In [?], Coulon and Howison construct the stack by approximating the distribution of the clusters of bids from each technology, but their approach relies heavily on numerical methods when it comes to derivative pricing. In [?], Aid *et al* simplify the stack construction by allowing only one heat rate (generator efficiency) per fuel type, a significant oversimplification of spot price dynamics for mathematical convenience. In [?], this work is extended to improve spot price dynamics and capture spikes, but at the expense of a static merit order, ruling out, among other things, the possibility that coal and gas to change positions in the stack in the future. The substantial drop in US natural gas prices in recent years due to shale gas discoveries provides a good example of the need to account for both future merit order changes and the overlap of bids from different technologies, particularly for longer term problems like plant valuation.

Our work extends the current status quo by providing closed-form formulae for the prices of a number of derivative products in a market driven by more than one underlying fuel and a variety of efficiencies. In particular, under only mild assumptions on the distribution under the pricing measure of the terminal value of the processes representing electricity demand and fuels, we obtain explicit formulae for spot prices, forwards and spark and dark spread options. This is computationally more efficient than the semi-closed form results obtained in [?] and [?]. Our formulae also capture very clearly the dependency of electricity derivatives upon the prices of forward contracts written on the fuels that are used in the production process. Our model of the bid stack allows the merit order, the ordering according to which fuels are arranged in the stack, to be dynamic: each fuel can become on the margin and set the market price of electricity. Alternatively several fuels can jointly set the price: Monitoring Analytics publishes hourly data on the types of fuels at the margin in PJM. Collecting this data from the period 2004-2010 reveals that the electricity price was fully set by a single technology (only one marginal fuel) in only 16.1% of the hours in this period. For the year 2010 alone, the number drops to less than 5%. This is different to [?], where the merit order is assumed to be static and only one fuel is allowed to be at the margin. While retaining mathematical tractability, our model of the bid stack adheres to many of the true features of the bid stack structure, and reproduces observed correlations and price dynamics, .

§**??** reviews the price formation mechanism in power markets and - in a very general setting - the definition of the spot price as the equilibrium between demand and supply. We introduce the explicit form of the bid stack in §**??** where we obtain our first closed form formulae. §**??** is dedicated to

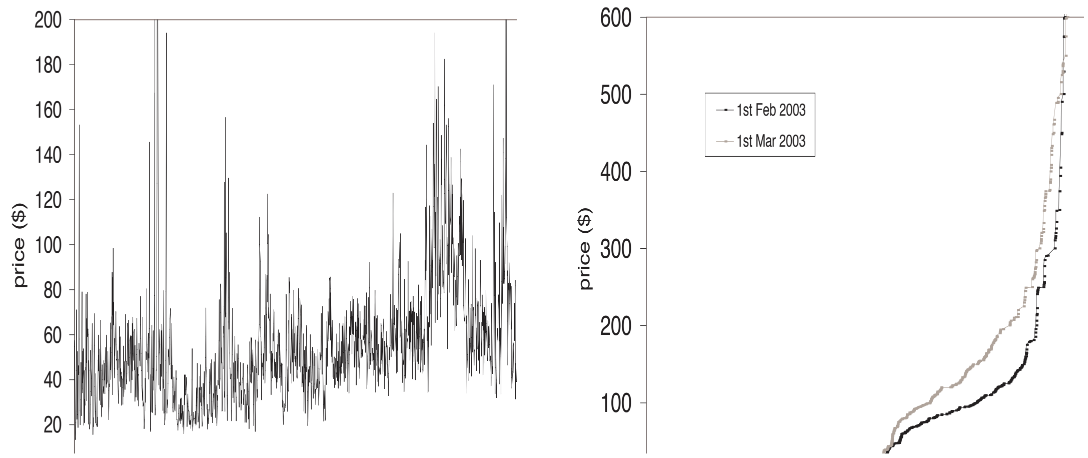
[1]We exclude here any discussion of other categories of models which are much less suitable to derivative pricing. These include unit commitment models which require large optimizations and/or very detailed market knowledge, as well as models of strategy bidding (c.f. [?]) and other equilibrium approaches (c.f. [?]).

forward contracts and in §?? we present our spread option analysis. §?? presents a simple extension of the model, which allows us to realistically capture spikes and §?? compares the results produced by our model to alternative approaches in the literature.

2. Structural approach to electricity pricing

In the following we work on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For a fixed time horizon $T \in \mathbb{R}_+$, we define the $(n+1)$-dimensional standard Wiener process[2] $(W_t^0, \mathbf{W}_t)_{t\in[0,T]}$, where $\mathbf{W} := (W^1, \ldots, W^n)$. Let $\mathcal{F}^0 := (\mathcal{F}_t^0)$ denote the filtration generated by W^0 and $\mathcal{F}^W := (\mathcal{F}_t^W)$ the filtration generated by $\mathbf{W}$. Further, we define the market filtration $\mathcal{F} := \mathcal{F}^0 \vee \mathcal{F}^W$. All relationships between random variables are to be understood in the almost surely sense.

2.1. **Price Setting in Electricity Markets.** We consider a market in which individual firms generate electricity. All firms submit day-ahead bids to a central market administrator, whose task is to allocate the production of electricity amongst them. Each firm's bids take the form of price-quantity pairs representing an amount of electricity the firm is willing to produce, and the price at which the firm is willing to sell it[3]. Firms differ in their characteristics such as their production and operation costs and their profit margins. Consequently, also the collection of bids submitted to the market administrator is expected to exhibit significant variation. An important part is therefore played by the merit order, a rule by which cheaper production units are called upon before more expensive ones in the electricity generation process. This ultimately guarantees that electricity is supplied at the lowest possible price. This assumption is in fact the result of price formation in a competitive equilibrium model. In the form we described, it is a simplification of the complicated unit commitment problem typically solved by optimization in order to satisfy various operational constraints of generators, as well as transmission constraints. Details vary from market to market and we do not address these issues here.



(a) *Historical daily peak prices in PJM* **(b)** *Sample bid stacks from PJM*

Figure 1. *Daily average real time peak prices (left) and a pair of sample bid stacks (right) for the PJM market in the North East US.*

Assumption 1. The market administrator arranges bids according to the merit order and hence in increasing order of costs of production.

[2]Throughout the paper, whenever a stochastic process is defined for $t \in [0, T]$, we will drop the subscript and simply use the bracket notation.

[3]Alternatively, firms may, in some markets, submit continuous bid curves, which map an amount of electricity to the price at which it is offered. For our purposes this distinction will however not be relevant.

We refer to the resulting mapping from the total supply of electricity and the factors that influence the bid levels to the price of the marginal unit as the *market bid stack* and assume that it can be represented by a measurable, function

$$b : [0, \bar{\xi}] \times \mathbb{R}^n \ni (\xi, x) \hookrightarrow b(\xi, x) \in \mathbb{R}.$$

which will be assumed to be strictly increasing in its first variable. Here, $\bar{\xi} \in \mathbb{R}_+$ represents the combined capacity of all generators in the market, henceforth the *market capacity* (measured in MW), and $x \in \mathbb{R}^n$ represents factors whose values are available to the firms when making their bids.

Demand for electricity is assumed to be price-inelastic and given exogenously by an $\mathcal{F}_t^0$-adapted process (D_t) (measured in MW). The market responds to this demand by generating electricity. It supplies an amount $\xi_t \in (0, \bar{\xi}]$. We choose the interval to be half open, as it does not make sense to define a price for zero supply. We assume that the market is in equilibrium with respect to the supply of and demand for electricity; i.e.

$$D_t = \xi_t, \quad \text{for } t \in [0, T]. \tag{1}$$

This implies that $D_t \in (0, \bar{\xi}]$ for $t \in [0, T]$ and (ξ_t) is $\mathcal{F}_t^0$-adapted.

The *market price of electricity* (P_t) is now defined as the price at which the last unit that is needed to satisfy demand sells its electricity; i.e. using (**??**),

$$P_t := b(D_t, \cdot), \quad \text{for } t \in [0, T]. \tag{2}$$

We emphasize the different roles played by the first (i.e. the demand) and all subsequent (i.e. factors influencing the bid levels) variables of the bid stack function b. Due to the inelasticity assumption, the level of demand fully determines the quantity of electricity that is being generated; all subsequent variables merely impact the merit order arrangement of the bids.

Remark 1. The price setting mechanism described above directly applies to day-ahead spot prices set by uniform auctions. However, we believe that it still offers a good approximation to the relationships driving real-time prices as well.

Figure **??** shows a sample of two bid stacks constructed from publicly available historical bid data from the PJM market in the US. One striking feature to notice is the difference between the flatness of the stack near zero and its steepness near 70,000 MW, clearly responsible for the heavy-tailed and spikey prices observed in the data. Secondly, note the change in curve between February and March 2003, a period which witnessed rapid natural gas price increases. While the bids shift upwards, this shift is confined to the upper part of the curve where gas typically features, while the lower portion remained unchanged.

2.2. **Mathematical Model of the Bid Stack.** From the previous subsection, it is clear that the price of electricity in a structural model like the one we are proposing depends critically on the construction of the function b. Before we explain how this is done in the current setting, we make the following assumption about the formation of firms' bids.

Assumption 2. Bids are driven by costs. In particular,

(A2.1) costs depend on fuel prices and firm-specific characteristics only;
(A2.2) firms' marginal costs are strictly increasing.

Remark 2. Although bids are not required to follow fuel costs, evidence in recent studies (c.f. [**?**]) suggests that this relationship is very strong, and that strategic bidding is often either limited, or does not substantially weaken fuel price dependence in the dynamics of bids.

Different generators use different technologies. Therefore, production costs are linked to different fuel prices (e.g. coal, natural gas, lignite, oil, etc.). Furthermore, within each fuel class, the cost of production may vary significantly, for example as old generators may have a higher heat rate (lower efficiency) than new units.

It is not our aim to provide a mathematical model that explains how to aggregate individual bids or captures strategic bidding. We group together generators that use the same fuel type and assume the resulting bid curve to be exogenously given and to satisfy Assumption **??**. Our strategy then is to apply the merit order to this collection of bid curves in order to construct the market bid stack.

The advantage of this approach is that it allows us to capture in a very tractable way the influence of fuel prices on the merit order.

Let I denote the index set of all fuel types (we assume there are n of them) that are being used in the market to generate electricity. With each $i \in I$ we associate an $\mathcal{F}_t^W$-adapted *fuel price process* (S_t^i) and we define the *fuel bid curve* for fuel i to be a measurable function

$$b_i : (0, \bar{\xi}_i] \times \mathbb{R} \ni (\xi, s) \hookrightarrow b_i(\xi, s) \in \mathbb{R},$$

where the argument ξ represents the amount of electricity supplied by generators utilizing fuel type i, s a possible value of the price S_t^i, and $\bar{\xi}^i \in \mathbb{R}_+$ the aggregate capacity of all the generators utilizing fuel type i. We assume that b_i is strictly increasing in its first argument. Further, also for $i \in I$, let the $\mathcal{F}_t$-adapted process . It follows that

$$\sum_{i \in I} \bar{\xi}^i = \bar{\xi}$$

and

$$D_t = \sum_{i \in I} \xi_t^i, \quad \text{for } t \in [0, T].$$

These definitions allow us to deduce P_t^i, the time t bid level at which the collection of generators that uses fuel type i is willing to sell the amount of electricity ξ_t^i; i.e.

$$P_t^i := b_i\left(\xi_t^i, S_t^i\right), \quad \text{for } t \in [0, T].$$

Note in particular that because b_i was assumed to be strictly increasing in its first argument the bid levels satisfy Assumption **??**.

In order to simplify the notation below, for $i \in I$, and for each $s \in \mathbb{R}$ we denote by $b_i(\,\cdot\,, s)^{-1}$ the generalized (right continuous) inverse of the function $\xi \hookrightarrow b_i(\xi, s)$. Recall, that it is defined as

$$b_i(\,\cdot\,, s)^{-1}(p) = \bar{\xi}^i \wedge \inf\{\xi \in (0, \bar{\xi}^i];\, b_i(\xi, s) > p\}$$

where we use the standard convention $\inf \emptyset = +\infty$. Using the notations

$$\underline{b}_i(s) := b_i(0, s) \quad \text{and} \quad \bar{b}_i(s) := b_i(\bar{\xi}^i, s)$$

and writing $\hat{b}_i^{-1}(p, s) = b_i(\,\cdot\,, s)^{-1}(p)$ to ease the notation, we see that $b_i^{-1}(p, s) = 0$ if $p \in (-\infty, \underline{b}_i(s))$, $b_i^{-1}(p, s) = \bar{\xi}^i$ if $p \in [\bar{b}_i(s), \infty)$, and $b_i^{-1}(p, s) \in [0, \bar{\xi}^i]$ if $p \in [\underline{b}_i(s), \bar{b}_i(s))$. For fuel $i \in I$ at price $S_t^i = s$, electricity prices below $\underline{b}_i(s)$ no capacity from the ith technology will be available. Similarly, once all resources from a technology are exhausted, increases in the electricity price will not lead to further production units being brought online. So defined, the inverse function $\hat{b}_i^{-1}$ maps a given price of electricity and the price of fuel i to the amount of electricity supplied by generators relying on this fuel type.

Proposition 1. *For a given vector $(D_t, \mathbf{S}_t)$, where $\mathbf{S}_t := (S_t^1, \ldots, S_t^n)$, the market price of electricity (P_t) is determined for $t \in [0, T]$ by:*

$$P_t = \inf\left\{p \in \mathbb{R} : \sum_{i \in I} \hat{b}_i^{-1}\left(p, S_t^i\right) = D_t\right\}. \tag{3}$$

Proof. By the definition of $\hat{b}_i^{-1}$, the function$\tilde{b}^{-1}$ defined by:

$$\tilde{b}^{-1}(p, s^1, \cdots, s^n) := \sum_{i \in I} \hat{b}_i^{-1}(p, s^i),$$

is when the prices of all the fuels are fixed, a non-decreasing map taking the electricity price to the corresponding amount of electricity generated by the market. As in the case of one fixed fuel price, for each fixed set of fuel prices, say $\mathbf{s} = (s^1, \cdots, s^n)$, we define the bidstack function $\xi \hookrightarrow b(\xi, \mathbf{s})$ as the generalized (right continuous) inverse of the function $\xi \hookrightarrow \tilde{b}^{-1}(p, s^1, \cdots, s^n)$ defined above, namely the function

$$b(\xi, \mathbf{s}) := \inf\{p \in \mathbb{R} : \sum_{i \in I} \hat{b}_i^{-1}\left(p, S^i\right) > \xi\}, \quad \text{for } (\xi, \mathbf{s}) \in \left[0, \bar{\xi}\right] \times \mathbb{R}^n. \tag{4}$$

The desired result follows from the definition of the market price of electricity in (**??**). $\square$

2.3. **Defining a Pricing Measure in the Structural Setting.** The results presented in this paper do not depend on a particular model for the evolution of the demand for electricity and the prices of fuels. In particular, the concrete bid stack model for the electricity spot price introduced in §**??** is simply a deterministic function of the exogenously given factors under the real world measure $\mathbb{P}$. However, for the pricing of derivatives in §**??** and §**??** we need to define a pricing measure $\mathbb{Q}$ and the distribution of the random factors at maturity under this new measure will be important for the results that we obtain later.

For an $\mathcal{F}_t$-adapted process $\boldsymbol{\theta}_t$, where $\boldsymbol{\theta}_t := (\theta_t^0, \theta_t^1, \ldots, \theta_t^n)$, a measure $\mathcal{Q} \sim \mathbb{P}$ is characterized by the Radon-Nikodym derivative

$$\frac{\mathrm{d}\mathcal{Q}}{\mathrm{d}\mathbb{P}} := \exp\left(-\int_0^T \boldsymbol{\theta}_u \cdot \, \mathrm{d}\mathbf{W}_u - \frac{1}{2}\int_0^T |\boldsymbol{\theta}_u|^2 \, \mathrm{d}u\right), \tag{5}$$

where we assume that $(\boldsymbol{\theta}_t)$ satisfies the so-called Novikov condition

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T |\boldsymbol{\theta}_u|^2 \, \mathrm{d}u\right)\right] < \infty.$$

We identify (θ_t^0) with the market price of demand risk and (θ_t^i), $i \in I$, with the market price of risk for fuel i.

We choose to avoid the difficulties of estimating the market price of risk in the most appropriate way (see for example [**?**] for several possibilities) and instead make the following assumption.

Assumption 3. The market chooses a pricing measure $\mathbb{Q} \sim \mathbb{P}$, such that

$$\mathbb{Q} \in \{\mathcal{Q} \sim \mathbb{P} : \text{all discounted prices of traded assets are local } \mathcal{Q}\text{-martingales}\}.$$

Note that we are not making any assumption regarding market completeness here. Because of the non-storability condition, certainly electricity cannot be considered a traded asset. Further, there are different approaches to modeling fuel prices; they may be treated as traded assets (hence local martingales under $\mathbb{Q}$) or - more realistically - assumed to exhibit mean reversion under the pricing measure. Either way, demand is a fundamental factor and the noise (W_t^0) associated with it means that the joined market of fuels and electricity is bound to be incomplete. For the results presented in this paper it does not matter, since all derivative products that we set out to value later on in the paper (forward contracts and spread options) are clearly traded assets and covered by Assumption **??**.

3. Exponential Bid Stack Model

Equation (**??**) in general cannot be solved explicitly. The reason for this is that any explicit solution essentially requires the inversion of the sum of inverses of individual fuel bid curves.

We now propose a specific form for the individual fuel bid curves, which allows us to obtain a closed form solution for the market bid stack b. Here and throughout the rest of the paper, for $i \in I$, we define b_i to be explicitly given by

$$b_i(\xi, s) := s \exp(k_i + m_i \xi), \quad \text{for } (\xi, s) \in [0, \bar{\xi}^i] \times \mathbb{R}_+, \tag{6}$$

where k_i and m_i are constants and m_i is strictly positive. Note that b_i clearly satisfies (**??**) and since it is strictly increasing on its domain of definition it also satisfies (**??**).

We want to briefly comment on our choice of b_i in comparison to three models already considered in the literature on the subject. First, if we excluded the dependency on ξ in the above definition of the fuel bid curves, i.e. if $m_i = 0$, then the model we propose collapses to the one introduced by Aid et al. (c.f. [**?**]), who work with a step function market bid stack. In other words, $m_i = 0$ in their approach. Second, our approach is related to the one suggested by Pirrong and Jermakyan (c.f. [**?**]), who also propose a market bid stack multiplicative in fuel price, but do not specify a parametric form for the function that multiplies the fuel price; i.e. the bid stack's dependency on the variable ξ. Further, they restrict their attention to a one fuel market. Third, compared to the work of Coulon and Howison (c.f. [**?**]), our explicit choice for the bid curves b_i allows us to avoid the numerical computation inherent in their approach and thus to save computation time. Note that the exponential functional form can be thought of in terms of clusters of bids like in [?] if each of these clusters is assumed to be distributed as the exponential of a uniform random variable.

For observed $(D_t, \mathbf{S}_t)$, let us define the sets $M, C \subseteq I$ by

$$M := \{i \in I : \text{generators using fuel } i \text{ are partially used}\}$$

and

$$C := \left\{i \in I : \text{the entire capacity } \bar{\xi}^i \text{ of generators using fuel } i \text{ is used}\right\}.$$

A possible procedure for establishing the members of M and C is to order all the values of $\underline{b}_i$ and $\bar{b}_i$ and determine the corresponding cumulative amounts of electricity that are supplied at these prices. Then find where demand lies in this ordering.

With the above definition of M and C we arrive at the following corollary to Proposition **??**.

Corollary 1. *For b_i of exponential form, as defined in (**??**), the market price of electricity is given explicitly by the left continuous version of*

$$P_t = \left(\prod_{i\in M} \left(S_t^i\right)^{\alpha_i}\right) \exp\left\{\beta + \gamma\left(D_t - \sum_{i\in C} \bar{\xi}^i\right)\right\}, \quad \text{for } t \in [0,T], \tag{7}$$

where

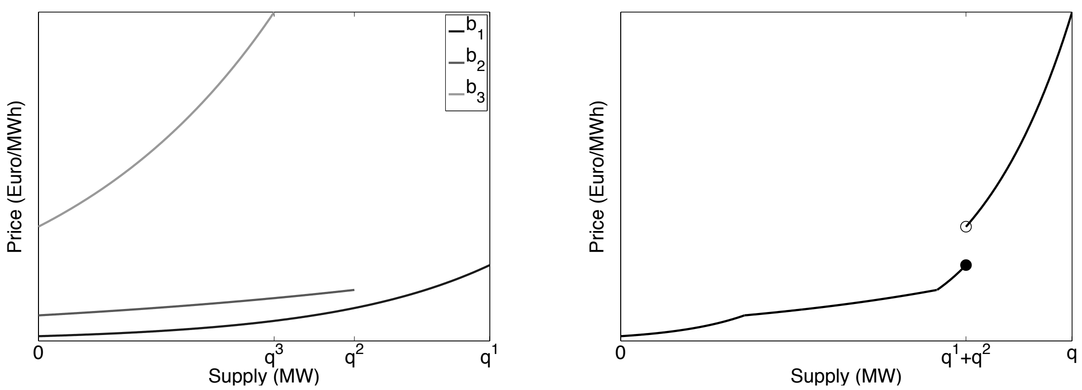
$$\alpha_i := \frac{1}{\zeta}\left(\prod_{j\in M, j\neq i} m_j\right), \quad \beta := \frac{1}{\zeta}\left(\sum_{l\in M} k_l \prod_{j\in M, j\neq l} m_j\right),$$

$$\gamma := \frac{1}{\zeta}\left(\prod_{j\in M} m_j\right) \quad \textit{and} \quad \zeta := \sum_{l\in M} \prod_{j\in M, j\neq l} m_j.$$

Proof. At any time $t \in [0,T]$ the electricity price depends on the composition of the sets M and C; i.e. the current set of marginal and fully utilized fuel types.

For $i \in M$, $\hat{b}_i^{-1} = b_i^{-1}$, for $i \in C$, $\hat{b}_i^{-1} = \bar{\xi}^i$ and for $i \in I \setminus \{M \cup C\}$, $\hat{b}_i^{-1} = 0$. Therefore, we replace I in (**??**) with M and take $\sum_{i\in C} \bar{\xi}^i$ to the right hand side of the equality. Notice that $\sum_{i\in M} \hat{b}_i^{-1}$ can now be inverted and that the inversion yields (**??**).

In order to choose the cheapest electricity price at points of discontinuity of b (as required by the infimum in (**??**)), we consider the left continuous version of (P_t). □



(a) *Fuel bid curves b_i.* **(b)** *Market bid stack b.*

Figure 2. *A schematic of individual fuel bid curves and the resulting market bid stack for $I := \{1,2,3\}$, $q := \bar{\xi}$.*

It is clear from equation (**??**) that the number of possible expressions for the electricity price is fully determined by the different configurations the sets M and C can take. In fact, fluctuations in demand and fuel prices can lead to

$$\sum_{i=1}^{n} \binom{n}{i} \left[\sum_{j=0}^{n-i} \binom{n-i}{j} \right] \tag{8}$$

distinct cases for (**??**). Nonetheless, the market bid stack is always a piece-wise exponential function of demand with constantly evolving shape as fuel prices move. This captures the complex dependency that exists between power and other energy prices.

3.1. **The Case of Two Fuels.** For the remainder of the paper, we restrict our attention to the case of a two-fuel market, consisting of coal and natural gas generators. Our results can in principle be extended to the general case of $n > 2$ fuels. However, the level of complexity of the formulas increases rapidly, as evidenced by the number of possible expressions given in (**??**). We also choose to omit the analysis of the one fuel case, which leads to far simpler expressions throughout, but cannot lead to merit order changes. From now on, we set $I := \{c, g\}$ and carry over all notation introduced in §**??** and §**??**.

From (**??**) we know that there are five possible expressions for the electricity spot price. We list them in Table **??**. Note that fixing D_t reduces this list to three, each of which — depending on the state of $\mathbf{S}_t$ — can set the electricity price. A similar reduction to three expressions occurs by fixing $\mathbf{S}_t$. We exploit this property to write formula (**??**) explicitly in the current two-fuel setting. To simplify the

P_t, for $t \in [0,T]$	Criterion	Composition of M	C
$S_t^c \exp\left(k_c + m_c D_t\right)$	$b_c\left(D_t, S_t^c\right) \le \underline{b}_g\left(S_t^g\right)$	$\{c\}$	$\{\varnothing\}$
$S_t^g \exp\left(k_g + m_g D_t\right)$	$b_g\left(D_t, S_t^g\right) \le \underline{b}_c\left(S_t^c\right)$	$\{g\}$	$\{\varnothing\}$
$S_t^c \exp\left(k_c + m_c\left(D_t - \bar{\xi}^g\right)\right)$	$b_c\left(D_t - \bar{\xi}^g, S_t^c\right) > \bar{b}_g\left(S_t^g\right)$	$\{c\}$	$\{g\}$
$S_t^g \exp\left(k_g + m_g\left(D_t - \bar{\xi}^c\right)\right)$	$b_g\left(D_t - \bar{\xi}^c, S_t^g\right) > \bar{b}_c\left(S_t^c\right)$	$\{g\}$	$\{c\}$
$\left(S_t^c\right)^{\alpha_c}\left(S_t^g\right)^{\alpha_g} \exp\left(\beta + \gamma D_t\right)$	otherwise	$\{c, g\}$	$\{\varnothing\}$

Table 1. *Distinct cases for the electricity price* (**??**) *in the two fuel case.*

presentation in the text below, we define

$$b_{cg}\left(\xi, \mathbf{S}\right) := \left(S^c\right)^{\alpha_c}\left(S^g\right)^{\alpha_g} \exp\left(\beta + \gamma \xi\right), \quad \text{for } \left(\xi, \mathbf{S}\right) \in \mathbb{R}_+^3,$$

where α_c, α_g, β and γ are defined in Corollary **??** and simplify for two fuels to

$$\alpha_c = \frac{m_g}{m_c + m_g}, \quad \alpha_g = 1 - \alpha_c = \frac{m_c}{m_c + m_g}, \quad \beta = \frac{k_c m_g + k_g m_c}{m_c + m_g}, \quad \gamma = \frac{m_c m_g}{m_c + m_g}$$

Further, we set:

$$i_- := \text{argmin}\{\bar{\xi}^c, \bar{\xi}^g\} \qquad \text{and} \qquad i_+ := \text{argmax}\{\bar{\xi}^c, \bar{\xi}^g\}.$$

Corollary 2. *With $I := \{c, g\}$, formula (**??**) can be written explicitly and, for $t \in [0,T]$, the electricity spot price is given by*

$$P_t = b_{low}\left(D_t, \mathbf{S}_t\right) \mathbb{I}_{\left(0, \bar{\xi}^i\right]}(D_t) + b_{mid}\left(D_t, \mathbf{S}_t\right) \mathbb{I}_{\left(\bar{\xi}^i, \bar{\xi}^j\right]}(D_t) + b_{high}\left(D_t, \mathbf{S}_t\right) \mathbb{I}_{\left(\bar{\xi}^j, \bar{\xi}\right]}(D_t),$$

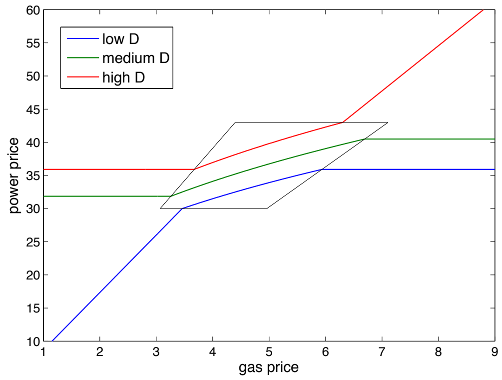
where, for $(\xi, \mathbf{S}) \in \mathbb{R}_+^2$:

$$\begin{aligned} b_{low}\left(\xi, \mathbf{S}\right) := {} & b_c\left(\xi, S^c\right) \mathbb{I}_{\left\{b_c(\xi, S^c) < \underline{b}_g(S^g)\right\}} + b_g\left(\xi, S^g\right) \mathbb{I}_{\left\{b_g(\xi, S^g) < \underline{b}_c(S^c)\right\}} \\ & + b_{cg}\left(\xi, \mathbf{S}\right) \mathbb{I}_{\left\{b_c(\xi, S^c) \ge \underline{b}_g(S^g), b_g(\xi, S^g) \ge \underline{b}_c(S^c)\right\}}, \end{aligned}$$

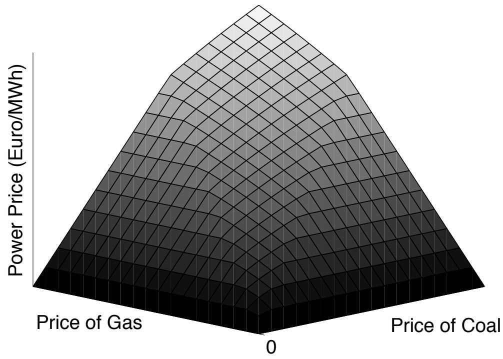
$$\begin{aligned} b_{mid}\left(\xi, \mathbf{S}\right) := {} & b_{i_+}\left(\xi, S^{i_+}\right) \mathbb{I}_{\left\{b_{i_+}(\xi, S^j) < \underline{b}_{i_-}\left(S^{i_-}\right)\right\}} + b_{i_+}\left(\xi - \bar{\xi}^{i_-}, S^{i_+}\right) \mathbb{I}_{\left\{b_{i_+}\left(\xi - \bar{\xi}^{i_+}, S^{i_+}\right) > \bar{b}_{i_-}\left(S^{i_-}\right)\right\}} \\ & + b_{cg}\left(\xi, \mathbf{S}\right) \mathbb{I}_{\left\{b_{i_+}(\xi, S^{i_+}) \ge \underline{b}_{i_-}\left(S^{i_-}\right), b_{i_+}\left(\xi - \bar{\xi}^{i_-}, S^{i_+}\right) \le \bar{b}_{i_-}\left(S^{i_-}\right)\right\}}, \end{aligned}$$

$$b_{high}\left(\xi,\mathbf{S}\right):=b_c\left(\xi-\bar{\xi}^g,S^c\right)\mathbb{I}_{\left\{b_c\left(\xi-\bar{\xi}^g,S^c\right)>\bar{b}_g(S^g)\right\}}+b_g\left(\xi-\bar{\xi}^c,S^g\right)\mathbb{I}_{\left\{b_g\left(\xi-\bar{\xi}^c,S^g\right)>\bar{b}_c(S^c)\right\}}$$
$$+\,b_{cg}\left(\xi,\mathbf{S}\right)\mathbb{I}_{\left\{b_c\left(\xi-\bar{\xi}^g,S^c\right)\le\bar{b}_g(S^g),b_g\left(\xi-\bar{\xi}^c,S^g\right)\le\bar{b}_c(S^c)\right\}}.$$

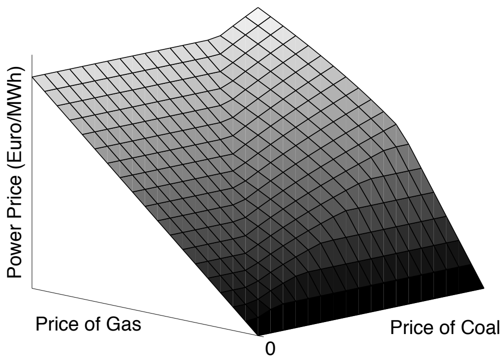
Proof. The expressions for $b_{\rm low}$, $b_{\rm mid}$, $b_{\rm high}$ are obtained from (**??**) by fixing D_t in the intervals $(0,\bar{\xi}^i]$, $(\bar{\xi}^i,\bar{\xi}^j]$, $(\bar{\xi}^i,\bar{\xi}]$ respectively and considering the different scenarios for M and C. □



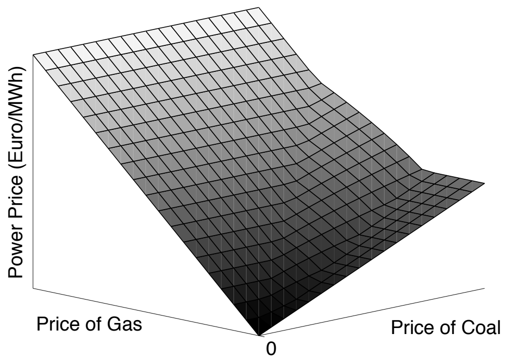
(a) *All demand levels, with coal fixed.*



(b) b_{low}.



(c) b_{mid}.

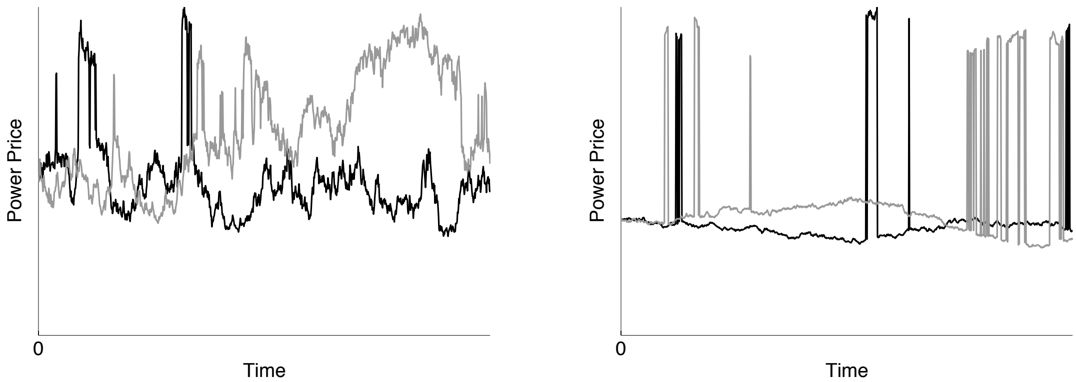


(d) b_{high}.

Figure 3. *Illustration of the dependence of the power spot price on fuel spot prices for three characteristic values of demand*

Figure **??** illustrates the typical dependence structure (for $\bar{\xi}^c>\bar{\xi}^g$) between electricity price and fuel price for each of the demand regimes characterized by Corollary **??**. Figures **(b)**-**(d)** show a two-dimensional surface plotting P_t against S_t^c and S_t^g, while **(a)** plots a representative demand level for all three regimes but with coal fixed. In all cases electricity is non-decreasing in fuel price, and is only constant against S_t^i if fuel i is not at the margin (i.e., $i\notin M$). The quadrilateral in the middle of **(a)** represents the region of gas-coal overlap, the last row of Table**??**. In each demand regimes, Figure **(a)** demonstrates that by increasing gas price from low to high, we move from a region of one fuel at the margin (with P_t linear the marginal fuel), to the overlap region (with P_t non-linear in both fuels) to one fuel at the margin again. Since $i_+=c$ here, coal is marginal both before and after the overlap region in this case.

Figure **??** provides a pair of sample simulated paths generated by the stack model for an arbitrary choice of stack parameters. In the left graph we set $m_c,m_g>0$ as required, while in the right graph we let these both approach zero, to compare with the case of a step function bid stack. Clearly a step function stack with only two steps produces unrealistic power spot price dynamics, while the two-fuel exponential stack model leads to reasonable simulations. Note that here both coal and gas have been taken to be exponential Ornstein-Uhlenbeck (OU) processes, and demand an OU process



(a) $m_c, m_g > 0$ **(b)** *no dependency on* ξ

Figure 4. *Simulation of the power price for typical parameters.*

truncated at 0 and $\bar{\xi}$. However the choice of model for these factors is irrelevant at this stage, as we are emphasizing here the consequence of our choice for the bid stack itself.

4. Forward Contracts

We now turn to the analysis of forward contracts in our structural framework. For the sake of simplicity, we ignore delivery periods and suggest that T be considered as a representative date in a typical monthly delivery period. The reader interested in finding out how to handle delivery over a fixed time period is referred to [**?**]. For the purpose of the present discussion, a *forward contract* with maturity T is defined by the payoff

$$P_T - F_t^p(T),$$

where $F_t^p(T)$ is the delivery price agreed at the initial date t, and paid by the holder of the long position at maturity. Simple arbitrage arguments (c.f. [**?**]) imply that

$$F_t^p(T) = \mathbb{E}^{\mathbb{Q}}\left[P_T | \mathcal{F}_t\right].$$

The result of Corollary **??** shows that the payoff of the forward is a function of demand and fuels, so that the electricity forward contract appears as a derivative on fuel prices and demand.

4.1. **Closed Form Expressions for Forward Prices.** For the explicit calculation of forward prices the following result will be useful. Let φ_1 denote the density of the standard univariate Gaussian distribution, and $\Phi_1(\cdot)$ and $\Phi_2(\cdot, \cdot; \rho)$ the cumulative distribution functions of the univariate and bivariate (with correlation ρ) standard Gaussian distributions respectively.

Lemma 1. *The following relationship holds between* φ_1, Φ_1 *and* Φ_2*:*

$$\int_{-\infty}^{a} \exp\left(l_1 + q_1 x\right) \varphi_1(x) \Phi_1(l_2 + q_2 x)\, \mathrm{d}x = \exp\left(l_1 + \frac{q_1^2}{2}\right) \Phi_2\left(v - q_1, \frac{l_2 + q_1 q_2}{\sqrt{1+q_2^2}}; \frac{-q_2}{\sqrt{1+q_2^2}}\right), \tag{9}$$

where $l_1, l_2, q_1, q_2 \in \mathbb{R}$ *and* $a \in \mathbb{R} \cup \{\infty\}$.

Proof. In equation (**??**) combine the explicit exponential term with the one contained in φ_1 and complete the square. Then, define the change of variable $(x, y) \to (z, w)$ by

$$x = z + q_1,$$
$$y = w\sqrt{1+q_2^2} + q_2(x - q_1).$$

The determinant of the Jacobian matrix J associated with this transformation is given by $|J| = \sqrt{1+q_2^2}$. Performing the change of variable immediately leads to the right hand side of (**??**). □

For the main result in this section we denote by $F_t^i(T)$, $i \in I$, the delivery price of a forward contract on fuel i with maturity T and write $\mathbf{F}_t(T) := (F_t^c(T), F_t^g(T))$.

Proposition 2. *Given $I = \{c, g\}$, if, under $\mathbb{Q}$, the random variables $(\log(S_T^c), \log(S_T^g))$ are jointly Gaussian with mean (μ_c, μ_g), variance (σ_c^2, σ_g^2) and correlation ρ, then for $t \in [0, T]$, the delivery price of a forward contract on electricity is given by:*

$$F_t^p(T) = \int_0^{\bar{\xi}^{i-}} f_{low}(D, \mathbf{F}_t(T))\, \phi_d(D)\ \mathrm{d}D + \int_{\bar{\xi}^{i-}}^{\bar{\xi}^{i+}} f_{mid}(D, \mathbf{F}_t(T))\, \phi_d(D)\ \mathrm{d}D + \int_{\bar{\xi}^{i+}}^{\bar{\xi}} f_{high}(D, \mathbf{F}_t(T))\, \phi_d(D)\ \mathrm{d}D, \tag{10}$$

where ϕ_d denotes the density of the random variable D_T and

$$f_{low}(\xi, \mathbf{F}) = \sum_{i \in I} b_i\left(\xi, F^i\right) \Phi_1\left(R_i(\xi, 0)/\sigma\right) + b_{cg}(\xi, \mathbf{F}) \exp\left(-\alpha_c\alpha_g\sigma^2/2\right)\left[1 - \sum_{i \in I} \Phi_1\left(R_i(\xi, 0)/\sigma + \alpha_j\sigma\right)\right],$$

$$\begin{aligned} f_{mid}(\xi, \mathbf{F}) &= b_{i_+}\left(\xi - \bar{\xi}^{i_-}, F^{i_+}\right) \Phi_1\left(-R_{i_+}\left(\xi - \bar{\xi}^{i_-}, \bar{\xi}^{i_-}\right)/\sigma\right) + b_{i_+}\left(\xi, F^{i_+}\right) \Phi_1\left(R_{i_+}(\xi, 0)/\sigma\right) \\ &+ b_{cg}(\xi, \mathbf{F}) \exp\left(-\alpha_c\alpha_g\sigma^2/2\right)\left[\Phi_1\left(R_{i_+}\left(\xi - \bar{\xi}^{i_-}, \bar{\xi}^{i_-}\right)/\sigma + \alpha_{i\_}\sigma\right) - \Phi_1\left(R_{i_+}(\xi, 0)/\sigma + \alpha_{i\_}\sigma\right)\right], \end{aligned}$$

$$f_{high}(\xi, \mathbf{F}) = \sum_{i \in I} b_i\left(\xi - \bar{\xi}^j, F^i\right) \Phi_1\left(-R_i\left(\xi - \bar{\xi}^j, \bar{\xi}^j\right)/\sigma\right) + b_{cg}(\xi, \mathbf{F}) \exp\left(-\alpha_c\alpha_g\sigma^2/2\right)\left[-1 + \sum_{i \in I} \Phi_1\left(R_i\left(\xi - \bar{\xi}^j, \bar{\xi}^j\right)/\sigma + \alpha_j\sigma\right)\right],$$

where $j = I \setminus \{i\}$ and

$$\begin{aligned} \sigma^2 &:= \sigma_c^2 - 2\rho\sigma_c\sigma_g + \sigma_g^2, \\ R_i(\xi_i, \xi_j) &:= k_j + m_j\xi_j - k_i - m_i\xi_i + \log\left(F_t^j\right) - \log\left(F_t^i\right) - \frac{1}{2}\sigma^2. \end{aligned}$$

The constants $\alpha_c, \alpha_g, \beta, \gamma$ are as defined in Corollary **??**.

Proof. By iterated conditioning, for $t \in [0, T]$, the price of the electricity forward $F_t^p(T)$ is given by

$$F_t^p(T) := \mathbb{E}^{\mathbb{Q}}\left[P_T \middle| \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[b(D_T, \mathbf{S}_T) \middle| \mathcal{F}_T^0 \vee \mathcal{F}_t^W\right] \middle| \mathcal{F}_t\right]. \tag{11}$$

The outer expectation can be written as the sum of three integrals corresponding to the events $\{D_T \in [0, \bar{\xi}^{i-}]\}$, $\{D_T \in [\bar{\xi}^{i-}, \bar{\xi}^{i+}]\}$ and $\{D_T \in [\bar{\xi}^{i+}, 0]\}$ respectively. We consider the first case and derive the f_{low} term. The other cases corresponding to f_{mid} and f_{high} are proven similarly.

From Corollary **??** we know that on the interval under consideration $b = b_{\text{low}}$. This expression for P_T is easily written in terms of independent standard Gaussian variables $\mathbf{Z} := (Z_1, Z_2)$ by using the identity

$$\begin{pmatrix} \log\left(S_T^c\right) \\ \log\left(S_T^g\right) \end{pmatrix} = \begin{pmatrix} \mu_c \\ \mu_g \end{pmatrix} + \begin{pmatrix} \sigma_c^2 & \rho\sigma_c\sigma_g \\ \rho\sigma_c\sigma_g & \sigma_g^2 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}.$$

With the definition $\hat{d} := \mathbb{E}^{\mathbb{Q}}[D_T | \mathcal{F}_T^0]$, the inner expectation can now be written in integral form as

$$\mathbb{E}\left[\tilde{b}_{\text{low}}\left(\hat{d}, \mathbf{Z}\right)\right] = I_c + I_g + I_{cg},$$

where $\tilde{b}_{\text{low}}(\xi, \mathbf{Z}) := b_{\text{low}}(\xi, \mathbf{S})$ and the expectation is computed with respect to the law of $\mathbf{Z}$. For example, after completing the square in z_1,

$$I_c = \int_{-\infty}^{\infty} \exp\left(l_1 + q_1 z_2\right) \phi_1(z_2) \Phi_1(l_2 + q_2 z_2)\ \mathrm{d}z_2,$$

with

$$\begin{aligned} l_1 &:= \mu_c + k_c + m_c \hat{d} + \frac{\sigma_c^4}{2}, \\ l_2 &:= -\sigma_c^2 - \frac{\mu_c + k_c + m_c \hat{d} + \mu_g}{\sigma_c(\sigma_c - \rho\sigma_g)}, \\ q_1 &:= \rho\sigma_c\sigma_g, \\ q_2 &:= \frac{\sigma_g(\sigma_g - \rho\sigma_c)}{\sigma_c(\sigma_c - \rho\sigma_g)}. \end{aligned}$$

Lemma **??** now applies with $a = \infty$. Similarly, I_g and I_{cg} are computed.

Using standard results we identify terms in I_c, I_g and I_{cg} with the prices of forward contracts on fuels. For $i \in I$,

$$F_t^i(T) = \mathbb{E}^{\mathbb{Q}}\left[S_T^i \middle| \mathcal{F}_t \right] = \exp\left(\mu_i + \frac{1}{2}(\sigma_i)^2 \right), \qquad \text{for } t \in [0,T]. \tag{12}$$

Substituting the resulting expression for the inner expectation in the first of the three integrals over demand corresponding to the outer expectation in (**??**) yields the first term in the Proposition. □

Remark 3. The assumption of lognormal fuel prices in Proposition **??** is a very common and natural choice for modeling energy (non-power) prices. The Geometric Brownian Motion modles including constant convenience yield, the classical exponential Ornstein-Uhlenbeck model of Schwartz [**?**], and the two-factor Schwartz-Smith model [**?**] all satisfy the lognormality assumption.

The above result does not depend upon any assumption on the distribution of the demand at maturity, and as a result, it can easily be computed numerically for any distribution. In markets where reasonably reliable load forecasts exist, one may consider demand to be a deterministic function, in which case the integrals appearing in formula (**??**) are not needed and the value of the forward contract becomes explicit. Another convenient special case, which allows us to obtain closed form formulae, is that of a Gaussian distribution truncated at zero and $\bar{\xi}$.

To simplify the notation we introduce the following shorthand notation for linear combinations of Gaussian distribution functions:

$$\Phi_2^{2\times 1}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, y; \rho \right) := \Phi_2(x_1, y; \rho) - \Phi_2(x_2, y; \rho).$$

Corollary 3. *In addition to the assumptions in Proposition* **??** *let demand at maturity satisfy*

$$D_T = \max\left(0, \min\left(\bar{\xi}, X\right)\right),$$

where $X \sim N(\mu_d, \sigma_d^2)$ is independent of $\mathbf{Z}$. Then for $t \in [0,T]$, the delivery price of a forward contract is given explicitly by

$$\begin{aligned} F_t^p(T) = {} & \Phi\left(\frac{-\mu_d}{\sigma_d}\right) \sum_{i\in I} b_i\left(0, F_t^i\right) \Phi\left(\frac{R_i(0,0)}{\sigma}\right) + \Phi\left(\frac{\mu_d - \bar{\xi}}{\sigma_d}\right) \sum_{i\in I} b_i\left(\bar{\xi}^i, F_t^i\right) \Phi\left(\frac{-R_i\left(\bar{\xi}^i, \bar{\xi}^i\right)}{\sigma}\right) \\ & + \sum_{i\in I} b_i\left(\mu_d, F_t^i\right) \exp\left(\frac{m_i^2\sigma_d^2}{2}\right) \Phi_2^{2\times 1}\left(\begin{bmatrix} \frac{\bar{\xi}^i - \mu_d}{\sigma_d} - m_i\sigma_d \\ \frac{-\mu_d}{\sigma_d} - m_i\sigma_d \end{bmatrix}, \frac{R_i(\mu_d, 0) - m_i^2\sigma_d^2}{\sigma_{i,d}}; \frac{m_i\sigma_d}{\sigma_{i,d}} \right) \\ & + \sum_{i\in I} b_i\left(\mu_d - \bar{\xi}^j, F_t^i\right) \exp\left(\frac{m_i^2\sigma_d^2}{2}\right) \Phi_2^{2\times 1}\left(\begin{bmatrix} \frac{\bar{\xi} - \mu_d}{\sigma_d} - m_i\sigma_d \\ \frac{\bar{\xi}^j - \mu_d}{\sigma_d} - m_i\sigma_d \end{bmatrix}, \frac{-R_i\left(\mu_d - \bar{\xi}^j, \bar{\xi}^j\right) + m_i^2\sigma_d^2}{\sigma_{i,d}}; \frac{-m_i\sigma_d}{\sigma_{i,d}} \right) \\ & + b_{cg}(\mu_d, \mathbf{F}_t) \exp(\eta) \Bigg\{ -\sum_{i\in I} \Phi_2^{2\times 1}\left(\begin{bmatrix} \frac{\bar{\xi}^i - \mu_d}{\sigma_d} - \gamma\sigma_d \\ \frac{-\mu_d}{\sigma_d} - \gamma\sigma_d \end{bmatrix}, \frac{R_i(\mu_d, 0) + \alpha_j\sigma^2 - \gamma m_i\sigma_d^2}{\sigma_{i,d}}; \frac{m_i\sigma_d}{\sigma_{i,d}} \right) \\ & + \sum_{i\in I} \Phi_2^{2\times 1}\left(\begin{bmatrix} \frac{\bar{\xi} - \mu_d}{\sigma_d} - \gamma\sigma_d \\ \frac{\bar{\xi}^j - \mu_d}{\sigma_d} - \gamma\sigma_d \end{bmatrix}, \frac{R_i\left(\mu_d - \bar{\xi}^j, \bar{\xi}^j\right) + \alpha_j\sigma^2 + \gamma m_i\sigma_d^2}{\sigma_{i,d}}; \frac{-m_i\sigma_d}{\sigma_{i,d}} \right) \Bigg\}, \end{aligned}$$

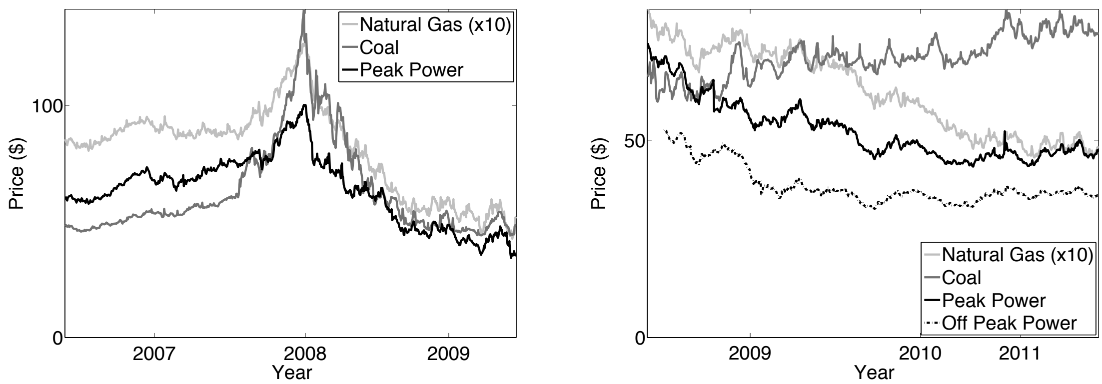
where $j = I \setminus \{i\}$ *and*

$$\sigma_{i,d}^2 := m_i^2 \sigma_d^2 + \sigma^2,$$
$$\eta := \frac{\gamma^2 \sigma_d^2 - \alpha_c \alpha_g \sigma^2}{2}.$$

Proof. The proof again relies on Lemma **??** this time with $a < \infty$. Each of the terms in f_{low}, f_{med} or f_{high} turns into the difference between two bivariate Gaussian cdf's, after integrate over demand. Then various terms can be combined to simplify to the result above.
TODO: The proof of this Corollary as well as the ones for the results on spreads in the next section are all similar to the proof of Proposition **??**. We need to decide how to present them in a concise yet clear way. We could possibly show the calculation details for one term, similarly to the proof of Proposition 2. □

Remark 4. It is interesting to note that maximum capacity $\bar{\xi}^i$ for each fuel enters linearly inside the bivariate Gaussian c.d.f.'s and exponentially outside the c.d.f.'s in each of the terms of Corollary **??**. This suggests that an extension to the case of stochastic capacity is possible if $\bar{\xi}^i$ is independent and Gaussian (truncated at 0). The procedure for calculating this expectation would be similar to the expectation taken over demand above, but result in trivariate Gaussians. On the other hand, if capacity from all fuels is closely correlated then the need for this addition complication is low, since instead a corresponding higher demand volatility σ_d would have a similar effect.

4.2. **Correlation Between Electricity and Fuel Forwards.** The PJM market in the US has a similar capacity from coal and gas generators. Moreover, historically, these are the two fuel types which are most likely to be at the margin and to set the price. Therefore, the region provides a suitable case study for analyzing the dependence structure suggested by our model. In Figure **??** we observe



(a) *Dec 2009 forward price dynamics.* **(b)** *Dec 2011 forward price dynamics.*

Figure 5. *Comparison of power, gas and coal futures prices for two delivery dates.*

the historical co-movement of forward (futures) prices in the PJM electricity market (both peak and off-peak), Henry Hub natural gas (scaled up by a factor of 10) and Central Appalachian coal market. We fix maturities December 2009 and December 2011, and plot the movement in futures prices over the two years just prior to maturity. The first figure covers the period 2007-09, characterized by a peak during the summer of 2008, when almost all commodities set new record highs. Gas, coal and power all moved fairly similarly during this two-year period, though the correlation between power and gas forward prices is most striking. The second figure shows the period 2009-2011, during which, due to shale gas discoveries, gas prices declined steadily over 2010, while coal prices held steady and even increased somewhat. As a result, this period is more revealing, as it corresponds to a time when gas bids merged increasingly with coal bids in the power market merit order. Historically, gas has been higher than coal in the merit order, but recently we have seen an increasing amount of overlap in the stack, as is suggested by the price movements shown.

Our bid stack model implies that the level of power prices should have been impacted both by the strengthening coal price and the falling gas price, leading to a relatively flat power price trajectory.

This is precisely what Figure **??** reveals, with very stable forward power prices during 2010-2011. The close correlation with gas is still visible, but power prices did not fall as much as gas, as they were supported by the price of coal. Finally, we can also see that the spread between peak and off-peak forwards for Dec 2011 delivery has narrowed somewhat, as we would also expect when there is more overlap between coal and gas bids in the stack. This subtle change in price dynamics is crucial for many companies exposed to multi-commodity risk, and is one which is very difficult to capture in a typical reduced-form approach, or indeed in a stack model without a flexible merit order and overlapping fuel types.

5. Spread Options

This section deals with the pricing of spread options in the structural setting presented above. We are concerned with spread options whose payoff is defined to be the difference between the market price of electricity and the cost of the amount of fuel needed by a particular power plant to generate one unit of electricity. If coal is the fuel that features in the payoff the resulting option is known as a *dark spread*, if it is gas the contract is called a *spark spread*. Denoting by $h_c, h_g \in \mathbb{R}_{++}$ the heat rate of coal and gas, dark and spark spread options with maturity T are defined by the payoffs

$$(P_T - h_c S_T^c)^+ \quad \text{and} \quad (P_T - h_g S_T^g)^+ ,$$

respectively. We only consider the dark spread but point out that if one interchanges the subscripts c and g in all results derived in this section they apply to the spark spread case. Further, since spread options are typically traded to hedge physical assets (generating units) the heat rates that feature in the option payoff are usually in line with the efficiency of power plants in the market. Our bid stack model implies a range of heat rates for coal generators. This imposes a restriction on h_c; specifically

$$\exp(k_c) \leq h_c \leq \exp\left(k_c + m_c \bar{\xi}^c\right) . \tag{13}$$

Then, as usually, the value (V_t) of a dark spread is obtained as the discounted conditional expectation under the pricing measure; i.e.

$$V_t = \mathrm{e}^{-r(T-t)} \mathbb{E}^{\mathbb{Q}} \left[P_T - h_c S_T^c \middle| \mathcal{F}_t \right],$$

which thanks to Corollary **??** is understood to be a derivative written on demand and fuels.

5.1. **Closed Form Expressions for Spread Option Prices.** The results derived in this section mirror the ones in Section **??** derived for the forward contract. First, conditioning on demand, we obtain an explicit formula for the price of the spread. Secondly, it is shown that this result can be extended to give a closed form formula in the case of truncated Gaussian demand.

We keep our earlier notation denoting by i_+ and i_- the dominant and the subordinate technology and define

$$\xi^h := \frac{\log h_c - k_c}{m_c},$$

where $0 \leq \xi^h \leq \bar{\xi}^c$. By its definition, ξ^h represents the amount of electricity that can be supplied from coal generators whose heat rate is smaller than or equal to h_c.

Proposition 3. *Given $I = \{c, g\}$, if, under $\mathbb{Q}$, the random variables $(\log(S_T^c), \log(S_T^g))$ are jointly Gaussian distributed with mean (μ_c, μ_g), variance (σ_c^2, σ_g^2) and correlation ρ, then , for $t \in [0, T]$, the* *price of dark spread with maturity T is given by*

(14)
$$
\begin{aligned}
V_t = e^{-r(T-t)} \Bigg\{ \int_0^{\min(\bar{\xi}^g,\xi^h)} v_{low,2}\left(D,\mathbf{F}_t(T)\right)\phi_d(D)\ \mathrm{d}D + \int_{\min(\bar{\xi}^g,\xi^h)}^{\bar{\xi}^{i_-}} v_{low,1}\left(D,\mathbf{F}_t(T)\right)\phi_d(D)\ \mathrm{d}D \\
+ \int_{\bar{\xi}^{i_-}}^{\max(\bar{\xi}^g,\xi^h)} v_{mid,3}\left(D,\mathbf{F}_t(T)\right)\phi_d(D)\ \mathrm{d}D + \int_{\max(\bar{\xi}^g,\xi^h)}^{\min(\bar{\xi}^g+\xi^h,\bar{\xi}^c)} v_{mid,2,i_+}\left(D,\mathbf{F}_t(T)\right)\phi_d(D)\ \mathrm{d}D \\
+ \int_{\min(\bar{\xi}^g+\xi^h,\bar{\xi}^c)}^{\bar{\xi}^{i_+}} v_{mid,1}\left(D,\mathbf{F}_t(T)\right)\phi_d(D)\ \mathrm{d}D + \int_{\bar{\xi}^{i_+}}^{\max(\bar{\xi}^c,\bar{\xi}^g+\xi^h)} v_{high,2}\left(D,\mathbf{F}_t(T)\right)\phi_d(D)\ \mathrm{d}D \\
+ \int_{\max(\bar{\xi}^c,\bar{\xi}^g+\xi^h)}^{\bar{\xi}} v_{high,1}\left(D,\mathbf{F}_t(T)\right)\phi_d(D)\ \mathrm{d}D \Bigg\}
\end{aligned}
$$

Proof. TODO: Need to decide how to present this and how much detail to include. For example, more details about why the regions split into various terms (more than for forwards), with possible plots to aid in understanding. However, we also want to keep things compact. As in the proof of Proposition **??**, by iterated conditioning, for $t \in [0,T]$, the price of the dark spread V_t is given by

$$V_t := \mathrm{e}^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[P_T - h_c S_T^c \middle| \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[b(D_T,\mathbf{S}_T) - h_c S_T^c \middle| \mathcal{F}_T^0 \vee \mathcal{F}_t^W\right] \mathcal{F}_t\right].$$

Again we write the outer expectation as the sum of integrals corresponding to the different forms the payoff can take depending on the value of D_T. The functional form of b is different for D_T lying in the intervals $[0,\bar{\xi}^{i_-}]$, $[\bar{\xi}^{i_-},\bar{\xi}^{i_+}]$, $[\bar{\xi}^{i_+},\bar{\xi}]$. In addition the functional form of the payoff now depends on whether $D_T \le \xi^h$ or $D_T \ge \xi^h$ and on the magnitude of ξ^h relative to $\bar{\xi}^c$ and $\bar{\xi}^g$. Therefore, the first case is subdivided into the intervals $[0,\min(\bar{\xi}^g,\xi^h)]$, $[\min(\bar{\xi}^g,\xi^h),\bar{\xi}^{i_-}]$; the second case is subdivided into $[\bar{\xi}^{i_-},\max(\bar{\xi}^g,\xi^h)]$, $[\max(\bar{\xi}^g,\xi^h),\min(\bar{\xi}^g+\xi^h,\bar{\xi}^c)]$, $[\min(\bar{\xi}^g+\xi^h,\bar{\xi}^c),\bar{\xi}^{i_+}]$; the third case is subdivided into $[\bar{\xi}^{i_+},\max(\bar{\xi}^c,\bar{\xi}^g+\xi^h)]$, $[\max(\bar{\xi}^c,\bar{\xi}^g+\xi^h),\bar{\xi}]$.

The integrands $v_{\dots}$ in (**??**) are obtained by calculating the inner expectation for each demand regime listed above, in a similar fashion as in Proposition **??**. The results are given in Appendix **??**. □

Note that (**??**) requires seven terms in order to cover all possible values of h_c within the range given by (**??**), as well as the two cases $c = i_+$ and $c = i_-$. However, only five of the seven terms appear at once, with only the second or third appearing (depending on $h_c \lesseqgtr \exp(k_c + m_c\bar{\xi}^g)$) and only the fifth or sixth (depending on $h_c \lesseqgtr \exp(k_c + m_c(\bar{\xi}^c - \bar{\xi}^g))$). These conditions can equivalently be written as $\xi_h \lesseqgtr \bar{\xi}^g$ and $\xi_h \lesseqgtr \bar{\xi}^c - \bar{\xi}^g$. Notice that if $c = i_-$, we can immediately deduce that $\xi_h < \bar{\xi}^g$ and $\xi_h > \bar{\xi}^c - \bar{\xi}^g$ irrespective of h_c, while for $c = i_+$ several cases are possible.

Following the same argument as for forwards earlier, we note that if demand is assumed deterministic, then the spread option price is given explicitly by choosing the appropriate integrand from Proposition **??**. To now obtain a convenient closed-form result for unknown demand, we extend our earlier notational tool for combining Gaussian cdfs. For any integer n, define

$$\Phi_2^{2\times n}\left(\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n}\end{bmatrix}, y; \rho\right) = \sum_{i=1}^{n}\left[\Phi_2(x_{1i},y;\rho) - \Phi_2(x_{2i},y;\rho)\right].$$

In addition, we introduce the following notation to capture all the relevant limits of integration. Define the vector $\mathbf{a} := (a_1,\dots,a_8)$ by

(15)
$$\mathbf{a} := \frac{1}{\sigma_d}\left[\begin{pmatrix} 0 \\ \min\left(\bar{\xi}^g,\xi^h\right) \\ \bar{\xi}^{i_-} \\ \max\left(\bar{\xi}^g,\xi^h\right) \\ \min\left(\bar{\xi}^c,\bar{\xi}^g+\xi^h\right) \\ \bar{\xi}^{i_+} \\ \max\left(\bar{\xi}^c,\bar{\xi}^g+\xi^h\right) \\ \bar{\xi} \end{pmatrix} - \mu_d\right].$$

Notice that the components of $\mathbf{a}$ are in increasing order and correspond to the limits of integration in equation (**??**). In the case that $c = i_+$, all of these values are needed, while the case $c = i_-$ is somewhat simpler because $a_3 = a_4$ and $a_5 = a_6$ (since by our assumption on h_c, $\xi^h < \bar{\xi}^c$). However, the result below is valid in both cases as various terms simply drop out in the latter case.

We now observe that all the terms in (**??**) have the same form as those in Proposition **??** for forwards, as demand appears linearly inside each Gaussian cdf and in the exponential function outside the cdf's. Hence, we can exploit Lemma **??** as before to price spread options in closed form for (truncated) Gaussian demand, leading to the result below.

Corollary 4. *In addition to the assumptions in Proposition* **??** *let demand at maturity satisfy*

$$D_T = \max\left(0, \min\left(\bar{\xi}, X\right)\right),$$

where $X \sim N(\mu_d, \sigma_d^2)$ is independent of $\mathbf{Z}$. Then for $t \in [0,T]$, the price of a dark spread is given explicitly by

$$\begin{aligned}
V_t = e^{-r(T-t)} \Bigg\{ & \Phi\left(-a_8\right) \sum_{i \in I} b_i\left(\bar{\xi}^i, F_t^i\right) \Phi\left(-R_i\left(\bar{\xi}^i, \bar{\xi}^i\right)/\sigma\right) - h_c F_t^c\left(1 - \Phi_1\left(a_7\right) + \Phi_1(a_6) - \Phi_1(a_5)\right) \\
& + b_c\left(\mu_d, F_t^c\right) \exp\left(\frac{m_c^2 \sigma_d^2}{2}\right) \Phi_2^{2\times 2}\left(\left[\begin{array}{cc} \bar{\xi}^c & a_3 \\ a_4 & a_2 \end{array}\right], \frac{R_c(\mu_d, 0) - m_c^2 \sigma_d^2}{\sigma_{c,d}}; \frac{m_c \sigma_d}{\sigma_{c,d}}\right) \\
& + b_c\left(\mu_d - \bar{\xi}^g, F_t^c\right) \exp\left(\frac{m_c^2 \sigma_d^2}{2}\right) \Phi_2^{2\times 2}\left(\left[\begin{array}{cc} a_8 & a_6 \\ a_7 & a_5 \end{array}\right], \frac{-R_c\left(\mu_d - \bar{\xi}^g, \bar{\xi}^g\right) + m_c^2 \sigma_d^2}{\sigma_{c,d}}; \frac{-m_c \sigma_d}{\sigma_{c,d}}\right) \\
& + b_g\left(\mu_d - \bar{\xi}^c, F_t^g\right) \exp\left(\frac{m_g^2 \sigma_d^2}{2}\right) \Phi_2^{2\times 1}\left(\left[\begin{array}{c} a_8 \\ \bar{\xi}^c \end{array}\right], \frac{-R_g\left(\mu_d - \bar{\xi}^c, \bar{\xi}^c\right) + m_g^2 \sigma_d^2}{\sigma_{g,d}}; \frac{-m_g \sigma_d}{\sigma_{g,d}}\right) \\
& - h_c F_t^c \Phi_2^{2\times 3}\left(\left[\begin{array}{ccc} a_7 & a_5 & a_3 \\ a_6 & a_4 & a_2 \end{array}\right], \frac{\tilde{R}_c\left(\left(\log h_c - \beta - \gamma \mu_d\right)/\alpha_g\right)}{\sigma_{g,\gamma}}; \frac{-\gamma \sigma_d}{\alpha_g \sigma_{g,\gamma}}\right) \\
& + b_{cg}\left(\mu_d, \mathbf{F}_t\right) \exp\left(\eta\right) \Bigg\{ \Phi_2^{2\times 2}\left(\left[\begin{array}{cc} \bar{\xi}^c & a_3 \\ a_4 & a_2 \end{array}\right], \frac{-R_c(\mu_d, 0) - \alpha_g \sigma^2 + \gamma m_c \sigma_d^2}{\sigma_{c,d}}; \frac{-m_c \sigma_d}{\sigma_{c,d}}\right) \\
& - \Phi_2^{2\times 2}\left(\left[\begin{array}{cc} a_8 & a_6 \\ a_7 & a_5 \end{array}\right], \frac{-R_c\left(\mu_d - \bar{\xi}^g, \bar{\xi}^g\right) - \alpha_g \sigma^2 + \gamma m_c \sigma_d^2}{\sigma_{c,d}}; \frac{-m_c \sigma_d}{\sigma_{c,d}}\right) \\
& + \Phi_2^{2\times 1}\left(\left[\begin{array}{c} a_8 \\ \bar{\xi}^c \end{array}\right], \frac{R_g\left(\mu_d - \bar{\xi}^c, \bar{\xi}^c\right) + \alpha_c \sigma^2 - \gamma m_g \sigma_d^2}{\sigma_{g,d}}; \frac{m_g \sigma_d}{\sigma_{g,d}}\right) \\
& - \Phi_2^{2\times 3}\left(\left[\begin{array}{ccc} a_7 & a_5 & a_3 \\ a_6 & a_4 & a_2 \end{array}\right], \frac{-\tilde{R}_c\left((\log H - \beta - \gamma\ \mu_d)/\alpha_g\right) - \alpha_g \sigma^2 - \gamma^2 \sigma_d^2/\alpha_g}{\sigma_{g,\gamma}}; \frac{\gamma \sigma_d}{\alpha_g \sigma_{g,\gamma}}\right) \Bigg\} \Bigg\},
\end{aligned}$$

where

$$\begin{aligned}
\tilde{R}_i(z) &:= z + \log\left(F_t^j\right) - \log\left(F_t^i\right) - \frac{1}{2}\sigma^2 \\
\sigma_{i,\gamma}^2 &:= \gamma^2 \sigma_d^2 / \alpha_i^2 + \sigma^2
\end{aligned}$$

Proof. TODO: Need to decide how to present this. ☐

6. Extension to Capture Spikes

While the bid stack model introduced in Section 3 may be sufficient for many markets and for many model applications, we suggest in this section a simple extension to more accurately capture the spot price density in some cases. In particular, in markets which are prone to dramatic price spikes during peak hours, or sudden negative prices off-peak, such a modification may prove very beneficial, and importantly does not impact the availability of closed-form solutions for forwards or spread options.

The extension proposed involves exploiting the fact that there is some positive probability that demand hits zero or maximum capacity in our model, corresponding to the events $\{X < 0\}$ and $\{X > \bar{\xi}\}$, where $X \sim N(\mu_d, \sigma_d^2)$. Hence, instead of fixing the power price at the endpoints of the two-fuel stack (truncating demand), we can redefine the price in these cases, using the notion of a 'spike regime', and/or a 'negative price regime', which can be interpreted as being set by a thin tail of bids in these extreme regions. These bids correspond to no particular technology and hence do not depend on a fuel price. Many variations are possible but we suggest the following structure which retains the pattern of exponential functions of demand.

$$P_t := \begin{cases} b(0, \mathbf{S_t}) - \exp(-m_n D_t) & \text{for } D_t < 0 \\ b(D_t, \mathbf{S_t}) & \text{for } 0 \le D_t \le \bar{\xi} \\ b(\bar{\xi}, \mathbf{S_t}) + \exp(m_s(D_t - \bar{\xi})) & \text{for } D_t > \bar{\xi} \end{cases} \tag{16}$$

where b is the market bid stack for the base model (of Section **??**) and m_n and m_s are constants which determine how volatile prices are in the additional regimes. Clearly the extended model is still strictly increasing in demand (with a discontinuity of \$1 at the top and bottom of the previous stack).

Remark 5. While it is possible to generate realistic positive spikes even in the base model by choosing one of the exponential fuel bid curves to be quite steep (large m_i), we note that this would be at the expense of realistically capturing changes in the merit order, by artificially stretching that technology's bids. An alternative is to use a three-fuel model where the third bid curve represents spikes (and hence has no fuel price multiplying the exponential), however we then face a challenge of calculating many more permutations of the three stack locations, losing some tractability. We instead favour the above approach, which doesn't specify a quantity of bids associated with spikes, but simply implies that wherever the bids from our regular fuels end, a thin layer of extra miscellaneous bids begins. Note also that the form of this regime (and by analogy the negative spike regime) could be made more heavy tailed (eg, a power function similar to that of Aid, Campi and Langrené's scarcity function in [**?**] or Barlow's stack in [**?**]), but we would then [**?**] capping the price at some maximum level to prevent unrealistically large spikes. A steep exponential seems simpler and preferable in our case.

Under the extended model, forward prices are given by the same expression as in Proposition **??** plus the following simple terms

$$\exp\left(m_s(\mu_d - \bar{\xi}) + \frac{1}{2}m_s^2\sigma_d^2\right)\Phi_1\left(\frac{\mu_d - \bar{\xi}}{\sigma_d} + m_s\sigma_d\right) + \exp\left(m_n\mu_d + \frac{1}{2}m_n^2\sigma_d^2\right)\Phi_1\left(\frac{-\mu_d}{\sigma_d} - m_n\sigma_d\right),$$

while spread options require only the addition of the first of these terms.[4]

TODO: Decide what to do with this section... shorten or lengthen it, move it to somewhere else in the paper, change the spike model, etc? Add simulations to compare with previous ones? Ideally, add a quick calibration to PJM, and use these parameter estimates it illustrate the impact of spikes (instead of arbitrary parameters), then reuse these parameters again for spread pricing in the next section.

7. Power plant valuation - Numerical Examples

TODO: Clean this section up. In particular, the current version has a bit of an unnatural break in it, since I changed the parameters after figure 8 ("We continue...") and reset everything to be symmetric (coal and gas) for simplicity. Also I added cointegration at this point. I'll change this to be symmetric throughout and cointegration included throughout. If we then have a second subsection using PJM-fitted parameters, then the two sections will make sense. Other changes: possibly look at varying other parameters as well as initial conditions of fuels. Finally, a few plots comparing prices of power plants in different models. Possibly could also consider plots of power to fuel correlation in stack model (using formulas in appendix), to illustrate its state dependent nature.

We now focus on the implications of the two-fuel exponential stack model on spread option prices, by analyzing the prices given by Corollary **??** for various different parameter choices, as well as fuel forward curve scenarios. The latter of these is an important consideration since one of the model's strengths is its ability to capture the probability of future merit order changes and the resulting impact on electricity prices. The fuel forward curves reveal crucial information about this probability. For example, if bids from coal and gas are currently at similar levels but one fuel is in backwardation, while the other is in contango, the future dynamics of power prices (under $\mathbb{Q}$) should reflect a high chance

[4] This is due to our assumption on the range of heat rates H, which guarantees that for the spike regime, the option will always be in the money, while for the negative price regime, it will never be. Of course, if we were to consider put spread options instead of calls, the second term would be needed instead of the first.

of the coal and gas bids separating. Hence, long-term spread option pricing benefits substantially from the ability to simply input the fuel forward prices directly into our formulae, avoiding the need to estimate a risk premium or calibrate our model for fuels.

An important application of spread options in electricity markets is the valuation of physical power plants, which can be approximated as a series of spread options on spot power with different maturities (and if fixed operating costs are low, $K = 0$ in the spread payoffs is a reasonable approximation). For comparison purposes, we also consider two other typical approaches to spark and dark spread option pricing. Firstly, we consider the classical Margrabe formula for spread options, where both electricity and gas or coal price are assumed to be lognormal. Letting $\log(P_T) \sim N(\mu_P, \sigma_P^2)$ and $\rho_{p,c}$ or $\rho_{p,g}$ the necessary correlation parameter, the price of a spread option on fuel i, with payoff $(P_T - h_i S_T^i)^+$ is

$$V_t^{\text{Marg}} = \mathrm{e}^{-r(T-t)} \left[F_t^p \Phi_1 \left(\frac{\log\left(\frac{F_t^p}{h_i F_t^i}\right) + \frac{1}{2}\sigma_{p,i}^2}{\sigma_{p,i}} \right) - h_i F_t^i \Phi_1 \left(\frac{\log\left(\frac{F_t^p}{h_i F_t^i}\right) - \frac{1}{2}\sigma_{p,i}^2}{\sigma_{p,i}} \right) \right]. \tag{17}$$

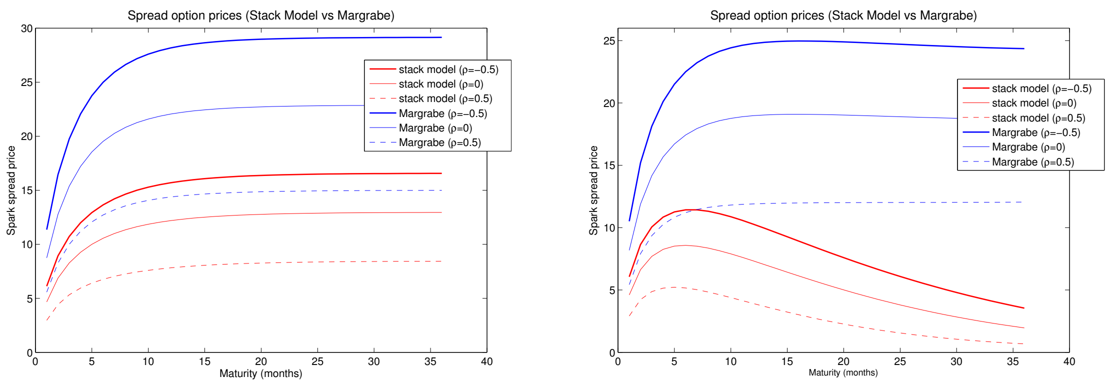
where $\sigma_{p,i}^2 = \sigma_p^2 - 2\rho_{p,i}\sigma_p\sigma_i + \sigma_i^2$.

The second alternative model that we consider is a simple cointegration model[5], where

$$P_t = w_c S_t^c + w_g S_t^g + Y_t$$

for some constant weights (cointegrating vector) w_c, w_g and some stationary process Y_t with a Gaussian distribution $N(\mu_y, \sigma_y^2)$. This is a rather naive model but one which retains a stronger link between the levels of electricity and fuels than that implied by Margrabe. However, unlike the stack model, the relative dependence on coal and gas is fixed initially by w_c, w_g, instead of dynamically adapting to fuel price movements (as well as demand). Note also that the spread prices for the cointegration model are calculated by simulation since no closed-form result is available.

For both Margrabe and the cointegration model, a meaningful comparison requires choosing the mean and variance appropriately to match that of the stack model. Matching the mean corresponds to calibrating to the power forward curve produced by the stack model, while matching the variance for a single maturity T involves choosing either σ_p or σ_y appropriately. Alternatively, to match the variance for all maturities at once, we can assume an exponential OU process for P_t in Margrabe (or an OU process for Y_t in the cointegration model) and find the best fit of mean reversion speed and volatility to approximately match all maturities.



(a) *Margrabe vs Stack Comparison (no input forwards)*

(b) *Margrabe vs Stack Comparison (gas contango, coal backwardated)*

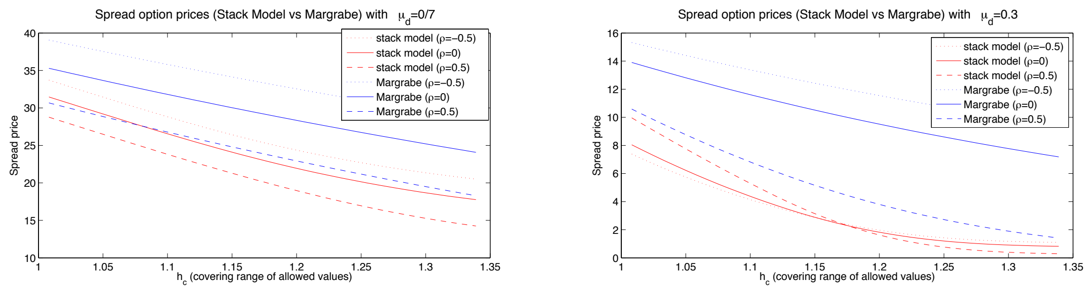
Figure 6. *Margrabe vs Stack Model: Plots of Spark Spread Prices for different correlation assumptions and given fuel forward curves*

It is to be expected that the stack model predicts lower spread option prices (especially for longer maturities), since the long-term levels of power and fuel prices are linked together very strongly by the stack structure, unlike simply correlating lognormals. This is illustrated for a spark spread in the left plot below of Figure **??** (for arbitrary stack parameters) for different levels of correlation parameter

[5]COMMENT - cite papers about evidence of correlation

in each of the models.[6] The right plot of Figure **??** shows the case of inputting a given (observed) forward curve for fuels, instead of taking the model implied curve. If the gas forward curve is in contango, while coal is in backwardation, then for long maturities, coal will almost always be below gas in the stack. Hence, a spark spread option has relatively little chance of being in the money (requires very high demand), lowering the option value. This is a good example of a dependency which can't be captured by Margrabe or other reduced-form models, but is automatically captured by the merit order built into the stack model.

While it is typical that increased correlation ρ between fuels in the stack model will lower spread option prices, this is not always true. For example, consider the plots of Figure **??**. Here $\xi_c = 0.6, \xi_g = 0.4$ and we consider a dark spread option with different values of h_c (with $T = 1$ fixed throughout). The left plot assumes a high mean demand $\mu_d = 0.7$ while the right plot considers low mean demand $\mu_d = 0.3$. In the first case, increasing correlation reduces price as we expect, but in the second plot the relationship is reversed for low values of h_c. The intuition here is that since demand is low, the price only gets set by gas if coal bids move above gas, but then the spread option is no longer in the money. When coal price is low so the spread option is in the money, typically coal is still setting the power price, so we don't get much benefit from high gas prices. So negative correlation doesn't help much. On the other hand, positive correlation helps reach some higher payoffs in states when both gas and coal prices rise together (and the exponential form of the stack leads to a higher payoff).



(a) *Margrabe vs Stack Comparison (high demand)*

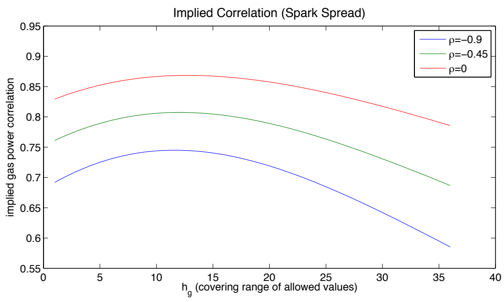
(b) *Margrabe vs Stack Comparison (low demand)*

Figure 7. *Margrabe vs Stack Model: Plots of Spark Spread Prices for different correlation assumptions and demand levels*

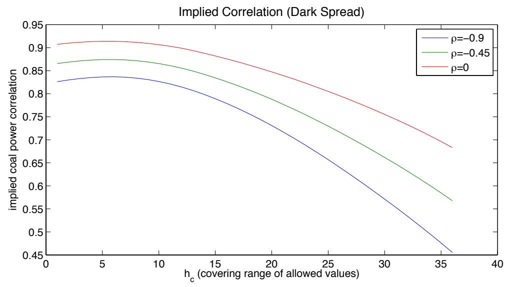
Another interesting analysis that can be performed is to look at implied correlation $\rho_{p,i}^{imp}$, meaning the value of $\rho_{p,i}$ in Margrabe's formula which reproduces the stack model price. Note that in each case we first choose the variance of power in the Margrabe formula to match the variance from the stack model (using formulas in appendix). As we can predict from Figures **??** and **??**, for high (positive) values of ρ, it may be impossible for Margrabe to reproduce the price, for any $\rho_{p,g} \in [-1, 1]$. In such cases, implied correlation does not exist. However, for negative values of ρ, it typically exists. Figure **??** illustrate the variation in implied correlation as a function of h_i, for both spark (left column) and dark (right column) spread options, using the same parameters as above. The second row shows the impact of moving from a high demand mean to a low demand mean.

We continue with further examples for that case that gas and coal are treated symmetrically (all parameters equal). The stack parameters we choose for our base case here are $k_c = k_g = 2, m_c = m_g = 1, \bar{\xi}^c = \bar{\xi}^g = 0.5$, while for the fuel process we choose exponential OU with mean level log(10), (and initial value log(10)), volatility 0.5 and mean reversion speed 1 (and correlation 0 in base case). The maturity is $T = 1$. For demand $\mu_d = 0.5$, $\sigma_d = 0.2$. In the left plot of Figure **??**, we investigate the impact on implied correlation of a dark spread option of varying $\bar{\xi}^g$. Note that it all cases Margrabe overprices the spread since $\rho^{imp} > 0$, but the difference is much larger with coal is the

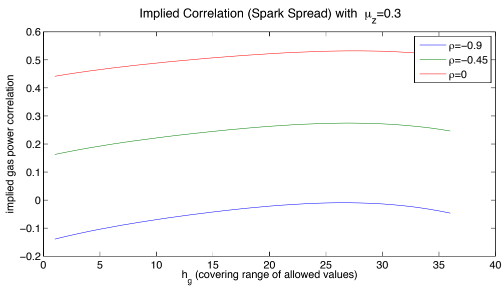
[6] Note that for Margrabe, this parameter $\rho_{p,g}$ correlates the power and gas, while for the stack model it's the coal to gas correlation parameter ρ. Power is of course not lognormal, but increased correlation between fuels serves to increase their correlation with power as well through the bid stack function. Hence in both cases, a higher correlation level typically implies a lower spark spread option price, since it reduces the volatility of the spread. This is always true for Margrabe, but some exceptions exist for the stack model, as illustrated in the next figure.



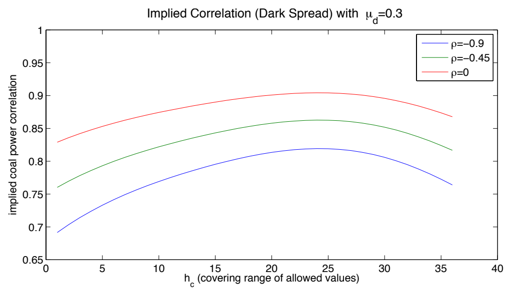
(a) *Implied Correlation - Spark (high demand)*



(b) *Implied Correlation - Dark (high demand)*



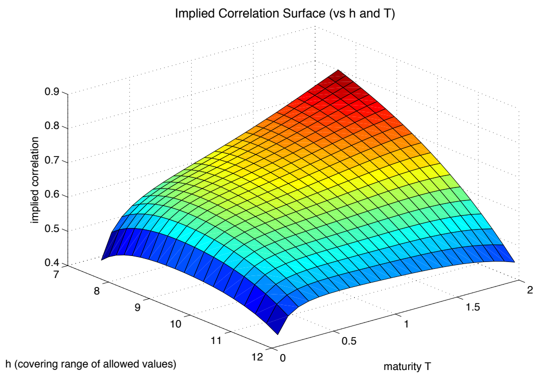
(c) *Implied Correlation - Spark (low demand)*



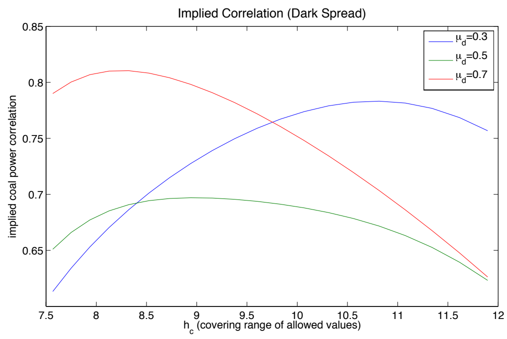
(d) *Implied Correlation - Dark (low demand)*

Figure 8. *Implied Correlation Plots for different correlation assumptions and demand levels*

dominant technology and hence power prices are most closely linked to coal, not gas. The right plot keeps $\bar{\xi}^c = \bar{\xi}^g = 0.5$, but instead varies μ_d (with σ_d now 0.12). We see that the implied correlation has a downward skew if demand is high, suggesting that Margrabe overprices by a larger amount for lower heat rates h_c (or h_g). In the case of $\mu_d = 0.5$ we observe a symmetric frown.



(a) *Implied correlation varying* $\bar{\xi}^g$

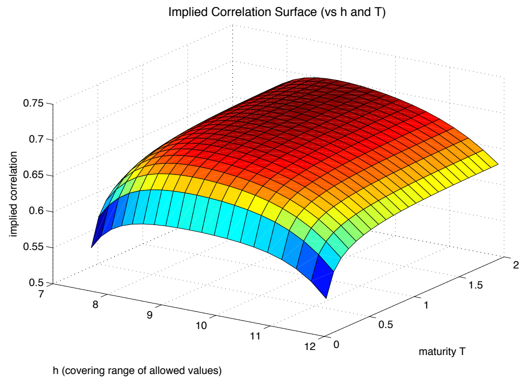


(b) *Implied correlation varying* μ_d

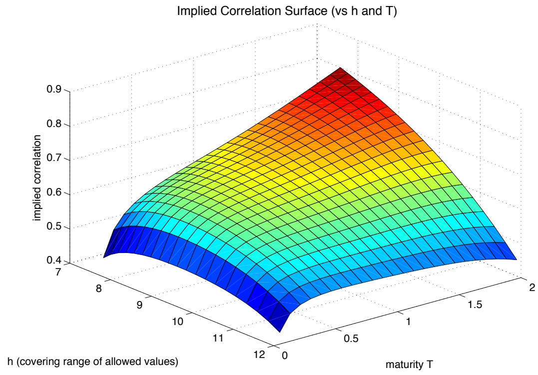
Figure 9. *Implied correlation analysis (plotted against* h*) for varying* $\bar{\xi}^g$ *and* μ_d

It is also interesting to investigate the implied correlation surface, where we vary both h and T. In the symmetric case, the result is illustrated in the left plot of Figure **??**. If instead we are given fuel forward curves as inputs, we can obtain a distinctive tilt in the surface for long maturities, even if everything else is symmetric. In the right plot below we let coal be in contango and gas in backwardation and plot $\rho_{p,g}^{imp}$ for a spark spread.

Next, we compare our model's results with both Margrabe and our simple cointegration model. While the choice of appropriate weights w_c, w_g in the cointegration model is in general unclear, in the case of symmetric coal and gas (and $\mu_d = 0.5$) we set $w_c = w_g = 1/2 \exp(k_c + 1/2 m_c \bar{\xi}^c)$, such that the power price is centred around the average of the median bid levels of coal and gas. The first row



(a) *Implied correlation surface (base)*



(b) *Implied correlation surface (forwards given)* μ_d

Figure 10. *Implied correlation surfaces, for symmetric case (left) and for spark spread in case of coal in contango, gas in backwardation (right)*

of plots in Figure **??** shows that for the symmetric base case, the cointegration model as expected comes much closer than Margrabe to agreeing with the stack model, but it still gives slightly higher prices for spreads. However, in the lower row of plots in Figure **??** we can see that for the case of inputting coal forwards in contango and gas in backwardation, the cointegration model sometimes predicts wildly different spread prices. This is particularly true for the case of a spark spread, where the high spread payoffs in the stack model occur when the power price is being set by coal alone. The big weakness of the cointegration approach is that the weights which govern the degree of correlation with each fuel (and are equal here) are assumed constant and hence cannot reflect future expected changes in the merit order.

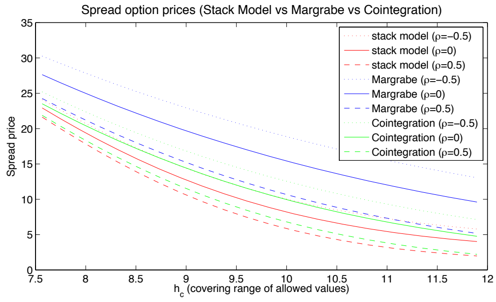
So far all of the plots above have assumed that the both the mean and variance of the power price distribution used in Margrabe and cointegration comparisons are always matched to future variance of P_T implied by the stack model.[7] However, an important consideration is that in practice we only have history to calibrate each model, and should not be borrowing extra information about the future from the stack model's structure when calibrating the other approaches. While it is somewhat reasonable to match the mean (as this is analogous to matching observed power forwards), matching the variance fully is less justifiable. Hence, in the following plots, we have calibrated our mean and variance to the scenario of fuel prices centred on their long-term mean levels (ie, no fuel forward inputs), but then priced the spread options with different assumptions about observed fuel forwards. In particular, we return our common example of coal in contango, gas in backwardation (and now $\mu_d = 0.7, \sigma_d = 0.12$). Figure **??** illustrates the large divergence in variance which occurs in this scenario, as the stack model's variance increases more rapidly when the fuel forward curves tend to separate the coal and gas bids in the future.

For this same example, we next look at the impact on the spread prices themselves in Figure **??**. We notice some significant changes as a result of not matching the mean and variance or power. In particular, for the spark spread option, the stack model price still prices below Margrabe for shorter maturities, but eventually begins to price above Margrabe for long maturity, as the impact of the greater variance of P_T begins to dominate.[8]

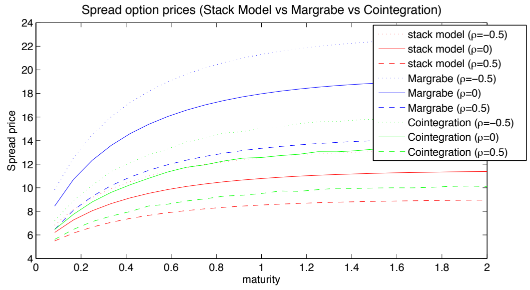
Possible conclusion to this section would be something on power plant valuation in particular, where we sum over the series of spreads for different maturities. (and compare across models again) Emphasise again the richer dependence structure that can be captured by the stack model.

[7] COMMENT - Should maybe discuss this earlier and in more detail. One point is that I am matching the variance for each different correlation value. ie, when correlation is 0.5 in Margrabe, I match with the variance from the case $\rho = 0.5$ in the stack, even though these correlations have different meanings (eg, gas or coal to power vs gas to coal). Maybe a better way?

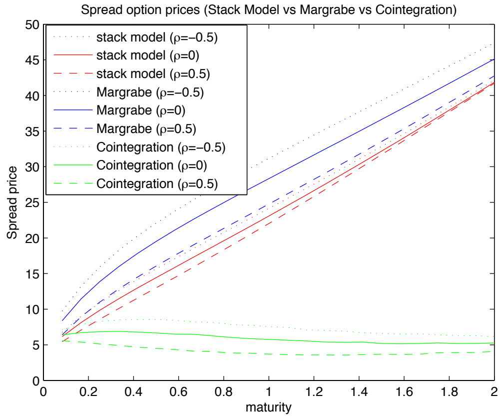
[8] COMMENT - It would be nice to bring in a mention somewhere here of the link to Geske's compound option model for equities, which predicts something similar to a stochastic vol model with negative correlation (ie, downwards skew). I feel that there should be an analagous result in our work too.



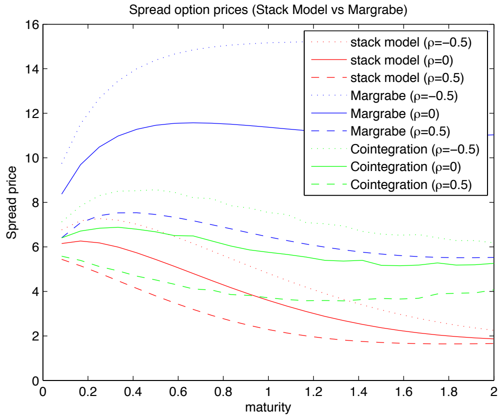
(a) *Symmetric case plotted against h*



(b) *Symmetric case plotted against T*

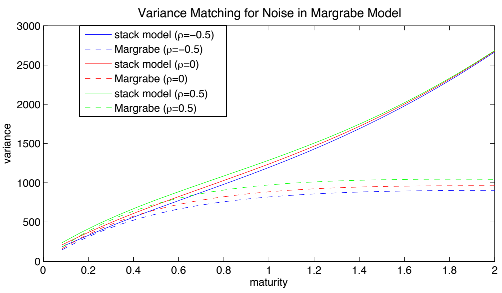


(c) *Fuel forward inputs (spark spread)*

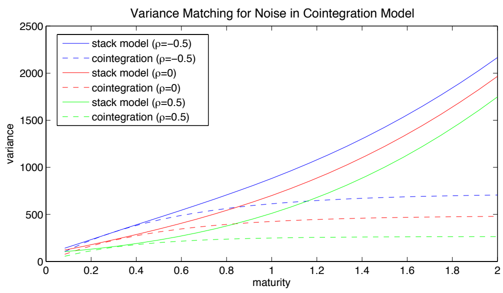


(d) *Fuel forward inputs (dark spread)*

Figure 11. *Comparisons with both Margrabe and Cointegration*

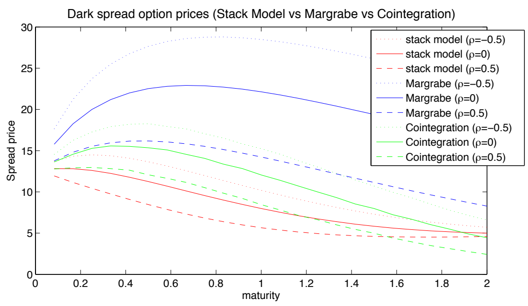


(a) *Variance of stack model vs Margrabe*

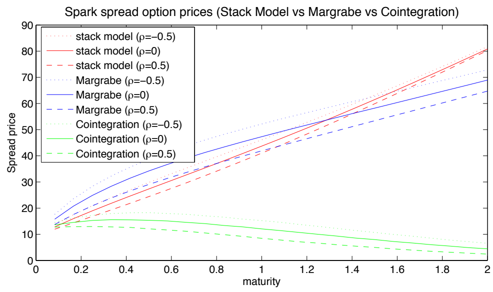


(b) *Variance of stack model vs cointegration*

Figure 12. *Illustration of variance divergence (vs T) for coal in contango, gas in backwardation*



(a) *Dark Spread Prices*



(b) *Spark Spread Prices*

Figure 13. *Spark and dark spread option prices (vs T) for coal in contango, gas in backwardation*

8. Conclusion

TODO: to be added very soon

Appendix A. Terms in Spread Formula

$$
\begin{aligned}
v_{\mathrm{low},1}\left(\xi,\mathbf{F}\right) &= b_c\left(\xi,F^c\right)\Phi_1\left(R_c(\xi,0)/\sigma\right) - h_cF^c\Phi_1\left(\tilde{R}_c\left(h(\xi)\right)/\sigma\right) \\
&\quad + b_{cg}\left(\xi,\mathbf{F}\right)\exp\left(\sigma^2_{\alpha_c\alpha_g}\right)\left[1-\Phi_1\left(R_c(\xi,0)/\sigma-\alpha_g\sigma\right)-\Phi_1\left(\tilde{R}_g\left(h(\xi)\right)/\sigma-\alpha_c\sigma\right)\right] \\
v_{\mathrm{low},2}\left(\xi,\mathbf{F}\right) &= 0 \\
v_{\mathrm{mid},1}\left(\xi,\mathbf{F}\right) &= b_{i_+}\left(\xi-\bar{\xi}^{i_-},F^{i_+}\right)\Phi_1\left(-R_{i_+}\left(\xi-\bar{\xi}^{i_-},\bar{\xi}^{i_-}\right)/\sigma\right) + b_{i_+}\left(\xi,F^{i_+}\right)\Phi_1\left(R_{i_+}\left(\xi,0\right)/\sigma\right) \\
&\quad - h_cF^c + b_{cg}\left(\xi,\mathbf{F}\right)\exp\left(-\frac{\alpha_c\alpha_g\sigma^2}{2}\right) \\
&\quad \times\left[\Phi_1\left(R_{i_+}\left(\xi-\bar{\xi}^{i_-},\bar{\xi}^{i_-}\right)/\sigma+\alpha_{i\_}\sigma\right)-\Phi_1\left(R_{i_+}(\xi,0)/\sigma+\alpha_{i\_}\sigma\right)\right], \\
v_{\mathrm{mid},2,c}\left(\xi,\mathbf{F}\right) &= v_{\mathrm{low},1}\left(\xi,\mathbf{F}\right) \\
v_{\mathrm{mid},3}\left(\xi,\mathbf{F}\right) &= 0 \\
v_{\mathrm{mid},2,g}\left(\xi,\mathbf{F}\right) &= v_{\mathrm{high},2}\left(\xi,\mathbf{F}\right) \\
v_{\mathrm{high},1}\left(\xi,\mathbf{F}\right) &= \sum_{i\in\mathbf{F}} b_i\left(\xi-\bar{\xi}^j,F^i\right)\Phi_1\left(-R_i\left(\xi-\bar{\xi}^j,\bar{\xi}^j\right)/\sigma\right) \\
&\quad - h_cF^c + b_{cg}\left(\xi,\mathbf{F}\right)\exp\left(-\frac{\alpha_c\alpha_g\sigma^2}{2}\right)\left[-1+\sum_{i\in\mathbb{F}}\Phi_1\left(R_i\left(\xi-\bar{\xi}^j,\bar{\xi}^j\right)/\sigma+\alpha_j\sigma\right)\right] \\
v_{\mathrm{high},2}\left(\xi,\mathbf{F}\right) &= b_g(\xi-\bar{\xi}^c,F^g)\Phi_1\left(-R_g\left(\xi-\bar{\xi}^c,\bar{\xi}^c\right)/\sigma\right) - h_cF^c\Phi_1\left(\tilde{R}_c\left(h(\xi)\right)/\sigma\right) \\
&\quad + b_{cg}\left(\xi,\mathbf{F}\right)\exp\left(-\frac{\alpha_c\alpha_g\sigma^2}{2}\right)\left[\Phi_1\left(R_g\left(\xi-\bar{\xi}^c,\bar{\xi}^c\right)/\sigma-\alpha_c\sigma\right)-\Phi_1\left(\tilde{R}_g\left(h(\xi)\right)/\sigma-\alpha_c\sigma\right)\right],
\end{aligned}
$$

where

$$
h(\xi) = \frac{1}{\alpha_g}\left(\log h_c - \beta - \gamma\xi\right).
$$

Appendix B. Moments and Covariances

If demand is given by a truncated Gaussian distribution with parameters μ_z and σ_z, then for $t\in[0,T]$, then the p-th moment of P_T is given explicitly by

$$
\begin{aligned}
\mathbb{E}_t[P_T^p] &= \Phi\left(\frac{-\mu_d}{\sigma_d}\right)\sum_{i\in\mathbb{F}} b_i^p(0,F_t^i)\mathrm{e}^{\frac{1}{2}(p^2-p)\sigma_i^2}\Phi\left(\frac{R_i^{(p)}(0,0)}{\sigma}\right) + \Phi\left(\frac{\mu_d-\bar{\xi}}{\sigma_d}\right)\sum_{i\in\mathbb{F}} b_i^p(\bar{\xi}^i,F_t^i)\Phi\left(\frac{-R_i^{(p)}(\bar{\xi}^i,\bar{\xi}^i)}{\sigma}\right) \\
&\quad + \sum_{i\in\mathbb{F}} b_i^p(\mu_d,F_t^i)\mathrm{e}^{\frac{1}{2}(p^2-p)\sigma_i^2+\frac{1}{2}p^2m_i^2\sigma_d^2}\Phi_2^{2\times1}\left(\begin{bmatrix}\frac{\bar{\xi}^i-\mu_d}{\sigma_d}-pm_i\sigma_d\\ \frac{-\mu_d}{\sigma_d}-pm_i\sigma_d\end{bmatrix},\frac{R_i^{(p)}(\mu_d,0)-pm_i^2\sigma_d^2}{\sigma_{i,d}};\frac{m_i\sigma_d}{\sigma_{i,d}}\right) \\
&\quad + \sum_{i\in\mathbb{F}} b_i^p(\mu_d-\bar{\xi}^j,F_t^i)\mathrm{e}^{\frac{1}{2}(p^2-p)\sigma_i^2+\frac{1}{2}p^2m_i^2\sigma_d^2}\Phi_2^{2\times1}\left(\begin{bmatrix}\frac{\bar{\xi}-\mu_d}{\sigma_d}-pm_i\sigma_d\\ \frac{\bar{\xi}^j-\mu_d}{\sigma_d}-pm_i\sigma_d\end{bmatrix},\frac{-R_i^{(p)}(\mu_d-\bar{\xi}^j,\bar{\xi}^j)+pm_i^2\sigma_d^2}{\sigma_{i,d}};\frac{-m_i\sigma_d}{\sigma_{i,d}}\right) \\
&\quad + b_{cg}^p(\mu_d,\mathbf{F}_t)\mathrm{e}^{\eta^{(p)}}\Bigg\{-\sum_{i\in\mathbb{F}}\Phi_2^{2\times1}\left(\begin{bmatrix}\frac{\bar{\xi}^i-\mu_d}{\sigma_d}-p\gamma\sigma_d\\ \frac{-\mu_d}{\sigma_d}-p\gamma\sigma_d\end{bmatrix},\frac{R_i(\mu_d,0)+p\alpha_j\sigma^2-p\gamma m_i\sigma_d^2}{\sigma_{i,d}};\frac{m_i\sigma_d}{\sigma_{i,d}}\right) \\
&\qquad + \sum_{i\in\mathbb{F}}\Phi_2^{2\times1}\left(\begin{bmatrix}\frac{\bar{\xi}-\mu_d}{\sigma_d}-p\gamma\sigma_d\\ \frac{\bar{\xi}^j-\mu_d}{\sigma_d}-p\gamma\sigma_d\end{bmatrix},\frac{R_i(\mu_d-\bar{\xi}^j,\bar{\xi}^j)+p\alpha_j\sigma^2+p\gamma m_i\sigma_d^2}{\sigma_{i,d}};\frac{-m_i\sigma_d}{\sigma_{i,d}}\right)\Bigg\},
\end{aligned}
$$

where $j = \mathbb{F} \setminus \{i\}$ as before and

$$R_i^{(p)}(\xi_i, \xi_j) := k_j + m_j\xi_j - k_i - m_i\xi_i + \log(F_t^j) - \log(F_t^i) - (p - \frac{1}{2})\sigma_i^2 - \frac{1}{2}\sigma_j^2 + p\rho\sigma_i\sigma_j,$$

$$\eta^{(p)} := \frac{p^2}{2}\left(\gamma^2\sigma_d^2 - \alpha_c\alpha_g\sigma^2\right).$$

A more general formula for $\mathbb{E}_t[P_T^p C_T^{p_c} G_T^{p_g}]$ can be obtained similarly, allowing us to calculate for example covariances between electricity and fuels. With all notation as above, we obtain:
TODO: Insert the formula here... or if we don't want this more general formula with p's (note notation improvement needed here - could replace with n or k but those are both used earlier too), we could just but in the case $\mathbb{E}_t[P_T C_T]$ or $\mathbb{E}_t[P_T G_T]$ as needed for covariance. More general usually better but not sure here.

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