PROBABILISTIC ANALYSIS OF MEAN-FIELD GAMES[*]

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Abstract. The purpose of this paper is to provide a complete probabilistic analysis of a large class of stochastic differential games with mean field interactions. We implement the Mean-Field Game strategy developed analytically by Lasry and Lions in a purely probabilistic framework, relying on tailor-made forms of the stochastic maximum principle. While we assume that the state dynamics are affine in the states and the controls, and the costs are convex, our assumptions on the nature of the dependence of all the coefficients upon the statistical distribution of the states of the individual players remains of a rather general nature. Our probabilistic approach calls for the solution of systems of forward-backward stochastic differential equations of a McKean–Vlasov type for which no existence result is known, and for which we prove existence and regularity of the corresponding value function. Finally, we prove that a solution of the Mean-Field Game problem as formulated by Lasry and Lions, does indeed provide approximate Nash equilibriums for games with a large number of players, and we quantify the nature of the approximation.

Key words. mean-field games, McKean–Vlasov forward-backward stochastic differential equations, propagation of chaos, stochastic maximum principle

AMS subject classifications. 93E20, 60H30, 60H10, 60F99

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1. Introduction. In a trailblazing contribution, Lasry and Lions [18, 19, 20] proposed a methodology to produce approximate Nash equilibriums for stochastic differential games with symmetric interactions and a large number of players. In their model, each player *feels* the presence and the behavior of the other players through the empirical distribution of their private states. This type of interaction was introduced and studied in statistical physics under the name of *mean-field* interaction, allowing for the derivation of effective equations in the limit of asymptotically large systems. Using intuition and mathematical results from propagation of chaos, Lasry and Lions propose to assign to each player, independently of what other players may do, a distributed closed loop strategy given by the solution of the limiting problem, arguing that the resulting game should be in an *approximate* Nash equilibrium. This streamlined approach is very attractive as large stochastic differential games are notoriously nontractable. They formulated the limiting problem as a system of two highly coupled nonlinear partial differential equations (PDE): the first one, of the Hamilton–Jacobi–Bellman type, takes care of the optimization part, while the second one, of the Kolmogorov type, guarantees the time consistency of the statistical distributions of the private states of the individual players. The issue of existence and uniqueness of solutions for such a system is a very delicate problem, as the solution of the former equation should propagate backward in time from a terminal condition while the solution of the latter should evolve forward in time from an initial condition. More than the nonlinearities, the conflicting directions of time compound the difficulties.

In a subsequent series of works [10, 9, 16, 17] with Ph.D. students and postdoctoral fellows, Lasry and Lions considered applications to domains as diverse as the management of exhaustible resources like oil, house insulation, and the analysis of pedestrian crowds. Motivated by problems in large communication networks, Caines, Huang, and Malhamé introduced, essentially at the same time [13], a similar strategy which they call the Nash Certainty Equivalence. They also studied practical applications to large populations behavior [12].

The goal of the present paper is to study the effective Mean-Field Game equations proposed by Lasry and Lions, from a probabilistic point of view. To this end, we recast the challenge as a fixed point problem in a space of flows of probability measures, show that these fixed points do exist and provide approximate Nash equilibriums for large games, and quantify the accuracy of the approximation.

We tackle the limiting stochastic optimization problems using the probabilistic approach of the stochastic maximum principle, thus reducing the problems to the solutions of Forward-Backward Stochastic Differential Equations (FBSDEs). The search for a fixed flow of probability measures turns the system of forward-backward stochastic differential equations into equations of the McKean–Vlasov type where the distribution of the solution appears in the coefficients. In this way, both the optimization and interaction components of the problem are captured by a single FBSDE, avoiding the twofold reference to Hamilton–Jacobi–Bellman equations on the one hand, and Kolmogorov equations on the other hand. As a by-product of this approach, the stochastic dynamics of the states could be degenerate. We give a general overview of this strategy in section 2. Motivated in part by the works of Lasry, Lions, and collaborators, Backward Stochastic Differential Equations (BSDEs) of the mean field type have recently been studied; see, for example, [3, 4]. However, existence and uniqueness results for BSDEs are much easier to come by than for FBSDEs, and here, we have to develop existence results from scratch.

Our first existence result is proven for bounded coefficients by means of a fixed point argument based on Schauder's theorem pretty much in the same spirit as in Cardaliaguet's notes [5]. Unfortunately, such a result does not apply to some of the linear-quadratic (LQ) games already studied [14, 1, 2, 7], and some of the most technical proofs of the papers are devoted to the extension of this existence result to coefficients with linear growth; see section 3. Our approximation and convergence arguments are based on probabilistic a priori estimates obtained from tailor-made versions of the stochastic maximum principle which we derive in section 2. The reader is referred to the book of Ma and Yong [21] for background material on adjoint equations, FBSDEs, and the stochastic maximum principle approach to stochastic optimization problems. As we rely on this approach, we find it natural to derive the compactness properties needed in our proofs from convexity properties of the coefficients of the game. The reader is also referred to the papers by Hu and Peng [11] and Peng and Wu [22] for general solvability properties of standard FBSDEs within the same framework of stochastic optimization.

The thrust of our analysis is not limited to existence of a solution to a rather general class of McKean–Vlasov FBSDEs, but also to the extension to this non-Markovian set-up of the construction of the FBSDE value function expressing the solution of the backward equation in terms of the solution of the forward dynamics. The existence of this value function is crucial for the formulation and the proofs of the results of the last part of the paper. In section 4, we indeed prove that the solutions of the *fixed point FBSDE* (which include a function $\hat{\alpha}$ minimizing the Hamiltonian of the system, three stochastic processes $(X_t, Y_t, Z_t)_{0 \leq t \leq T}$ solving the FBSDE, and the

FBSDE value function u) provide a set of distributed strategies which, when used by the players of a N-player game, form an ϵ_N-approximate Nash equilibrium, and we quantify the speed at which ϵ_N tends to 0 when $N \to +\infty$. This type of argument has been used for simpler models in [2] or [5]. Here, we use convergence estimates which are part of the standard theory of propagation of chaos (see, for example, [25, 15]) and the Lipschitz continuity and linear growth the FBSDE value function u which we prove earlier in the paper.

2. General notation and assumptions. Here, we introduce the notation and the basic tools from stochastic analysis which we use throughout the paper. We also remind the reader of the general assumptions under which the converse of the stochastic maximum principle applies to standard optimization problems. This set of assumptions will be strengthened in section 3 in order to tackle the mean-field interaction in the specific case of mean-field games.

2.1. The N player game. We consider a stochastic differential game with N players, each player $i \in \{1, \ldots, N\}$ controlling his own private state $U_t^i \in \mathbb{R}^d$ at time $t \in [0, T]$ by taking an action β_t^i in a set $A \subset \mathbb{R}^k$. We assume that the dynamics of the private states of the individual players are given by Itô's stochastic differential equations of the form

$$(2.1) \quad dU_t^i = b^i(t, U_t^i, \bar{\nu}_t^N, \beta_t^i)dt + \sigma^i(t, U_t^i, \bar{\nu}_t^N, \beta_t^i)dW_t^i, \qquad 0 \le t \le T, \quad i = 1, \ldots, N,$$

where the $W^i = (W_t^i)_{0 \le t \le T}$ are m-dimensional independent Wiener processes, $(b^i, \sigma^i) : [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \times A \hookrightarrow \mathbb{R}^d \times \mathbb{R}^{d \times m}$ are deterministic measurable functions satisfying the set of assumptions (A.1)–(A.4) spelled out below, and $\bar{\nu}_t^N$ denotes the empirical distribution of $U_t = (U_t^1, \ldots, U_t^N)$ defined as

$$\bar{\nu}_t^N(dx') = \frac{1}{N}\sum_{i=1}^{N}\delta_{U_t^i}(dx').$$

Here and in the following, we use the notation δ_x for the Dirac measure (unit point mass) at x, and $\mathcal{P}(E)$ for the space of probability measures on E whenever E is a topological space equipped with its Borel σ-field. In this framework, $\mathcal{P}(E)$ itself is endowed with the Borel σ-field generated by the topology of weak convergence of measures.

Each player chooses a strategy in the space $\mathbb{A} = \mathbb{H}^{2,k}$ of progressively measurable A-valued stochastic processes $\beta = (\beta_t)_{0 \le t \le T}$ satisfying the admissibility condition:

$$(2.2) \qquad\qquad \mathbb{E}\left[\int_0^T |\beta_t|^2 dt\right] < +\infty.$$

The choice of a strategy is driven by the desire to minimize an expected cost over the period $[0, T]$, each individual cost being a combination of running and terminal costs. For each $i \in \{1, \ldots, N\}$, the running cost to player i is given by a measurable function $f^i : [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \times A \hookrightarrow \mathbb{R}$ and the terminal cost by a measurable function $g^i : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \hookrightarrow \mathbb{R}$ in such a way that if the N players use the strategy $\beta = (\beta^1, \ldots, \beta^N) \in \mathbb{A}^N$, the expected total cost to player i is

$$(2.3) \qquad\qquad J^i(\beta) = \mathbb{E}\left[g^i(U_T^i, \bar{\nu}_T^N) + \int_0^T f^i(t, U_t^i, \bar{\nu}_t^N, \beta_t^i)dt\right].$$

Here $\mathbb{A}^N$ denotes the product of N copies of $\mathbb{A}$. Later in the paper, we let $N \to \infty$ and use the notation $J^{N,i}$ in order to emphasize the dependence upon N. Notice that even though only β_t^i appears in the formula giving the cost to player i, this cost depends upon the strategies used by the other players indirectly, as these strategies affect not only the private state U_t^i, but also the empirical distribution $\bar{\nu}_t^N$ of all the private states. As explained in the introduction, our model requires that the behaviors of the players be *statistically identical*, imposing that the coefficients b^i, σ^i, f^i, and g^i do not depend upon i. We denote them by b, σ, f, and g.

In solving the game, we are interested in the notion of optimality given by the concept of Nash equilibrium. Recall that a set of admissible strategies $\alpha^* = (\alpha^{*1}, \ldots, \alpha^{*N}) \in \mathbb{A}^N$ is said to be a Nash equilibrium for the game if

$$\forall i \in \{1, \ldots, N\}, \ \forall \alpha^i \in \mathbb{A}, \qquad J^i(\alpha^*) \leq J^i(\alpha^{*-i}, \alpha^i),$$

where we use the standard notation (α^{*-i}, α^i) for the set of strategies $(\alpha^{*1}, \ldots, \alpha^{*N})$ where α^{*i} has been replaced by α^i.

2.2. The mean-field problem. In the case of large symmetric games, some form of averaging is expected when the number of players tends to infinity. The Mean-Field Game (MFG) philosophy of Lasry and Lions is to search for approximate Nash equilibriums through the solution of effective equations appearing in the limiting regime $N \to \infty$, and assigning to each player the strategy α provided by the solution of the effective system of equations they derive. In the present context, the implementation of this idea involves the solution of the following fixed point problem which we break down in three steps for pedagogical reasons:

(i) Fix a deterministic function $[0, T] \ni t \hookrightarrow \mu_t \in \mathcal{P}(\mathbb{R}^d)$.
(ii) Solve the standard stochastic control problem

$$(2.4) \quad \inf_{\alpha \in \mathbb{A}} \mathbb{E}\left[\int_0^T f(t, X_t, \mu_t, \alpha_t)dt + g(X_T, \mu_T)\right]$$

$$\text{subject to} \quad dX_t = b(t, X_t, \mu_t, \alpha_t)dt + \sigma(t, X_t, \mu_t, \alpha_t)dW_t; \quad X_0 = x_0.$$

(iii) Determine the function $[0, T] \ni t \hookrightarrow \hat{\mu}_t \in \mathcal{P}(\mathbb{R}^d)$ so that $\forall t \in [0, T]$, $\mathbb{P}_{X_t} = \hat{\mu}_t$. Once these three steps have been taken successfully, if the *fixed-point* optimal control α identified in step (ii) is in feedback form; i.e., of the form $\alpha_t = \hat{\alpha}(t, X_t, \mathbb{P}_{X_t})$ for some function $\hat{\alpha}$ on $[0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)$, denoting by $\hat{\mu}_t = \mathbb{P}_{X_t}$ the *fixed-point* marginal distributions, the prescription $\hat{\alpha}_t^{i*} = \hat{\alpha}(t, X_t^i, \hat{\mu}_t)$, if used by the players $i = 1, \ldots, N$ of a large game, should form an approximate Nash equilibrium. We prove this fact rigorously in section 4, and we quantify the accuracy of the approximation.

2.3. The Hamiltonian. For the sake of simplicity, we assume that $A = \mathbb{R}^k$, and in order to lighten the notation and to avoid many technicalities, that the volatility is an uncontrolled constant matrix $\sigma \in \mathbb{R}^{d \times m}$. The fact that the volatility is uncontrolled allows us to use a simplified version for the Hamiltonian:

$$(2.5) \qquad H(t, x, \mu, y, \alpha) = \langle b(t, x, \mu, \alpha), y \rangle + f(t, x, \mu, \alpha),$$

for $t \in [0, T]$, $x, y \in \mathbb{R}^d$, $\alpha \in \mathbb{R}^k$, and $\mu \in \mathcal{P}(\mathbb{R}^d)$. In anticipation of the application of the stochastic maximum principle, assumptions (A.1) and (A.2) are chosen to make possible the minimization of the Hamiltonian and provide enough regularity for the minimizer. Indeed, our first task will be to minimize the Hamiltonian with respect

to the control parameter, and understand how minimizers depend upon the other variables.

(A.1) The drift b is an affine function of α in the sense that it is of the form

$$(2.6) \qquad b(t, x, \mu, \alpha) = b_1(t, x, \mu) + b_2(t)\alpha,$$

where the mapping $[0, T] \ni t \hookrightarrow b_2(t) \in \mathbb{R}^{d \times k}$ is measurable and bounded, and the mapping $[0, T] \ni (t, x, \mu) \hookrightarrow b_1(t, x, \mu) \in \mathbb{R}^d$ is measurable and bounded on bounded subsets of $[0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$.

Here and in the following, whenever E is a separable Banach space and p is an integer greater than 1, $\mathcal{P}_p(E)$ stands for the subspace of $\mathcal{P}(E)$ of probability measures of order p, i.e., having a finite moment of order p, so that $\mu \in \mathcal{P}_p(E)$ if $\mu \in \mathcal{P}(E)$ and

$$(2.7) \qquad M_{p,E}(\mu) = \left(\int_E \|x\|_E^p d\mu(x) \right)^{1/p} < +\infty.$$

We write M_p for $M_{p,\mathbb{R}^d}$. Below, bounded subsets of $\mathcal{P}_p(E)$ are defined as sets of probability measures with uniformly bounded moments of order p.

(A.2) There exist two positive constants λ and c_L such that for any $t \in [0, T]$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, the function $\mathbb{R}^d \times \mathbb{R}^k \ni (x, \alpha) \hookrightarrow f(t, x, \mu, \alpha) \in \mathbb{R}$ is once continuously differentiable with Lipschitz-continuous derivatives (so that $f(t, \cdot, \mu, \cdot)$ is $C^{1,1}$), the Lipschitz constant in x, and α being bounded by c_L (so that it is uniform in t and μ). Moreover, it satisfies the convexity assumption

$$(2.8) \quad f(t, x', \mu, \alpha') - f(t, x, \mu, \alpha) - \langle (x' - x, \alpha' - \alpha), \partial_{(x,\alpha)} f(t, x, \mu, \alpha) \rangle \geq \lambda |\alpha' - \alpha|^2.$$

The notation $\partial_{(x,\alpha)} f$ stands for the gradient in the joint variables (x, α). Finally, f, $\partial_x f$, and $\partial_\alpha f$ are locally bounded over $[0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^k$.

The minimization of the Hamiltonian is taken care of by the following result.

LEMMA 2.1. *If we assume that assumptions* (A.1)–(A.2) *are in force, then, for all* $(t, x, \mu, y) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d$, *there exists a unique minimizer* $\hat{\alpha}(t, x, \mu, y)$ *of H. Moreover, the function* $[0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \ni (t, x, \mu, y) \hookrightarrow \hat{\alpha}(t, x, \mu, y)$ *is measurable, locally bounded and Lipschitz-continuous with respect to (x, y), uniformly in* $(t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$, *the Lipschitz constant depending only upon λ, the supremum norm of b_2 and the Lipschitz constant of $\partial_\alpha f$ in x.*

Proof. For any given (t, x, μ, y), the function $\mathbb{R}^k \ni \alpha \hookrightarrow H(t, x, \mu, y, \alpha)$ is once continuously differentiable and strictly convex so that $\hat{\alpha}(t, x, \mu, y)$ appears as the unique solution of the equation $\partial_\alpha H(t, x, \mu, y, \hat{\alpha}(t, x, \mu, y)) = 0$. By strict convexity, measurability of the minimizer $\hat{\alpha}(t, x, \mu, y)$ is a consequence of the gradient descent algorithm. Local boundedness of $\hat{\alpha}(t, x, \mu, y)$ also follows from the strict convexity (2.8). Indeed,

$$H(t, x, \mu, y, 0) \geq H(t, x, \mu, y, \hat{\alpha}(t, x, \mu, y))$$
$$\geq H(t, x, \mu, y, 0) + \langle \hat{\alpha}(t, x, \mu, y), \partial_\alpha H(t, x, \mu, y, 0) \rangle + \lambda |\hat{\alpha}(t, x, \mu, y)|^2,$$

so that

$$(2.9) \qquad |\hat{\alpha}(t, x, \mu, y)| \leq \lambda^{-1} \left(|\partial_\alpha f(t, x, \mu, 0)| + |b_2(t)| \, |y| \right).$$

Inequality (2.9) will be used repeatedly. Moreover, by the implicit function theorem, $\hat{\alpha}$ is Lipschitz-continuous with respect to (x, y), the Lipschitz-constant being controlled by the uniform bound on b_2 and by the Lipschitz-constant of $\partial_{(x,\alpha)} f$. $\quad\square$

2.4. Stochastic maximum principle. Going back to the program (i)–(iii) outlined in subsection 2.2, the first two steps therein consist in solving a standard minimization problem when the distributions $(\mu_t)_{0 \le t \le T}$ are *frozen*. Then, one could express the value function of the optimization problem (2.4) as the solution of the corresponding Hamilton–Jacobi–Bellman (HJB) equation. This is the keystone of the analytic approach to the MFG theory, the matching problem (iii) being resolved by coupling the HJB equation with a Kolmogorov equation intended to identify the $(\mu_t)_{0 \le t \le T}$ with the marginal distributions of the optimal state of the problem. The resulting system of PDEs can be written as

(2.10)
$$
\begin{cases}
\partial_t v(t,x) + \dfrac{\sigma^2}{2}\Delta_x v(t,x) + H\big(t,x,\mu_t, \nabla_x v(t,x), \hat{\alpha}(t,x,\mu_t, \nabla_x v(t,x))\big) = 0, \\[2mm]
\partial_t \mu_t - \dfrac{\sigma^2}{2}\Delta_x \mu_t + \mathrm{div}_x\big(b\big(t,x,\mu_t, \hat{\alpha}(t,x,\mu_t, \nabla_x v(t,x))\big)\mu_t\big) = 0
\end{cases}
$$

in $[0,T] \times \mathbb{R}^d$, with $v(T,\cdot) = g(\cdot,\mu_T)$ and $\mu_0 = \delta_{x_0}$ as boundary conditions, the first equation being the HJB equation of the stochastic control problem when the flow $(\mu_t)_{0 \le t \le T}$ is frozen, the second equation being the Kolmogorov equation giving the time evolution of the flow $(\mu_t)_{0 \le t \le T}$ of measures dictated by the dynamics (2.4) of the *state of the system*. These two equations are coupled by the fact that the Hamiltonian appearing in the HJB equation is a function of the measure μ_t at time t and the drift appearing in the Kolmogorov equation is a function of the gradient of the value function v. Notice that the first equation is a backward equation to be solved from a terminal condition while the second equation is forward in time starting from an initial condition. The resulting system thus reads as a two-point boundary value problem, the general structure of which is known to be intricate.

Instead, the strategy we have in mind relies on a probabilistic description of the optimal states of the optimization problem (2.4) as provided by the so-called stochastic maximum principle. Indeed, the latter provides a necessary condition for the optimal states of the problem (2.4): Under suitable conditions, the optimally controlled diffusion processes satisfy the forward dynamics in a characteristic FBSDE, referred to as *the adjoint system* of the stochastic optimization problem. Moreover, the stochastic maximum principle provides a sufficient condition since, under additional convexity conditions, the forward dynamics of any solution to the adjoint system are optimal. In what follows, we use the sufficiency condition for proving the existence of solutions to the limit problem (i)–(iii) stated in subsection 2.2. This requires additional assumptions. In addition to (A.1)–(A.2) we will also assume:

(A.3) The function $[0,T] \ni t \hookrightarrow b_1(t,x,\mu)$ is affine in x; i.e., it has the form $[0,T] \ni t \hookrightarrow b_0(t,\mu) + b_1(t)x$, where b_0 and b_1 are $\mathbb{R}^d$ and $\mathbb{R}^{d \times d}$ valued, respectively, and bounded on bounded subsets of their respective domains. In particular, b reads

(2.11) $$b(t,x,\mu,\alpha) = b_0(t,\mu) + b_1(t)x + b_2(t)\alpha.$$

(A.4) The function $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \ni (x,\mu) \hookrightarrow g(x,\mu)$ is locally bounded. Moreover, for any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, the function $\mathbb{R}^d \ni x \hookrightarrow g(x,\mu)$ is once continuously differentiable and convex, and has a c_L-Lipschitz-continuous first order derivative.

In order to make the paper self-contained, we state and briefly prove the form of the sufficiency part of the stochastic maximum principle as it applies to (ii) when the flow of measures $(\mu_t)_{0 \le t \le T}$ are frozen. Instead of the standard version given for example in Chapter IV of the textbook by Yong and Zhou [26], we shall use the following theorem.

THEOREM 2.2. *Under assumptions* (A.1)–(A.4), *if the mapping* $[0, T] \ni t \hookrightarrow \mu_t \in \mathcal{P}_2(\mathbb{R}^d)$ *is measurable and bounded, and the cost functional* J *is defined by*

$$(2.12) \qquad J(\beta; \mu) = \mathbb{E}\left[g(U_T, \mu_T) + \int_0^T f(t, U_t, \mu_t, \beta_t) dt\right]$$

for any progressively measurable process $\beta = (\beta_t)_{0 \le t \le T}$ *satisfying the admissibility condition* (2.2) *where* $U = (U_t)_{0 \le t \le T}$ *is the corresponding controlled diffusion process*

$$U_t = x_0 + \int_0^t b(s, U_s, \mu_s, \beta_s) ds + \sigma W_t, \quad t \in [0, T]$$

for $x_0 \in \mathbb{R}^d$, *if the forward-backward system*
(2.13)
$$\begin{cases} dX_t = b(t, X_t, \mu_t, \hat{\alpha}(t, X_t, \mu_t, Y_t)) dt + \sigma dW_t, \quad X_0 = x_0 \\ dY_t = -\partial_x H(t, X_t, \mu_t, Y_t, \hat{\alpha}(t, X_t, \mu_t, Y_t)) dt + Z_t dW_t, \quad Y_T = \partial_x g(X_T, \mu_T) \end{cases}$$

has a solution $(X_t, Y_t, Z_t)_{0 \le t \le T}$ *such that*

$$(2.14) \qquad \mathbb{E}\left[\sup_{0 \le t \le T} (|X_t|^2 + |Y_t|^2) + \int_0^T |Z_t|^2 dt\right] < +\infty,$$

and if we set $\hat{\alpha}_t = \hat{\alpha}(t, X_t, \mu_t, Y_t)$, *then for any* $\beta = (\beta_t)_{0 \le t \le T}$ *satisfying* (2.2), *it holds that*

$$J(\hat{\alpha}; \mu) + \lambda \mathbb{E} \int_0^T |\beta_t - \hat{\alpha}_t|^2 dt \le J(\beta; \mu).$$

Proof. By Lemma 2.1, $\hat{\alpha} = (\hat{\alpha}_t)_{0 \le t \le T}$ satisfies (2.2), and the standard proof of the stochastic maximum principle (see, for example, Theorem 6.4.6 in Pham [23]) gives

$$J(\beta; \mu) \ge J(\hat{\alpha}; \mu) + \mathbb{E} \int_0^T \Big[H(t, U_t, \mu_t, Y_t, \beta_t) - H(t, X_t, \mu_t, Y_t, \hat{\alpha}_t)$$
$$- \langle U_t - X_t, \partial_x H(t, X_t, \mu_t, Y_t, \hat{\alpha}_t) \rangle - \langle \beta_t - \hat{\alpha}_t, \partial_\alpha H(t, X_t, \mu_t, Y_t, \hat{\alpha}_t) \rangle \Big] dt.$$

By linearity of b and assumption (A.2) on f, the Hessian of H satisfies (2.8), so that the required convexity assumption is satisfied. The result easily follows. $\square$

Remark 2.3. As the proof shows, the result of Theorem 2.2 above still holds if the control $\beta = (\beta_t)_{0 \le t \le T}$ is merely adapted to a larger filtration as long as the Wiener process $W = (W_t)_{0 \le t \le T}$ remains a Brownian motion for this filtration.

Remark 2.4. Theorem 2.2 has interesting consequences. First, it says that the optimal control, if it exists, must be unique. Second, it also implies that, given two solutions (X, Y, Z) and (X', Y', Z') to (2.13), $d\mathbb{P} \otimes dt$ almost everywhere (a.e.) it holds that

$$\hat{\alpha}(t, X_t, \mu_t, Y_t) = \hat{\alpha}(t, X'_t, \mu_t, Y'_t),$$

so that X and X' coincide by the Lipschitz property of the coefficients of the forward equation. As a consequence, (Y, Z) and (Y', Z') coincide as well.

It should be noticed that in some sense, the bound provided by Theorem 2.2 is sharp within the realm of convex models as shown, for example, by the following slight variation on the same theme. We shall use this form repeatedly in the proof of our main result.

PROPOSITION 2.5. *Under the same assumptions and notation as in Theorem* 2.2 *above, if we consider, in addition, another measurable and bounded mapping* $[0,T] \ni t \hookrightarrow \mu'_t \in \mathcal{P}_2(\mathbb{R}^d)$ *and the controlled diffusion process* $U' = (U'_t)_{0 \leq t \leq T}$ *defined by*

$$U'_t = x'_0 + \int_0^t b(s, U'_s, \mu'_s, \beta_s)ds + \sigma W_t, \quad t \in [0,T]$$

for an initial condition $x'_0 \in \mathbb{R}^d$ *possibly different from* x_0; *then,*

(2.15)
$$J(\hat{\alpha}; \mu) + \langle x'_0 - x_0, Y_0 \rangle + \lambda \mathbb{E} \int_0^T |\beta_t - \hat{\alpha}_t|^2 dt$$
$$\leq J([\beta, \mu']; \mu) + \mathbb{E}\left[\int_0^T \langle b_0(t, \mu'_t) - b_0(t, \mu_t), Y_t \rangle dt\right],$$

where

(2.16)
$$J([\beta, \mu']; \mu) = \mathbb{E}\left[g(U'_T, \mu_T) + \int_0^T f(t, U'_t, \mu_t, \beta_t)dt\right].$$

The parameter $[\beta, \mu']$ *in the cost* $J([\beta, \mu']; \mu)$ *indicates that the flow of measures in the drift of* U' *is* $(\mu'_t)_{0 \leq t \leq T}$ *whereas the flow of measures in the cost functions is* $(\mu_t)_{0 \leq t \leq T}$. *In fact, we should also indicate that the initial condition* x'_0 *might be different from* x_0, *but we prefer not to do so since there is no risk of confusion in what follows. Also, when* $x'_0 = x_0$ *and* $\mu'_t = \mu_t$ *for any* $t \in [0,T]$, $J([\beta, \mu']; \mu) = J(\beta; \mu)$.

Proof. The idea is to go back to the original proof of the stochastic maximum principle and, using Itô's formula, expand

$$\left(\langle U'_t - X_t, Y_t \rangle + \int_0^t \left[f(s, U'_s, \mu_s, \beta_s) - f(s, X_s, \mu_s, \hat{\alpha}_s)\right]ds\right)_{0 \leq t \leq T}.$$

Since the initial conditions x_0 and x'_0 are possibly different, we get the additional term $\langle x'_0 - x_0, Y_0 \rangle$ in the left-hand side of (2.15). Similarly, since the drift of U' is driven by $(\mu'_t)_{0 \leq t \leq T}$, we get the additional difference of the drifts in order to account for the fact that the drifts are driven by the different flows of probability measures. $\quad\square$

3. The mean-field FBSDE. In order to solve the standard stochastic control problem (2.4) using the Pontryagin maximum principle, we minimize the Hamiltonian H with respect to the control variable α, and inject the minimizer $\hat{\alpha}$ into the forward equation of the state as well as the adjoint backward equation. Since the minimizer $\hat{\alpha}$ depends upon both the forward state X_t and the adjoint process Y_t, this creates a strong coupling between the forward and backward equations leading to the FBSDE (2.13). The MFG matching condition (iii) of subsection 2.2 then reads: Seek a family of probability distributions $(\mu_t)_{0 \leq t \leq T}$ of order 2 such that the process X solving the forward equation of (2.13) admits $(\mu_t)_{0 \leq t \leq T}$ as flow of marginal distributions.

In a nutshell, the probabilistic approach to the solution of the mean-field game problem results in the solution of a FBSDE of the McKean–Vlasov type

(3.1)
$$\begin{cases} dX_t = b\big(t, X_t, \mathbb{P}_{X_t}, \hat{\alpha}(t, X_t, \mathbb{P}_{X_t}, Y_t)\big)dt + \sigma dW_t, \\ dY_t = -\partial_x H\big(t, X_t, \mathbb{P}_{X_t}, Y_t, \hat{\alpha}(t, X_t, \mathbb{P}_{X_t}, Y_t)\big)dt + Z_t dW_t, \end{cases}$$

with the initial condition $X_0 = x_0 \in \mathbb{R}^d$, and terminal condition $Y_T = \partial_x g(X_T, \mathbb{P}_{X_T})$. To the best of our knowledge, this type of FBSDE has not been considered in the

existing literature. However, our experience with the classical theory of FBSDEs tells us that existence and uniqueness are expected to hold in short time when the coefficients driving (3.1) are Lipschitz-continuous in the variables x, α, and μ from standard contraction arguments. This strategy can also be followed in the McKean–Vlasov setting, taking advantage of the Lipschitz regularity of the coefficients upon the parameter μ for the 2-Wasserstein distance, exactly as in the theory of McKean–Vlasov (forward) SDEs; see Sznitman [25]. However, the short time restriction is not really satisfactory for many reasons, and, in particular, for practical applications. Throughout the paper, all the regularity properties with respect to μ are understood in the sense of the 2-Wasserstein's distance W_2. Whenever E is a separable Banach space, for any $p \geq 1$, $\mu, \mu' \in \mathcal{P}_p(E)$, the distance $W_p(\mu, \mu')$ is defined by

$$W_p(\mu, \mu')$$
$$= \inf \left\{ \left[\int_{E \times E} |x - y|_E^p \, \pi(dx, dy) \right]^{1/p} ; \ \pi \in \mathcal{P}_p(E \times E) \text{ with marginals } \mu \text{ and } \mu' \right\}.$$

Below, we develop an alternative approach and prove existence of a solution over arbitrarily prescribed time duration T. The crux of the proof is to take advantage of the convexity of the coefficients. Indeed, in optimization theory, convexity often leads to compactness. Our objective is then to take advantage of this compactness in order to solve the matching problem (iii) in (2.4) by applying Schauder's fixed point theorem in an appropriate space of finite measures on $\mathcal{C}([0, T]; \mathbb{R}^d)$.

For the sake of convenience, we restate the general FBSDE (3.1) of McKean–Vlasov type in the special set-up of the present paper. It reads

$$(3.2) \qquad \begin{aligned} dX_t &= \left[b_0(t, \mathbb{P}_{X_t}) + b_1(t)X_t + b_2(t)\hat{\alpha}(t, X_t, \mathbb{P}_{X_t}, Y_t) \right] dt + \sigma dW_t, \\ dY_t &= -\left[b_1^\dagger(t)Y_t + \partial_x f\big(t, X_t, \mathbb{P}_{X_t}, \hat{\alpha}(t, X_t, \mathbb{P}_{X_t}, Y_t)\big) \right] dt + Z_t dW_t, \end{aligned}$$

where $a^\dagger$ denotes the transpose of the matrix a.

Remark 3.1. We can compare the system of PDEs (2.10) with the mean-field FBSDE (3.2). Formally, the adjoint variable Y_t at time t reads as $\nabla_x v(t, X_t)$, so that the dynamics of Y are directly connected with the dynamics of the gradient of the value function v in (3.2); similarly, the distribution of X_t identifies with μ_t in (3.2).

3.1. Standing assumptions and main result. In addition to (A.1)–(A.4), we shall rely on the following assumptions in order to solve the matching problem (iii) in (2.4):

(A.5) The functions $[0, T] \ni t \hookrightarrow f(t, 0, \delta_0, 0)$, $[0, T] \ni t \hookrightarrow \partial_x f(t, 0, \delta_0, 0)$ and $[0, T] \ni t \hookrightarrow \partial_\alpha f(t, 0, \delta_0, 0)$ are bounded by c_L, and $\forall t \in [0, T]$, $x, x' \in \mathbb{R}^d$, $\alpha, \alpha' \in \mathbb{R}^k$, and $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$, it holds that

$$\begin{aligned} \big| (f, g)(t, x', \mu', \alpha') &- (f, g)(t, x, \mu, \alpha) \big| \\ &\leq c_L \big[1 + |(x', \alpha')| + |(x, \alpha)| + M_2(\mu) + M_2(\mu') \big] \big[|(x', \alpha') - (x, \alpha)| + W_2(\mu', \mu) \big]. \end{aligned}$$

Moreover, b_0, b_1, and b_2 in (2.11) are bounded by c_L and b_0 satisfies for any $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$: $|b_0(t, \mu') - b_0(t, \mu)| \leq c_L W_2(\mu, \mu')$.

(A.6) For all $t \in [0, T]$, $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, $|\partial_\alpha f(t, x, \mu, 0)| \leq c_L$.

(A.7) For all $(t, x) \in [0, T] \times \mathbb{R}^d$, $\langle x, \partial_x f(t, 0, \delta_x, 0) \rangle \geq -c_L(1 + |x|)$, $\langle x, \partial_x g(0, \delta_x) \rangle \geq -c_L(1 + |x|)$.

THEOREM 3.2. *Under* (A.1–7), *the forward-backward system* (3.1) *has a solution. Moreover, for any solution* $(X_t, Y_t, Z_t)_{0 \le t \le T}$ *to* (3.1), *there exists a function* $u : [0,T] \times \mathbb{R}^d \hookrightarrow \mathbb{R}^d$ (*referred to as the FBSDE value function*), *satisfying the growth and Lipschitz properties*

$$(3.3) \qquad \forall t \in [0,T], \quad \forall x, x' \in \mathbb{R}^d, \quad \begin{cases} |u(t,x)| \le c(1 + |x|), \\ |u(t,x) - u(t,x')| \le c|x - x'| \end{cases}$$

for some constant $c \ge 0$, *and such that,* $\mathbb{P}$-*almost surely* (a.s.), $\forall t \in [0,T]$, $Y_t = u(t, X_t)$. *In particular, for any* $\ell \ge 1$, $\mathbb{E}[\sup_{0 \le t \le T} |X_t|^\ell] < +\infty$.

(A.5) provides Lipschitz continuity while condition (A.6) controls the smoothness of the running cost f with respect to α uniformly in the other variables. The most unusual assumption is certainly condition (A.7). We refer to it as a *weak mean-reverting* condition as it looks like a standard mean-reverting condition for recurrent diffusion processes. Moreover, as shown by the proof of Theorem 3.2, its role is to control the expectation of the forward component in (3.1) and to establish an a priori bound for it. This is of crucial importance in order to make the compactness strategy effective. We use the terminology *weak* as no convergence is expected for large time.

Remark 3.3. An interesting example which we should keep in mind is the so-called linear-quadratic model in which b_0, f, and g have the form

$$b_0(t,\mu) = b_0(t)\overline{\mu}, \ f(t,x,\mu,\alpha) = \frac{1}{2}\big|m(t)x + \bar{m}(t)\overline{\mu}\big|^2 + \frac{1}{2}|n(t)\alpha|^2, \ g(x,\mu) = \frac{1}{2}\big|qx + \bar{q}\overline{\mu}\big|^2,$$

where q, $\bar{q}$, $m(t)$, and $\bar{m}(t)$ are elements of $\mathbb{R}^{d \times d}$, $n(t)$ is an element of $\mathbb{R}^{k \times k}$, and $\overline{\mu}$ stands for the mean of μ. Assumptions (A.1)–(A.7) are then satisfied when $b_0(t) \equiv 0$ (so that b_0 is bounded as required in (A.5)) and $\bar{q}^\dagger q \ge 0$ and $\bar{m}(t)^\dagger m(t) \ge 0$ in the sense of quadratic forms (so that (A.7) holds). In particular, in the one-dimensional case $d = m = 1$, (A.7) says that $q\bar{q}$ and $m(t)\bar{m}(t)$ must be nonnegative. As shown in [7], these conditions are not optimal for existence when $d = m = 1$, as (3.2) is indeed shown to be solvable when $[0,T] \ni t \hookrightarrow b_0(t)$ is a (possibly nonzero) continuous function and $q(q+\bar{q}) \ge 0$ and $m(t)(m(t)+\bar{m}(t)) \ge 0$. Obviously, the gap between these conditions is the price to pay for treating general systems within a single framework.

Another example investigated in [7] is $b_0 \equiv 0$, $b_1 \equiv 0$, $b_2 \equiv 1$, $f \equiv \alpha^2/2$, with $d = m = 1$. When $g(x,\mu) = rx\overline{\mu}$, with $r \in \mathbb{R}^*$, Assumptions (A.1)–(A.7) are satisfied when $r > 0$ (so that (A.7) holds). The optimal condition given in [7] is $1 + rT \ne 0$. When $g(x,\mu) = x\gamma(\overline{\mu})$, for a bounded Lipschitz-continuous function γ from $\mathbb{R}$ into itself, Assumptions (A.1)–(A.7) are satisfied.

Remark 3.4. Uniqueness of the solution to (3.1) is a natural but challenging question. We address it in subsection 3.3.

3.2. Definition of the matching problem. The proof of Theorem 3.2 is split into four main steps. The first one consists of making the statement of the matching problem (iii) in (2.4) rigorous. To this end, we need the following lemma.

LEMMA 3.5. *Given* $\mu \in \mathcal{P}_2(\mathcal{C}([0,T]; \mathbb{R}^d))$ *with marginal distributions* $(\mu_t)_{0 \le t \le T}$, *the FBSDE* (2.13) *is uniquely solvable. If* $(X_t^{x_0;\mu}, Y_t^{x_0;\mu}, Z_t^{x_0;\mu})_{0 \le t \le T}$ *denotes its solution, then there exist a constant* $c > 0$, *only depending upon the parameters of* (A.1)–(A.7), *and a locally bounded measurable function* $u^\mu : [0,T] \times \mathbb{R}^d \hookrightarrow \mathbb{R}^d$ *such that*

$$\forall x, x' \in \mathbb{R}^d, \quad |u^\mu(t,x') - u^\mu(t,x)| \le c|x' - x|,$$

and $\mathbb{P}$-*a.s.,* $\forall t \in [0,T]$, $Y_t^{x_0;\mu} = u^\mu(t, X_t^{x_0;\mu})$.

Proof. Since $\partial_x H$ reads $\partial_x H(t, x, \mu, y, \alpha) = b_1^\dagger(t)y + \partial_x f(t, x, \mu, \alpha)$, by Lemma 2.1, the driver $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d \ni (t, x, y) \hookrightarrow \partial_x H(t, x, \mu_t, \hat{\alpha}(t, x, \mu_t, y))$ of the backward equation in (2.13) is Lipschitz continuous in the variables (x, y), uniformly in t. Therefore, by Theorem 1.1 in [8], existence and uniqueness hold for small time. In other words, when T is arbitrary, there exists $\delta > 0$, depending on the Lipschitz constant of the coefficients in the variables x and y such that unique solvability holds on $[T - \delta, T]$, that is when the initial condition x_0 of the forward process is prescribed at some time $t_0 \in [T - \delta, T]$. The solution is then denoted by $(X_t^{t_0, x_0}, Y_t^{t_0, x_0}, Z_t^{t_0, x_0})_{t_0 \leq t \leq T}$. Following the proof of Theorem 2.6 in [8], existence and uniqueness can be established on the whole $[0, T]$ by iterating the unique solvability property in short time provided we have

$$(3.4) \qquad \forall x_0, x_0' \in \mathbb{R}^d, \quad \left| Y_{t_0}^{t_0, x_0} - Y_{t_0}^{t_0, x_0'} \right|^2 \leq c|x_0 - x_0'|^2,$$

for some constant c independent of t_0 and δ. Notice that, by Blumenthal's Zero-One Law, the random variables $Y_{t_0}^{t_0, x_0}$ and $Y_{t_0}^{t_0, x_0'}$ are deterministic. By (2.15), we have

$$(3.5) \qquad \hat{J}^{t_0, x_0} + \langle x_0' - x_0, Y_{t_0}^{t_0, x_0} \rangle + \lambda \mathbb{E} \int_{t_0}^T |\hat{\alpha}_t^{t_0, x_0} - \hat{\alpha}_t^{t_0, x_0'}|^2 dt \leq \hat{J}^{t_0, x_0'},$$

where $\hat{J}^{t_0, x_0} = J((\hat{\alpha}_t^{t_0, x_0})_{t_0 \leq t \leq T}; \mu)$ and $\hat{\alpha}_t^{t_0, x_0} = \hat{\alpha}(t, X_t^{t_0, x_0}, \mu_t, Y_t^{t_0, x_0})$ (with similar definitions for $\hat{J}^{t_0, x_0'}$ and $\hat{\alpha}_t^{t_0, x_0'}$ by replacing x_0 by x_0'). Exchanging the roles of x_0 and x_0' and adding the resulting inequality with (3.5), we deduce that

$$(3.6) \qquad 2\lambda \mathbb{E} \int_{t_0}^T |\hat{\alpha}_t^{t_0, x_0} - \hat{\alpha}_t^{t_0, x_0'}|^2 dt \leq \langle x_0' - x_0, Y_{t_0}^{t_0, x_0'} - Y_{t_0}^{t_0, x_0} \rangle.$$

Moreover, by standard SDE estimates first and then by standard BSDE estimates (see Theorem 3.3, Chapter 7 in [26]), there exists a constant c independent of t_0 and δ, such that

$$\mathbb{E} \left[\sup_{t_0 \leq t \leq T} |X_t^{t_0, x_0} - X_t^{t_0, x_0'}|^2 \right] + \mathbb{E} \left[\sup_{t_0 \leq t \leq T} |Y_t^{t_0, x_0} - Y_t^{t_0, x_0'}|^2 \right] \leq c \mathbb{E} \int_{t_0}^T |\hat{\alpha}_t^{t_0, x_0} - \hat{\alpha}_t^{t_0, x_0'}|^2 dt.$$

Plugging (3.6) into the above inequality completes the proof of (3.4).

The function u^μ is then defined as $u^\mu : [0, T] \times \mathbb{R}^d \ni (t, x) \hookrightarrow Y_t^{t, x}$. The representation property of Y in terms of X directly follows from Corollary 1.5 in [8]. Local boundedness of u^μ follows from the Lipschitz continuity in the variable x together with the obvious inequality: $\sup_{0 \leq t \leq T} |u^\mu(t, 0)| \leq \sup_{0 \leq t \leq T} [\mathbb{E}[|u^\mu(t, X_t^{0,0}) - u^\mu(t, 0)|] + \mathbb{E}[|Y_t^{0,0}|]] < +\infty$. $\quad \square$

We now set the following definition.

DEFINITION 3.6. *To each $\mu \in \mathcal{P}_2(\mathcal{C}([0, T]; \mathbb{R}^d))$ with marginal distributions $(\mu_t)_{0 \leq t \leq T}$, we associate the measure $\mathbb{P}_{X^{x_0; \mu}}$, where $X^{x_0; \mu}$ is the solution of (2.13) with initial condition x_0. The resulting mapping $\mathcal{P}_2(\mathcal{C}([0, T]; \mathbb{R}^d)) \ni \mu \hookrightarrow \mathbb{P}_{X^{x_0; \mu}} \in \mathcal{P}_2(\mathcal{C}([0, T]; \mathbb{R}^d))$ is denoted by Φ, and we call solution of the matching problem (iii) in (2.4) any fixed point μ of Φ. For such a fixed point μ, $X^{x_0; \mu}$ satisfies (3.1).*

Definition 3.6 captures the essence of the approach of Lasry and Lions who freeze the probability measure at the optimal value when optimizing the cost. This is not the case in the study of the control of McKean–Vlasov dynamics investigated in [6] as in such a setting, optimization is also performed with respect to the measure argument. See also [7] and [2] for the linear quadratic case.

3.3. Uniqueness. With Definition 3.6 at hand, we can address the issue of uniqueness in the same conditions as Lasry and Lions (see section 3 in [5]).

PROPOSITION 3.7. *If, in addition to* (A.1)–(A.7), *we assume that* f *has the form*

$$f(t, x, \mu, \alpha) = f_0(t, x, \mu) + f_1(t, x, \alpha), \quad t \in [0, T], \ x \in \mathbb{R}^d, \ \alpha \in \mathbb{R}^k, \ \mu \in \mathcal{P}_2(\mathbb{R}^d),$$

f_0 *and* g *satisfying the monotonicity property:*

(3.7)
$$\int_{\mathbb{R}^d} \big(f_0(t, x, \mu) - f_0(t, x, \mu')\big) d\big(\mu - \mu'\big)(x) \geq 0,$$
$$\int_{\mathbb{R}^d} \big(g(x, \mu) - g(x, \mu')\big) d\big(\mu - \mu'\big)(x) \geq 0$$

for any $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$ *and* $t \in [0, T]$, *then* (3.1) *has at most one solution.*

Proof. Given two flows of measures $\mu = (\mu_t)_{0 \leq t \leq T}$ and $\mu' = (\mu'_t)_{0 \leq t \leq T}$ solving the matching problem as in Definition 3.6, we denote by $(\hat{\alpha}_t)_{0 \leq t \leq T}$ and $(\hat{\alpha}'_t)_{0 \leq t \leq T}$ the associated controls and by $(X_t)_{0 \leq t \leq T}$ and $(X'_t)_{0 \leq t \leq T}$ the associated controlled trajectories. Then by Proposition 2.5,

$$J(\hat{\alpha}; \mu) + \lambda \mathbb{E} \int_0^T |\hat{\alpha}_t - \hat{\alpha}'_t|^2 dt \leq J\big([\hat{\alpha}', \mu']; \mu\big) = \mathbb{E}\bigg[g(X'_T, \mu_T) + \int_0^T f(t, X'_t, \mu_t, \hat{\alpha}'_t) dt\bigg].$$

Therefore,

$$J(\hat{\alpha}; \mu) - J(\hat{\alpha}'; \mu') + \lambda \mathbb{E} \int_0^T |\hat{\alpha}_t - \hat{\alpha}'_t|^2 dt$$

$$\leq \mathbb{E}\bigg[g(X'_T, \mu_T) - g(X'_T, \mu'_T) + \int_0^T \big(f(t, X'_t, \mu_t, \hat{\alpha}'_t) - f(t, X'_t, \mu'_t, \hat{\alpha}'_t)\big) dt\bigg]$$

$$= \int_{\mathbb{R}^d} \big(g(x, \mu_T) - g(x, \mu'_T)\big) d\mu'_T(x) + \int_0^T \int_{\mathbb{R}^d} \big(f_0(t, x, \mu_t) - f_0(t, x, \mu'_t)\big) d\mu'_t(x) dt.$$

By exchanging the roles of μ and μ' and then by summing the resulting inequality with that above, the monotonicity property (3.7) implies that

$$\mathbb{E} \int_0^T |\hat{\alpha}_t - \hat{\alpha}'_t|^2 dt \leq 0,$$

from which uniqueness follows. ∎

3.4. Existence under additional boundedness conditions. We first prove existence under an extra boundedness assumption.

PROPOSITION 3.8. *The system* (3.1) *is solvable if, in addition to* (A.1)–(A.7), *we also assume that* $\partial_x f$ *and* $\partial_x g$ *are uniformly bounded; i.e., for some constant* $c_B > 0$

(3.8) $\forall t \in [0, T], \ x \in \mathbb{R}^d, \ \mu \in \mathcal{P}_2(\mathbb{R}^d), \ \alpha \in \mathbb{R}^k, \quad |\partial_x g(x, \mu)|, \ |\partial_x f(t, x, \mu, \alpha)| \leq c_B.$

Notice that (3.8) implies (A.7).

Proof. We apply Schauder's fixed point theorem in the space $\mathcal{M}_1(\mathcal{C}([0, T]; \mathbb{R}^d))$ of finite signed measure ν of order 1 on $\mathcal{C}([0, T]; \mathbb{R}^d)$ endowed with the Kantorovich–Rubinstein norm:

$$\|\nu\|_{\mathrm{KR}} = \sup\bigg\{\bigg|\int_{\mathcal{C}([0,T];\mathbb{R}^d)} F(w) d\nu(w)\bigg| ; \ F \in \mathrm{Lip}_1\big(\mathcal{C}([0, T]; \mathbb{R}^d)\big)\bigg\}$$

for $\nu \in \mathcal{M}_1(\mathcal{C}([0,T];\mathbb{R}^d))$, which is known to coincide with the Wasserstein distance W_1 on $\mathcal{P}_1(\mathcal{C}([0,T];\mathbb{R}^d))$. In what follows, we prove existence by proving that there exists a closed convex subset $\mathcal{E} \subset \mathcal{P}_2(\mathcal{C}([0,T];\mathbb{R}^d)) \subset \mathcal{M}_1(\mathcal{C}([0,T];\mathbb{R}^d))$, which is stable for Φ, with a relatively compact range, Φ being continuous on $\mathcal{E}$.

Step 1. We first establish several a priori estimates for the solution of (2.13). The coefficients $\partial_x f$ and $\partial_x g$ being bounded, the terminal condition in (2.13) is bounded and the growth of the driver is of the form

$$|\partial_x H(t, x, \mu_t, y, \hat{\alpha}(t, x, \mu_t, y))| \leq c_B + c_L|y|.$$

By expanding $(|Y_t^{x_0;\mu}|^2)_{0 \leq t \leq T}$ as the solution of a one-dimensional BSDE, we can compare it with the deterministic solution of a deterministic BSDE with a constant terminal condition; see Theorem 6.2.2 in [23]. This implies that there exists a constant c, only depending upon c_B, c_L, and T, such that, for any $\mu \in \mathcal{P}_2(\mathcal{C}([0,T];\mathbb{R}^d))$

$$(3.9) \qquad \forall t \in [0,T], \quad |Y_t^{x_0;\mu}| \leq c$$

holds $\mathbb{P}$-a.s. By (2.9) in the proof of Lemma 2.1 and by (A.6), we deduce that (the value of c possibly varying from line to line)

$$(3.10) \qquad \forall t \in [0,T], \quad \hat{\alpha}(t, X_t^{x_0;\mu}, \mu_t, Y_t^{x_0;\mu}) \leq c.$$

Plugging this bound into the forward part of (2.13), standard L^p estimates for SDEs imply that there exists a constant c', only depending upon c_B, c_L, and T, such that

$$(3.11) \qquad \mathbb{E}\left[\sup_{0 \leq t \leq T} |X_t^{x_0;\mu}|^4\right] \leq c'.$$

We consider the restriction of Φ to the subset $\mathcal{E}$ of probability measures of order 4 whose fourth moment is not greater than c', i.e.,

$$\mathcal{E} = \left\{\mu \in \mathcal{P}_4(\mathcal{C}([0,T];\mathbb{R}^d)) : M_{4,\mathcal{C}([0,T];\mathbb{R}^d)}(\mu) \leq c'\right\},$$

$\mathcal{E}$ is convex and closed for the 1-Wasserstein distance and Φ maps $\mathcal{E}$ into itself.

Step 2. The family of processes $((X_t^{x_0;\mu})_{0 \leq t \leq T})_{\mu \in \mathcal{E}}$ is tight in $\mathcal{C}([0,T];\mathbb{R}^d)$. Indeed, by the form (2.11) of the drift and (3.10), there exists a constant c'' such that, for any $\mu \in \mathcal{E}$ and $0 \leq s \leq t \leq T$,

$$|X_t^{x_0;\mu} - X_s^{x_0;\mu}| \leq c''\left[(t-s)\left(1 + \sup_{0 \leq r \leq T} |X_r^{x_0;\mu}|\right) + |B_t - B_s|\right],$$

so that tightness follows from (3.11). By (3.11) again, $\Phi(\mathcal{E})$ is actually relatively compact for the 1-Wasserstein distance on $\mathcal{C}([0,T];\mathbb{R}^d)$. Indeed, tightness says that it is relatively compact for the topology of weak convergence of measures and (3.11) says that any weakly convergent sequence $(\mathbb{P}_{X^{x_0;\mu_n}})_{n \geq 1}$, with $\mu_n \in \mathcal{E}$ for any $n \geq 1$, is convergent for the 1-Wasserstein distance.

Step 3. We finally check that Φ is continuous on $\mathcal{E}$. Given another measure $\mu' \in \mathcal{E}$, we deduce from (2.15) in Proposition 2.5 that

$$(3.12) \quad J(\hat{\alpha}; \mu) + \lambda \mathbb{E}\int_0^T |\hat{\alpha}_t' - \hat{\alpha}_t|^2 dt \leq J([\hat{\alpha}', \mu']; \mu) + \mathbb{E}\int_0^T \langle b_0(t, \mu_t') - b_0(t, \mu_t), Y_t \rangle dt,$$

where $\hat{\alpha}_t = \hat{\alpha}(t, X_t^{x_0;\mu}, \mu_t, Y_t^{x_0;\mu})$, for $t \in [0, T]$, with a similar definition for $\hat{\alpha}'_t$ by replacing μ by μ'. By optimality of $\hat{\alpha}'$ for the cost functional $J(\cdot; \mu')$, we claim

$$J([\hat{\alpha}', \mu']; \mu) \leq J(\hat{\alpha}; \mu') + J([\hat{\alpha}', \mu']; \mu) - J(\hat{\alpha}'; \mu'),$$

so that (3.12) yields

(3.13)
$$\lambda \mathbb{E} \int_0^T |\hat{\alpha}'_t - \hat{\alpha}_t|^2 dt \leq J(\hat{\alpha}; \mu') - J(\hat{\alpha}; \mu) + J([\hat{\alpha}', \mu']; \mu) - J(\hat{\alpha}'; \mu')$$
$$+ \mathbb{E} \int_0^T \langle b_0(t, \mu'_t) - b_0(t, \mu_t), Y_t \rangle dt.$$

We now compare $J(\hat{\alpha}; \mu')$ with $J(\hat{\alpha}; \mu)$ (and similarly $J(\hat{\alpha}'; \mu')$ with $J([\hat{\alpha}', \mu']; \mu)$). We notice that $J(\hat{\alpha}; \mu)$ is the cost associated with the flow of measures $(\mu_t)_{0 \leq t \leq T}$ and the diffusion process $X^{x_0;\mu}$ whereas $J(\hat{\alpha}; \mu')$ is the cost associated with the flow of measures $(\mu'_t)_{0 \leq t \leq T}$ and the controlled diffusion process U satisfying

$$dU_t = [b_0(t, \mu'_t) + b_1(t)U_t + b_2(t)\hat{\alpha}_t] dt + \sigma dW_t, \quad t \in [0, T]; \quad U_0 = x_0.$$

By Gronwall's lemma, there exists a constant c such that

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |X_t^{x_0,\mu} - U_t|^2\right] \leq c \int_0^T W_2^2(\mu_t, \mu'_t) dt.$$

Since μ and μ' are in $\mathcal{E}$, we deduce from (A.5), (3.10), and (3.11) that

$$J(\hat{\alpha}; \mu') - J(\hat{\alpha}; \mu) \leq c\left(\int_0^T W_2^2(\mu_t, \mu'_t) dt\right)^{1/2},$$

with a similar bound for $J([\hat{\alpha}', \mu']; \mu) - J(\hat{\alpha}'; \mu')$ (the argument is even simpler as the costs are driven by the same processes), so that, from (3.13) and (3.9) again, together with Gronwall's lemma to go back to the controlled SDEs,

$$\mathbb{E} \int_0^T |\hat{\alpha}'_t - \hat{\alpha}_t|^2 dt + \mathbb{E}\left[\sup_{0 \leq t \leq T} |X_t^{x_0;\mu} - X_t^{x_0;\mu'}|^2\right] \leq c\left(\int_0^T W_2^2(\mu_t, \mu'_t) dt\right)^{1/2}.$$

As probability measures in $\mathcal{E}$ have bounded moments of order 4, Cauchy-Schwarz inequality yields (keep in mind that $W_1(\Phi(\mu), \Phi(\mu')) \leq \mathbb{E}[\sup_{0 \leq t \leq T} |X_t^{x_0;\mu} - X_t^{x_0;\mu'}|])$

$$W_1(\Phi(\mu), \Phi(\mu')) \leq c\left(\int_0^T W_2^2(\mu_t, \mu'_t) dt\right)^{1/4} \leq c\left(\int_0^T W_1^{1/2}(\mu_t, \mu'_t) dt\right)^{1/4},$$

which shows that Φ is continuous on $\mathcal{E}$ with respect to the 1-Wasserstein distance W_1 on $\mathcal{P}_1(\mathcal{C}([0, T]; \mathbb{R}^d))$. □

3.5. Approximation procedure. Examples of functions f and g, which are convex in x and such that $\partial_x f$ and $\partial_x g$ are bounded, are rather limited in number and scope. For instance, boundedness of $\partial_x f$ and $\partial_x g$ fails in the typical case when f and g are quadratic with respect to x. In order to overcome this limitation, we propose to approximate the cost functions f and g by two sequences $(f^n)_{n \geq 1}$ and $(g^n)_{n \geq 1}$, referred to as approximated cost functions, satisfying (A.1)–(A.7) uniformly with respect to $n \geq 1$, and such that, for any $n \geq 1$, (3.1), with

$(\partial_x f, \partial_x g)$ replaced by $(\partial_x f^n, \partial_x g^n)$, has a solution (X^n, Y^n, Z^n). In this framework, Proposition 3.8 says that such approximated FBSDEs are indeed solvable when $\partial_x f^n$ and $\partial_x g^n$ are bounded for any $n \geq 1$. Our approximation procedure relies on the following lemma.

LEMMA 3.9. *If there exist two sequences* $(f^n)_{n \geq 1}$ *and* $(g^n)_{n \geq 1}$ *such that*

(i) *there exist two parameters* c'_L *and* $\lambda' > 0$ *such that, for any* $n \geq 1$, f^n *and* g^n *satisfy* (A.1)–(A.7) *with respect to* λ' *and* c'_L;

(ii) f^n *(resp.,* g^n*) converges towards* f *(resp., g) uniformly on any bounded subset of* $[0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^k$ *(resp.* $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$*);*

(iii) *for any* $n \geq 1$, (3.1), *with* $(\partial_x f, \partial_x g)$ *replaced by* $(\partial_x f^n, \partial_x g^n)$, *has a solution which we denote by* (X^n, Y^n, Z^n);

then, (3.1) *is solvable.*

Proof. We establish tightness of the processes $(X^n)_{n \geq 1}$ in order to extract a convergent subsequence. For any $n \geq 1$, we consider the approximated Hamiltonian

$$H^n(t, x, \mu, y, \alpha) = \langle b(t, x, \mu, \alpha), y \rangle + f^n(t, x, \mu, \alpha),$$

together with its minimizer $\hat{\alpha}^n(t, x, \mu, y) = \text{argmin}_\alpha H^n(t, x, \mu, y, \alpha)$. Setting $\hat{\alpha}^n_t = \hat{\alpha}^n(t, X^n_t, \mathbb{P}_{X^n_t}, Y^n_t)$ for any $t \in [0, T]$ and $n \geq 1$, our first step will be to prove that

$$(3.14) \qquad \sup_{n \geq 1} \mathbb{E}\left[\int_0^T |\hat{\alpha}^n_s|^2 ds \right] < +\infty.$$

Since X^n is the diffusion process controlled by $(\hat{\alpha}^n_t)_{0 \leq t \leq T}$, we use Theorem 2.2 to compare its behavior to the behavior of a *reference controlled process* U^n whose dynamics are driven by a specific control β^n. We shall consider two different versions for U^n corresponding to the following choices for β^n:

$$(3.15) \qquad \text{(i) } \beta^n_s = \mathbb{E}(\hat{\alpha}^n_s) \quad \text{for } 0 \leq s \leq T; \quad \text{(ii) } \beta^n \equiv 0.$$

For each of these controls, we compare the cost to the optimal cost by using the version of the stochastic maximum principle which we proved earlier, and subsequently, derive useful information on the optimal control $(\hat{\alpha}^n_s)_{0 \leq s \leq T}$.

Step 1. We first consider (i) in (3.15). In this case

$$(3.16) \quad U^n_t = x_0 + \int_0^t \left[b_0(s, \mathbb{P}_{X^n_s}) + b_1(s) U^n_s + b_2(s) \mathbb{E}(\hat{\alpha}^n_s) \right] ds + \sigma W_t, \quad t \in [0, T].$$

Notice that taking expectations on both sides of (3.16) shows that $\mathbb{E}(U^n_s) = \mathbb{E}(X^n_s)$, for $0 \leq s \leq T$, and that

$$\left[U^n_t - \mathbb{E}(U^n_t) \right] = \int_0^t b_1(s) \left[U^n_s - \mathbb{E}(U^n_s) \right] ds + \sigma W_t, \quad t \in [0, T],$$

from which it easily follows that $\sup_{n \geq 1} \sup_{0 \leq s \leq T} \text{Var}(U^n_s) < +\infty$.

By Theorem 2.2, with $g^n(\cdot, \mathbb{P}_{X^n_T})$ as terminal cost and $(f^n(t, \cdot, \mathbb{P}_{X^n_t}, \cdot))_{0 \leq t \leq T}$ as running cost, we get

$$(3.17) \begin{aligned} \mathbb{E}\left[g^n\left(X^n_T, \mathbb{P}_{X^n_T}\right) \right] &+ \mathbb{E} \int_0^T \left[\lambda' |\hat{\alpha}^n_s - \beta^n_s|^2 + f^n\left(s, X^n_s, \mathbb{P}_{X^n_s}, \hat{\alpha}^n_s\right) \right] ds \\ &\leq \mathbb{E}\left[g^n\left(U^n_T, \mathbb{P}_{X^n_T}\right) + \int_0^T f^n\left(s, U^n_s, \mathbb{P}_{X^n_s}, \beta^n_s\right) ds \right]. \end{aligned}$$

Using the fact that $\beta_s^n = \mathbb{E}(\hat{\alpha}_s^n)$, the convexity condition in (A.2)–(A.4) and Jensen's inequality, we obtain

(3.18)
$$g^n\big(\mathbb{E}(X_T^n), \mathbb{P}_{X_T^n}\big) + \int_0^T \big[\lambda'\mathrm{Var}(\hat{\alpha}_s^n) + f^n\big(s, \mathbb{E}(X_s^n), \mathbb{P}_{X_s^n}, \mathbb{E}(\hat{\alpha}_s^n)\big)\big]ds$$
$$\leq \mathbb{E}\bigg[g^n\big(U_T^n, \mathbb{P}_{X_T^n}\big) + \int_0^T f^n\big(s, U_s^n, \mathbb{P}_{X_s^n}, \mathbb{E}(\hat{\alpha}_s^n)\big)ds\bigg].$$

By (A.5), we deduce that there exists a constant c, depending only on λ, c_L, x_0, and T, such that (the actual value of c possibly varying from line to line)

$$\int_0^T \mathrm{Var}(\hat{\alpha}_s^n)ds \leq c\big(1 + \mathbb{E}\big[|U_T^n|^2\big]^{1/2} + \mathbb{E}\big[|X_T^n|^2\big]^{1/2}\big)\mathbb{E}\big[|U_T^n - \mathbb{E}(X_T^n)|^2\big]^{1/2}$$
$$+ c\int_0^T \big(1 + \mathbb{E}\big[|U_s^n|^2\big]^{1/2} + \mathbb{E}\big[|X_s^n|^2\big]^{1/2} + \mathbb{E}\big[|\hat{\alpha}_s^n|^2\big]^{1/2}\big)\mathbb{E}\big[|U_s^n - \mathbb{E}(X_s^n)|^2\big]^{1/2}ds.$$

Since $\mathbb{E}(X_t^n) = \mathbb{E}(U_t^n)$ for any $t \in [0, T]$, we deduce from the uniform boundedness of the variance of $(U_s^n)_{0 \leq s \leq T}$ that

(3.19)
$$\int_0^T \mathrm{Var}(\hat{\alpha}_s^n)ds \leq c\bigg[1 + \sup_{0 \leq s \leq T} \mathbb{E}[|X_s^n|^2]^{1/2} + \bigg(\mathbb{E}\int_0^T |\hat{\alpha}_s^n|^2ds\bigg)^{1/2}\bigg].$$

From this, the linearity of the dynamics of X^n and Gronwall's inequality, we deduce

(3.20)
$$\sup_{0 \leq s \leq T} \mathrm{Var}(X_s^n) \leq c\bigg[1 + \bigg(\mathbb{E}\int_0^T |\hat{\alpha}_s^n|^2ds\bigg)^{1/2}\bigg],$$

since

(3.21)
$$\sup_{0 \leq s \leq T} \mathbb{E}\big[|X_s^n|^2\big] \leq c\bigg[1 + \mathbb{E}\int_0^T |\hat{\alpha}_s^n|^2ds\bigg].$$

Bounds like (3.20) allow us to control for any $0 \leq s \leq T$, the Wasserstein distance between the distribution of X_s^n and the Dirac mass at the point $\mathbb{E}(X_s^n)$.

Step 2. We now compare X^n to the process controlled by the null control. So we consider case (ii) in (3.15), and now

$$U_t^n = x_0 + \int_0^t \big[b_0(s, \mathbb{P}_{X_s^n}) + b_1(s)U_s^n\big]ds + \sigma W_t, \quad t \in [0, T].$$

Since no confusion is possible, we still denote the solution by U^n although it is different from the one in the first step. By the boundedness of b_0 in (A.5), it holds that $\sup_{n \geq 1} \mathbb{E}[\sup_{0 \leq s \leq T} |U_s^n|^2] < +\infty$. Using Theorem 2.2 as before in the derivation of (3.17) and (3.18), we get

$$g^n\big(\mathbb{E}(X_T^n), \mathbb{P}_{X_T^n}\big) + \int_0^T \big[\lambda'\mathbb{E}(|\hat{\alpha}_s^n|^2) + f^n\big(s, \mathbb{E}(X_s^n), \mathbb{P}_{X_s^n}, \mathbb{E}(\hat{\alpha}_s^n)\big)\big]ds$$
$$\leq \mathbb{E}\bigg[g^n\big(U_T^n, \mathbb{P}_{X_T^n}\big) + \int_0^T f^n\big(s, U_s^n, \mathbb{P}_{X_s^n}, 0\big)ds\bigg].$$

By convexity of f^n with respect to α (see (A.2)) together with (A.6), we have

$$g^n\big(\mathbb{E}(X_T^n),\mathbb{P}_{X_T^n}\big) + \int_0^T \big[\lambda'\mathbb{E}\big(|\hat{\alpha}_s^n|^2\big) + f^n\big(s,\mathbb{E}(X_s^n),\mathbb{P}_{X_s^n},0\big)\big]\,ds$$

$$\leq \mathbb{E}\bigg[g^n\big(U_T^n,\mathbb{P}_{X_T^n}\big) + \int_0^T f^n\big(s,U_s^n,\mathbb{P}_{X_s^n},0\big)ds\bigg] + c\mathbb{E}\int_0^T |\hat{\alpha}_s^n|ds$$

for some constant c, independent of n. Using (A.5), we obtain

$$g^n\big(\mathbb{E}(X_T^n),\delta_{\mathbb{E}(X_T^n)}\big) + \int_0^T \big[\lambda'\mathbb{E}\big(|\hat{\alpha}_s^n|^2\big) + f^n\big(s,\mathbb{E}(X_s^n),\delta_{\mathbb{E}(X_s^n)},0\big)\big]\,ds$$

$$\leq g^n\big(0,\delta_{\mathbb{E}(X_T^n)}\big) + \int_0^T f^n\big(s,0,\delta_{\mathbb{E}(X_s^n)},0\big)ds + c\mathbb{E}\int_0^T |\hat{\alpha}_s^n|ds$$

$$+ c\bigg(1 + \sup_{0\leq s\leq T} \big[\mathbb{E}\big[|X_s^n|^2\big]^{1/2}\big]\bigg)\bigg(1 + \sup_{0\leq s\leq T} \big[\mathrm{Var}(X_s^n)\big]^{1/2}\bigg),$$

the value of c possibly varying from line to line. From (3.21), Young's inequality yields

$$g^n\big(\mathbb{E}(X_T^n),\delta_{\mathbb{E}(X_T^n)}\big) + \int_0^T \bigg[\frac{\lambda'}{2}\mathbb{E}\big(|\hat{\alpha}_s^n|^2\big) + f^n\big(s,\mathbb{E}(X_s^n),\delta_{\mathbb{E}(X_s^n)},0\big)\bigg]\,ds$$

$$\leq g^n\big(0,\delta_{\mathbb{E}(X_T^n)}\big) + \int_0^T f^n\big(s,0,\delta_{\mathbb{E}(X_s^n)},0\big)ds + c\bigg(1 + \sup_{0\leq s\leq T}\big[\mathrm{Var}(X_s^n)\big]\bigg).$$

By (3.20), we obtain

$$g^n\big(\mathbb{E}(X_T^n),\delta_{\mathbb{E}(X_T^n)}\big) + \int_0^T \bigg[\frac{\lambda'}{2}\mathbb{E}\big(|\hat{\alpha}_s^n|^2\big) + f^n\big(s,\mathbb{E}(X_s^n),\delta_{\mathbb{E}(X_s^n)},0\big)\bigg]\,ds$$

$$\leq g^n\big(0,\delta_{\mathbb{E}(X_T^n)}\big) + \int_0^T f^n\big(s,0,\delta_{\mathbb{E}(X_s^n)},0\big)ds + c\bigg(1 + \bigg[\int_0^T \mathbb{E}\big(|\hat{\alpha}_s^n|^2\big)ds\bigg]^{1/2}\bigg).$$

Young's inequality and the convexity in x of g^n and f^n from (A.2)–(A.4) give

$$\big\langle\mathbb{E}(X_T^n),\partial_x g^n\big(0,\delta_{\mathbb{E}(X_T^n)}\big)\big\rangle + \int_0^T \bigg[\frac{\lambda'}{4}\mathbb{E}\big(|\hat{\alpha}_s^n|^2\big) + \big\langle\mathbb{E}(X_s^n),\partial_x f^n\big(s,0,\delta_{\mathbb{E}(X_s^n)},0\big)\big\rangle\bigg]\,ds \leq c.$$

By (A.7), we have $\mathbb{E}\int_0^T |\hat{\alpha}_s^n|^2 ds \leq c\big(1 + \sup_{0\leq s\leq T}\mathbb{E}\big[|X_s^n|^2\big]^{1/2}\big)$, and the bound (3.14) now follows from (3.21), and as a consequence

$$(3.22) \qquad\qquad \mathbb{E}\bigg[\sup_{0\leq s\leq T}|X_s^n|^2\bigg] \leq c.$$

Using (3.14) and (3.22), we can prove that the processes $(X^n)_{n\geq 1}$ are tight. Indeed, there exists a constant c', independent of n, such that, for any $0 \leq s \leq t \leq T$,

$$|X_t^n - X_s^n| \leq c'(t-s)^{1/2}\bigg[1 + \bigg(\int_0^T \big[|X_r^n|^2 + |\hat{\alpha}_r^n|^2\big]dr\bigg)^{1/2}\bigg] + c'|W_t - W_s|,$$

so that tightness follows from (3.14) and (3.22).

Step 3. Let μ be the limit of a convergent subsequence $(\mathbb{P}_{X^{n_p}})_{p\geq 1}$. By (3.22), $M_{2,\mathcal{C}([0,T];\mathbb{R}^d)}(\mu) < +\infty$. Therefore, by Lemma 3.5, FBSDE (2.13) has a unique

solution $(X_t, Y_t, Z_t)_{0 \leq t \leq T}$. Moreover, there exists $u : [0, T] \times \mathbb{R}^d \hookrightarrow \mathbb{R}^d$, which is c-Lipschitz in the variable x for the same constant c as in the statement of the lemma, such that $Y_t = u(t, X_t)$ for any $t \in [0, T]$. In particular,

$$(3.23) \qquad \sup_{0 \leq t \leq T} |u(t, 0)| \leq \sup_{0 \leq t \leq T} \left[\mathbb{E}\big[|u(t, X_t) - u(t, 0)|\big] + \mathbb{E}\big[|Y_t|\big] \right] < +\infty.$$

We deduce that there exists a constant c' such that $|u(t, x)| \leq c'(1 + |x|)$ for $t \in [0, T]$ and $x \in \mathbb{R}^d$. By (2.9) and (A.6), we deduce that (for a possibly new value of c') $|\hat{\alpha}(t, x, \mu_t, u(t, x))| \leq c'(1 + |x|)$. Plugging this bound into the forward SDE satisfied by X in (2.13), we deduce that

$$(3.24) \qquad \forall \ell \geq 1, \quad \mathbb{E}\left[\sup_{0 \leq t \leq T} |X_t|^\ell \right] < +\infty,$$

and, thus,

$$(3.25) \qquad \mathbb{E} \int_0^T |\hat{\alpha}_t|^2 dt < +\infty,$$

with $\hat{\alpha}_t = \hat{\alpha}(t, X_t, \mu_t, Y_t)$ for $t \in [0, T]$. We can now apply the same argument to any $(X_t^n)_{0 \leq t \leq T}$ for any $n \geq 1$. We claim

$$(3.26) \qquad \forall \ell \geq 1, \quad \sup_{n \geq 1} \mathbb{E}\left[\sup_{0 \leq t \leq T} |X_t^n|^\ell \right] < +\infty.$$

Indeed, the constant c in the statement of Lemma 3.5 does not depend on n. Moreover, the second-order moments of $\sup_{0 \leq t \leq T} |X_t^n|$ are bounded, uniformly in $n \geq 1$ by (3.22). By (A.5), the driver in the backward component in (2.13) is at most of linear growth in (x, y, α), so that by (3.14) and standard L^2 estimates for BSDEs (see Theorem 3.3, Chapter 7 in [26]), the second-order moments of $\sup_{0 \leq t \leq T} |Y_t^n|$ are uniformly bounded as well. This shows (3.26) by repeating the proof of (3.24). By (3.24) and (3.26), we get that $\sup_{0 \leq t \leq T} W_2(\mu_t^{n_p}, \mu_t) \to 0$ as p tends to $+\infty$, with $\mu^{n_p} = \mathbb{P}_{X^{n_p}}$.

Repeating the proof of (3.13), we have

$$(3.27) \qquad \begin{aligned} \lambda' \mathbb{E} \int_0^T |\hat{\alpha}_t^n - \hat{\alpha}_t|^2 dt &\leq J^n\big(\hat{\alpha}; \mu^n\big) - J\big(\hat{\alpha}; \mu\big) + J\big([\hat{\alpha}^n, \mu^n]; \mu\big) - J^n\big(\hat{\alpha}^n; \mu^n\big) \\ &\quad + \mathbb{E} \int_0^T \langle b_0(t, \mu_t^n) - b_0(t, \mu_t), Y_t \rangle dt, \end{aligned}$$

where $J(\cdot; \mu)$ is given by (2.12) and $J^n(\cdot; \mu^n)$ is defined in a similar way, but with (f, g) and $(\mu_t)_{0 \leq t \leq T}$ replaced by (f^n, g^n) and $(\mu_t^n)_{0 \leq t \leq T}$; $J([\hat{\alpha}^n, \mu^n]; \mu)$ is defined as in (2.16). With these definitions at hand, we notice that

$$\begin{aligned} &J^n\big(\hat{\alpha}; \mu^n\big) - J\big(\hat{\alpha}; \mu\big) \\ &= \mathbb{E}\big[g^n(U_T^n, \mu_T^n) - g(X_T, \mu_T)\big] + \mathbb{E} \int_0^T \big[f^n\big(t, U_t^n, \mu_t^n, \hat{\alpha}_t\big) - f\big(t, X_t, \mu_t, \hat{\alpha}_t\big)\big] dt, \end{aligned}$$

where U^n is the controlled diffusion process

$$dU_t^n = \big[b_0(t, \mu_t^n) + b_1(t) U_t^n + b_2(t) \hat{\alpha}_t\big] dt + \sigma dW_t, \quad t \in [0, T]; \quad U_0^n = x_0.$$

By Gronwall's lemma and by convergence of μ^{n_p} towards μ for the 2-Wasserstein distance, we claim that $U^{n_p} \to X$ as $p \to +\infty$ for the norm $\mathbb{E}[\sup_{0\leq s\leq T} |\cdot_s|^2]^{1/2}$. Using on one hand the uniform convergence of f^n and g^n towards f and g on bounded subsets of their respective domains, and on the other hand the convergence of μ^{n_p} towards μ together with the bounds (3.24)–(3.26), we deduce that $J^{n_p}(\hat{\alpha}; \mu^{n_p}) \to J(\hat{\alpha}; \mu)$ as $p \to +\infty$. Similarly, using the bounds (3.14) and (3.24)–(3.26), the other differences in the right-hand side in (3.27) tend to 0 along the subsequence $(n_p)_{p\geq 1}$ so that $\hat{\alpha}^{n_p} \to \hat{\alpha}$ as $p \to +\infty$ in $L^2([0,T] \times \Omega, dt \otimes d\mathbb{P})$. We deduce that X is the limit of the sequence $(X^{n_p})_{p\geq 1}$ for the norm $\mathbb{E}[\sup_{0\leq s\leq T} |\cdot_s|^2]^{1/2}$. Therefore, μ matches the law of X exactly, proving that (3.1) is solvable. □

3.6. Choice of the approximating sequence. In order to complete the proof of Theorem 3.2, we must specify the choice of the approximating sequence in Lemma 3.9. Actually, the choice is performed in two steps. We first consider the case when the cost functions f and g are strongly convex in the variables x,

LEMMA 3.10. *Assume that, in addition to* (A.1)–(A.7), *there exists a constant $\gamma > 0$ such that the functions f and g satisfy* (*compare with* (2.8)):

$$
\begin{aligned}
&f(t, x', \mu, \alpha') - f(t, x, \mu, \alpha) \\
(3.28) \quad &\quad - \langle (x' - x, \alpha' - \alpha), \partial_{(x,\alpha)} f(t, x, \mu, \alpha) \rangle \geq \gamma |x' - x|^2 + \lambda |\alpha' - \alpha|^2, \\
&g(x', \mu) - g(x, \mu) - \langle x' - x, \partial_x g(x, \mu) \rangle \geq \gamma |x' - x|^2.
\end{aligned}
$$

Then, there exist two positive constants λ' and c'_L, depending only upon λ, c_L, and γ, and two sequences of functions $(f^n)_{n\geq 1}$ and $(g^n)_{n\geq 1}$ such that

(i) *for any $n \geq 1$, f^n and g^n satisfy* (A.1)–(A.7) *with respect to the parameters λ' and c'_L and $\partial_x f^n$ and $\partial_x g^n$ are bounded,*

(ii) *for any bounded subsets of $[0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^k$, there exists an integer n_0, such that, for any $n \geq n_0$, f^n and g^n coincide with f and g, respectively.*

The proof of Lemma 3.10 is a pure technical exercise in convex analysis, and for this reason, we postpone its proof to the appendix in section 5.

3.7. Proof of Theorem 3.2. Equation (3.1) is solvable when, in addition to (A.1)–(A.7), f and g satisfy the convexity condition (3.28). Indeed, by Lemma 3.10, there exists an approximating sequence $(f^n, g^n)_{n\geq 1}$ satisfying (i) and (ii) in the statement of Lemma 3.9, and also (iii) by Proposition 3.8. When f and g satisfy (A.1)–(A.7) only, the assumptions of Lemma 3.9 are satisfied with the following approximating sequence:

$$
f_n(t, x, \mu, \alpha) = f(t, x, \mu, \alpha) + \frac{1}{n}|x|^2; \quad g_n(x, \mu) = g(x, \mu) + \frac{1}{n}|x|^2
$$

for $(t, x, \mu, \alpha) \in [0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \times \mathbb{R}^k$ and $n \geq 1$. Therefore, (3.1) is solvable under (A.1)–(A.7). Moreover, given an arbitrary solution to (3.1), the existence of a function u, as in the statement of Theorem 3.2, follows from Lemma 3.5 and (3.23). Boundedness of the moments of the forward process is then proven as in (3.24). □

4. Propagation of chaos and approximate Nash equilibriums. While the rationale for the mean-field strategy proposed by Lasry and Lions is clear given the nature of Nash equilibriums (as opposed to other forms of optimization suggesting the optimal control of stochastic dynamics of the McKean–Vlasov type as studied in [6]), it may not be obvious how the solution of the FBSDE introduced and solved in the previous sections provides approximate Nash equilibriums for large games. In this

section, we prove just that. The proof relies on the Lipschitz property of the FBSDE value function, standard arguments in propagation of chaos theory, and the following specific result due to Horowitz and Karandikar (see, for example, section 10 in [24]) which we state as a lemma for future reference.

LEMMA 4.1. *Given* $\mu \in \mathcal{P}_{d+5}(\mathbb{R}^d)$, *there exists a constant c depending only upon* d *and* $M_{d+5}(\mu)$ *(see the notation* (2.7)*), such that*

$$\mathbb{E}\big[W_2^2(\bar{\mu}^N, \mu)\big] \leq CN^{-2/(d+4)},$$

where $\bar{\mu}^N$ *denotes the empirical measure of any sample of size N from* μ.

Throughout this section, assumptions (A.1)–(A.7) are in force. We let $(X_t, Y_t, Z_t)_{0 \leq t \leq T}$ be a solution of (3.1) and let u be the associated FBSDE value function. We denote by $(\mu_t)_{0 \leq t \leq T}$ the flow of marginal probability measures $\mu_t = \mathbb{P}_{X_t}$ for $0 \leq t \leq T$. We also denote by J the optimal cost of the limiting mean-field problem

$$(4.1) \qquad J = \mathbb{E}\bigg[g(X_T, \mu_T) + \int_0^T f\big(t, X_t, \mu_t, \hat{\alpha}(t, X_t, \mu_t, Y_t)\big)dt\bigg],$$

where as before, $\hat{\alpha}$ is the minimizer function constructed in Lemma 2.1. For convenience, we fix a sequence $((W_t^i)_{0 \leq t \leq T})_{i \geq 1}$ of independent m-dimensional Brownian motions, and for each integer N, we consider the solution $(X_t^1, \ldots, X_t^N)_{0 \leq t \leq T}$ of the system of N stochastic differential equations

$$(4.2) \quad dX_t^i = b\big(t, X_t^i, \bar{\mu}_t^N, \hat{\alpha}(t, X_t^i, \mu_t, u(t, X_t^i))\big)dt + \sigma dW_t^i, \qquad \bar{\mu}_t^N = \frac{1}{N}\sum_{j=1}^N \delta_{X_t^j},$$

with $t \in [0, T]$ and $X_0^i = x_0$. Equation (4.2) is well posed since u satisfies the regularity property (3.3) and the minimizer $\hat{\alpha}(t, x, \mu_t, y)$ was proven, in Lemma 2.1, to be Lipschitz continuous and at most of linear growth in the variables x and y, uniformly in $t \in [0, T]$. The processes $(X^i)_{1 \leq i \leq N}$ give the dynamics of the private states of the N players in the stochastic differential game of interest when the players use the strategies

$$(4.3) \qquad \bar{\alpha}_t^{N,i} = \hat{\alpha}(t, X_t^i, \mu_t, u(t, X_t^i)), \qquad 0 \leq t \leq T, \ \ i \in \{1, \ldots, N\}.$$

These strategies are in closed loop form. They are even *distributed* since at each time $t \in [0, T]$, a player need only know the state of his own private state in order to compute the value of the control to apply at that time. By boundedness of b_0 and by (2.9) and (3.3), it holds that

$$(4.4) \qquad \sup_{N \geq 1} \max_{1 \leq i \leq N} \bigg[\mathbb{E}\bigg[\sup_{0 \leq t \leq T} |X_t^i|^2\bigg] + \mathbb{E}\int_0^T |\bar{\alpha}_t^{N,i}|^2 dt\bigg] < +\infty.$$

For the purpose of comparison, we recall the notation we use when the players choose a generic set of strategies, say $((\beta_t^i)_{0 \leq t \leq T})_{1 \leq i \leq N}$. In this case, the dynamics of the private state U^i of player $i \in \{1, \ldots, N\}$ are given by

$$(4.5) \qquad dU_t^i = b\big(t, U_t^i, \bar{\nu}_t^N, \beta_t^i\big)dt + \sigma dW_t^i, \qquad \bar{\nu}_t^N = \frac{1}{N}\sum_{j=1}^N \delta_{U_t^j},$$

with $t \in [0, T]$ and $U_0^i = x_0$, and where $((\beta_t^i)_{0 \leq t \leq T})_{1 \leq i \leq N}$ are N square-integrable $\mathbb{R}^k$-valued processes that are progressively measurable with respect to the filtration generated by $(W^1, \ldots, W^N)$. For each $1 \leq i \leq N$, we denote by

$$(4.6) \qquad \bar{J}^{N,i}(\beta^1, \ldots, \beta^N) = \mathbb{E}\left[g\left(U_T^i, \bar{\nu}_T^N\right) + \int_0^T f(t, U_t^i, \bar{\nu}_t^N, \beta_t^i)dt\right],$$

the cost to the ith player. Our goal is to construct approximate Nash equilibriums for the N-player game. We follow the approach used by Bensoussan et al. [2] in the linear-quadratic case. See also [5].

THEOREM 4.2. *Under assumptions* (A.1)–(A.7), *the strategies* $(\bar{\alpha}_t^{N,i})_{0 \leq t \leq T, \, 1 \leq i \leq N}$ *defined in* (4.3) *form an approximate Nash equilibrium of the N-player game* (4.5)–(4.6). *More precisely, there exists a constant $c > 0$ and a sequence of positive numbers* $(\epsilon_N)_{N \geq 1}$ *such that, for each $N \geq 1$,*
- (i) $\epsilon_N \leq cN^{-1/(d+4)}$;
- (ii) *for any player $i \in \{1, \ldots, N\}$ and any progressively measurable strategy $\beta^i = (\beta_t^i)_{0 \leq t \leq T}$, such that $\mathbb{E}\int_0^T |\beta_t^i|^2 dt < +\infty$, one has*

$$(4.7) \qquad \bar{J}^{N,i}(\bar{\alpha}^{N,1}, \ldots, \bar{\alpha}^{N,i-1}, \beta^i, \bar{\alpha}^{N,i+1}, \ldots, \bar{\alpha}^{N,N}) \geq \bar{J}^{N,i}(\bar{\alpha}^{N,1}, \ldots, \bar{\alpha}^{N,N}) - \epsilon_N.$$

Proof. By symmetry (invariance under permutation) of the coefficients of the private states dynamics and costs, we need only prove (4.7) for $i = 1$. Given a progressively measurable process $\beta^1 = (\beta_t^1)_{0 \leq t \leq T}$ satisfying $\mathbb{E}\int_0^T |\beta_t^1|^2 dt < +\infty$, let us use the quantities defined in (4.5) and (4.6) with $\beta_t^i = \bar{\alpha}_t^{N,i}$ for $i \in \{2, \ldots, N\}$ and $t \in [0, T]$. By boundedness of b_0, b_1, and b_2 and by Gronwall's inequality, we get

$$(4.8) \qquad \mathbb{E}\left[\sup_{0 \leq t \leq T} |U_t^1|^2\right] \leq c\left(1 + \mathbb{E}\int_0^T |\beta_t^1|^2 dt\right).$$

Using the fact that the strategies $(\bar{\alpha}_t^{N,i})_{0 \leq t \leq T}$ satisfy the square integrability condition of admissibility, the same argument gives

$$(4.9) \qquad \mathbb{E}\left[\sup_{0 \leq t \leq T} |U_s^i|^2\right] \leq c,$$

for $2 \leq i \leq N$, which clearly implies after summation that

$$(4.10) \qquad \frac{1}{N}\sum_{j=1}^N \mathbb{E}\left[\sup_{0 \leq t \leq T} |U_t^j|^2\right] \leq c\left(1 + \frac{1}{N}\mathbb{E}\int_0^T |\beta_t^1|^2 dt\right).$$

For the next step of the proof we introduce the system of decoupled independent and identically distributed states

$$d\bar{X}_t^i = b\left(t, \bar{X}_t^i, \mu_t, \hat{\alpha}(t, \bar{X}_t^i, \mu_t, u(t, \bar{X}_t^i))\right)dt + \sigma dW_t^i, \quad 0 \leq t \leq T.$$

Notice that the stochastic processes $\bar{X}^i$ are independent copies of X and, in particular, $\mathbb{P}_{\bar{X}_t^i} = \mu_t$ for any $t \in [0, T]$ and $i \in \{1, \cdots, N\}$. We shall use the notation

$$\hat{\alpha}_t^i = \hat{\alpha}\left(t, \bar{X}_t^i, \mu_t, u(t, \bar{X}_t^i)\right), \quad t \in [0, T], \quad i \in \{1, \ldots, N\}.$$

Using the regularity of the FBSDE value function u and the uniform boundedness of the family $(M_{d+5}(\mu_t))_{0 \leq t \leq T}$ derived in Theorem 3.2 together with the estimate

recalled in Lemma 4.1, we can follow Sznitman's proof [25] (see also Theorem 1.3 of [15]) and get

$$(4.11) \qquad \max_{1 \leq i \leq N} \mathbb{E}\left[\sup_{0 \leq t \leq T} |X_t^i - \bar{X}_t^i|^2\right] \leq cN^{-2/(d+4)},$$

(recall that $(X^1, \ldots, X^N)$ solves (4.2)), and this implies

$$(4.12) \qquad \sup_{0 \leq t \leq T} \mathbb{E}\left[W_2^2(\bar{\mu}_t^N, \mu_t)\right] \leq cN^{-2/(d+4)}.$$

Indeed, for each $t \in [0, T]$,

$$(4.13) \qquad W_2^2(\bar{\mu}_t^N, \mu_t) \leq \frac{2}{N} \sum_{i=1}^{N} |X_t^i - \bar{X}_t^i|^2 + 2W_2^2\left(\frac{1}{N}\sum_{i=1}^{N} \delta_{\bar{X}_t^i}, \mu_t\right),$$

so that, taking expectations on both sides and using (4.11) and Lemma 4.1, we get the desired estimate (4.12). Using the local-Lipschitz regularity of the coefficients g and f together with Cauchy–Schwarz inequality, we get, for each $i \in \{1, \ldots, N\}$,

$$\left|J - \bar{J}^{N,i}(\bar{\alpha}^{N,1}, \ldots, \bar{\alpha}^{N,N})\right|$$
$$= \left|\mathbb{E}\left[g(\bar{X}_T^i, \mu_T) + \int_0^T f(t, \bar{X}_t^i, \mu_t, \hat{\alpha}_t^i)dt - g(X_T^i, \bar{\mu}_T^N) - \int_0^T f(t, X_t^i, \bar{\mu}_t^N, \bar{\alpha}_t^{N,i})dt\right]\right|$$
$$\leq c\mathbb{E}\left[\left(1 + |\bar{X}_T^i|^2 + |X_T^i|^2 + \frac{1}{N}\sum_{j=1}^{N}|X_T^j|^2\right)\right]^{1/2} \mathbb{E}\left[|\bar{X}_T^i - X_T^i|^2 + W_2^2(\mu_T, \bar{\mu}_T^N)\right]^{1/2}$$
$$+ c\int_0^T \left\{\mathbb{E}\left[\left(1 + |\bar{X}_t^i|^2 + |X_t^i|^2 + |\hat{\alpha}_t^i|^2 + |\bar{\alpha}_t^{N,i}|^2 + \frac{1}{N}\sum_{j=1}^{N}|X_t^j|^2\right)\right]^{1/2}\right.$$
$$\left. \times \mathbb{E}\left[|\bar{X}_t^i - X_t^i|^2 + |\hat{\alpha}_t^i - \bar{\alpha}_t^{N,i}|^2 + W_2^2(\mu_t, \bar{\mu}_t^N)\right]^{1/2}\right\} dt$$

for some constant $c > 0$ which can change from line to line. By (4.4), we deduce

$$\left|J - \bar{J}^{N,i}(\bar{\alpha}^{N,1}, \ldots, \bar{\alpha}^{N,N})\right| \leq c\mathbb{E}\left[|\bar{X}_T^i - X_T^i|^2 + W_2^2(\mu_T, \bar{\mu}_T^N)\right]^{1/2}$$
$$+ c\left(\int_0^T \mathbb{E}\left[|\bar{X}_t^i - X_t^i|^2 + |\hat{\alpha}_t^i - \bar{\alpha}_t^{N,i}|^2 + W_2^2(\mu_t, \bar{\mu}_t^N)\right]dt\right)^{1/2}.$$

Now, by the Lipschitz property of the minimizer $\hat{\alpha}$ proven in Lemma 2.1 and by the Lipschitz property of u in (3.3), we notice that

$$|\hat{\alpha}_t^i - \bar{\alpha}_t^{N,i}| = \left|\hat{\alpha}\left(t, \bar{X}_t^i, \mu_t, u(t, \bar{X}_t^i)\right) - \hat{\alpha}\left(t, X_t^i, \mu_t, u(t, X_t^i)\right)\right| \leq c|\bar{X}_t^i - X_t^i|.$$

Using (4.11) and (4.12), this proves that, for any $1 \leq i \leq N$,

$$(4.14) \qquad \bar{J}^{N,i}(\bar{\alpha}^{1,N}, \ldots, \bar{\alpha}^{N,N}) = J + O(N^{-1/(d+4)}).$$

This suggests that, in order to prove inequality (4.7) for $i = 1$, we could restrict ourselves to compare $\bar{J}^{N,1}(\beta^1, \bar{\alpha}^{2,N}, \ldots, \bar{\alpha}^{N,N})$ to J. Using the argument which led

to (4.8), (4.9), and (4.10), together with the definitions of U^j and X^j for $j = 1, \dots, N$, we get, for any $t \in [0, T]$,

$$\mathbb{E}\left[\sup_{0 \leq s \leq t} |U_t^1 - X_t^1|^2\right] \leq \frac{c}{N} \int_0^t \sum_{j=1}^N \mathbb{E}\left[\sup_{0 \leq r \leq s} |U_r^j - X_r^j|^2\right] ds + c\mathbb{E}\int_0^T |\beta_t^1 - \bar{\alpha}_t^{N,1}|^2 dt,$$

$$\mathbb{E}\left[\sup_{0 \leq s \leq t} |U_t^i - X_t^i|^2\right] \leq \frac{c}{N} \int_0^t \sum_{j=1}^N \mathbb{E}\left[\sup_{0 \leq r \leq s} |U_r^j - X_r^j|^2\right] ds, \quad 2 \leq i \leq N.$$

Therefore, using Gronwall's inequality, we get

$$(4.15) \qquad \frac{1}{N} \sum_{j=1}^N \mathbb{E}\left[\sup_{0 \leq t \leq T} |U_t^j - X_t^j|^2\right] \leq \frac{c}{N} \mathbb{E}\int_0^T |\beta_t^1 - \bar{\alpha}_t^{N,1}|^2 dt,$$

so that

$$(4.16) \qquad \sup_{0 \leq t \leq T} \mathbb{E}\left[|U_t^i - X_t^i|^2\right] \leq \frac{c}{N}\mathbb{E}\int_0^T |\beta_t^1 - \bar{\alpha}_t^{N,1}|^2 dt, \quad 2 \leq i \leq N.$$

Putting together (4.4), (4.11), and (4.16), we see that, for any $A > 0$, there exists a constant c_A depending on A such that

$$(4.17) \qquad \mathbb{E}\int_0^T |\beta_t^1|^2 dt \leq A \quad \implies \quad \max_{2 \leq i \leq N} \sup_{0 \leq t \leq T} \mathbb{E}\left[|U_t^i - \bar{X}_t^i|^2\right] \leq c_A N^{-2/(d+4)}.$$

Let us fix $A > 0$ (to be determined later) and assume that $\mathbb{E}\int_0^T |\beta_t^1|^2 dt \leq A$. Using (4.17) we see that

$$(4.18) \qquad \frac{1}{N-1} \sum_{j=2}^N \mathbb{E}\left[|U_t^j - \bar{X}_t^j|^2\right] \leq c_A N^{-2/(d+4)}$$

for a constant c_A depending upon A, and whose value can change from line to line. Now by the triangle inequality for the Wasserstein distance,

(4.19)

$$\mathbb{E}\left[W_2^2(\bar{\nu}_t^N, \mu_t)\right] \leq c\left\{\mathbb{E}\left[W_2^2\left(\frac{1}{N} \sum_{j=1}^N \delta_{U_t^j}, \frac{1}{N-1} \sum_{j=2}^N \delta_{U_t^j}\right)\right]\right.$$

$$\left. + \frac{1}{N-1} \sum_{j=2}^N \mathbb{E}\left[|U_t^j - \bar{X}_t^j|^2\right] + \mathbb{E}\left[W_2^2\left(\frac{1}{N-1} \sum_{j=2}^N \delta_{\bar{X}_t^j}, \mu_t\right)\right]\right\}.$$

We note that

$$\mathbb{E}\left[W_2^2\left(\frac{1}{N} \sum_{j=1}^N \delta_{U_t^j}, \frac{1}{N-1} \sum_{j=2}^N \delta_{U_t^j}\right)\right] \leq \frac{1}{N(N-1)} \sum_{j=2}^N \mathbb{E}\left[|U_t^1 - U_t^j|^2\right],$$

which is $O(N^{-1})$ because of (4.8) and (4.10). Plugging this inequality into (4.19), and using (4.18) to control the second term and Lemma 4.1 to estimate the third term therein, we conclude that

$$(4.20) \qquad \mathbb{E}\left[W_2^2(\bar{\nu}_t^N, \mu_t)\right] \leq c_A N^{-2/(d+4)}.$$

For the final step of the proof we define $(\bar{U}_t^1)_{0 \le t \le T}$ as the solution of the SDE

$$d\bar{U}_t^1 = b(t, \bar{U}_t^1, \mu_t, \beta_t^1)dt + \sigma dW_t^1, \quad 0 \le t \le T; \quad \bar{U}_0^1 = x,$$

so that, from the definition (4.5) of U^1, we get

$$U_t^1 - \bar{U}_t^1 = \int_0^t [b_0(s, \bar{\nu}_s^N) - b_0(s, \mu_s)]ds + \int_0^t b_1(s)[U_s^1 - \bar{U}_s^1]ds.$$

Using the Lipschitz property of b_0, (4.20), and the boundedness of b_1 and applying Gronwall's inequality, we get

(4.21)
$$\sup_{0 \le t \le T} \mathbb{E}\big[|U_t^1 - \bar{U}_t^1|^2\big] \le c_A N^{-2/(d+4)},$$

so that, going over the computation leading to (4.14) once more and using (4.20), (4.8), (4.9), and (4.10),

$$\bar{J}^{N,1}(\beta^1, \bar{\alpha}^{N,2}, \dots, \bar{\alpha}^{N,N}) \ge J(\beta^1) - c_A N^{-1/(d+4)},$$

where $J(\beta^1)$ stands for the *mean-field cost* of β^1:

(4.22)
$$J(\beta^1) = \mathbb{E}\bigg[g(\bar{U}_T^1, \mu_T) + \int_0^T f\big(t, \bar{U}_t^1, \mu_t, \beta_t^1\big)dt\bigg].$$

Since $J \le J(\beta^1)$ (notice that, even though β^1 is adapted to a larger filtration than the filtration of W^1, the stochastic maximum principle still applies as pointed out in Remark 2.3), we get in the end

(4.23)
$$\bar{J}^{N,1}(\beta^1, \bar{\alpha}^{N,2}, \dots, \bar{\alpha}^{N,N}) \ge J - c_A N^{-1/(d+4)},$$

and from (4.14) and (4.23), we easily derive the desired inequality (4.7). Actually, the combination of (4.14) and (4.23) shows that $(\bar{\alpha}^{N,1}, \dots, \bar{\alpha}^{N,N})$ is an ϵ-Nash equilibrium for N large enough, with a precise quantification (though not optimal) of the relationship between N and ϵ. But for the proof to be complete in full generality, we need to explain how we choose A, and discuss what happens when $\mathbb{E}\int_0^T |\beta_t^1|^2 dt > A$.

Using the convexity in x of g around $x = 0$ and the convexity of f in (x, α) around $x = 0$ and $\alpha = 0$ (see (2.8)), we get

$$\bar{J}^{N,1}(\beta^1, \bar{\alpha}^{N,2}, \dots, \bar{\alpha}^{N,N})$$

$$\ge \mathbb{E}\bigg[g(0, \bar{\nu}_T^N) + \int_0^T f(t, 0, \bar{\nu}_t^N, 0)dt\bigg] + \lambda \mathbb{E}\int_0^T |\beta_t^1|^2 dt$$

$$+ \mathbb{E}\bigg[\langle U_T^1, \partial_x g(0, \bar{\nu}_T^N)\rangle + \int_0^T \big(\langle U_t^1, \partial_x f(t, 0, \bar{\nu}_t^N, 0)\rangle + \langle \beta_t^1, \partial_\alpha f(t, 0, \bar{\nu}_t^N, 0)\rangle\big)dt\bigg].$$

The local-Lipschitz assumption with respect to the Wasserstein distance and the definition of the latter imply the existence of a constant $c > 0$ such that for any $t \in [0, T]$,

$$\mathbb{E}\big[|f(t, 0, \bar{\nu}_t^N, 0) - f(t, 0, \delta_0, 0)|\big] \le c\mathbb{E}\big[1 + M_2^2(\bar{\nu}_t^N)\big] = c\bigg[1 + \bigg(\frac{1}{N}\sum_{i=1}^N \mathbb{E}\big[|U_t^i|^2\big]\bigg)\bigg].$$

with a similar inequality for g. From this, we deduce

$$\bar{J}^{N,1}(\beta^1, \bar{\alpha}^{N,2}, \ldots, \bar{\alpha}^{N,N}) \geq g(0, \delta_0) + \int_0^T f(t, 0, \delta_0, 0)dt$$

$$+ \mathbb{E}\left[\langle U_T^1, \partial_x g(0, \bar{\nu}_T^N)\rangle + \int_0^T \left(\langle U_t^1, \partial_x f(t, 0, \bar{\nu}_t^N, 0)\rangle + \langle \beta_t^1, \partial_\alpha f(t, 0, \bar{\nu}_t^N, 0)\rangle\right)dt\right]$$

$$+ \lambda \mathbb{E}\int_0^T |\beta_t^1|^2 dt - c\left[1 + \left(\frac{1}{N}\sum_{i=1}^N \sup_{0 \leq t \leq T} \mathbb{E}\left[|U_t^i|^2\right]\right)\right].$$

By (A.5), we know that $\partial_x g$, $\partial_x f$, and $\partial_\alpha f$ are, at most, of linear growth in the measure parameter (for the L^2-norm), so that, for any $\delta > 0$, there exists a constant c_δ such that

(4.24)

$$\bar{J}^{N,1}(\beta^1, \bar{\alpha}^{N,2}, \ldots, \bar{\alpha}^{N,N}) \geq g(0, \delta_0) + \int_0^T f(t, 0, \delta_0, 0)dt + \frac{\lambda}{2}\mathbb{E}\int_0^T |\beta_t^1|^2 dt$$

$$- \delta \sup_{0 \leq t \leq T} \mathbb{E}\left[|U_t^1|^2\right] - c_\delta\left(1 + \frac{1}{N}\sum_{i=1}^N \sup_{0 \leq t \leq T} \mathbb{E}\left[|U_t^i|^2\right]\right).$$

Estimates (4.8) and (4.9) show that one can choose δ small enough in (4.24) and c so that

$$\bar{J}^{N,1}(\beta^1, \bar{\alpha}^{N,2}, \ldots, \bar{\alpha}^{N,N}) \geq -c + \left(\frac{\lambda}{4} - \frac{c}{N}\right)\mathbb{E}\int_0^T |\beta_t^1|^2 dt.$$

This proves that there exists an integer N_0 such that, for any integer $N \geq N_0$ and constant $\bar{A} > 0$, one can choose $A > 0$ such that

(4.25) $$\mathbb{E}\int_0^T |\beta_t^1|^2 dt \geq A \quad \Longrightarrow \quad \bar{J}^{N,1}(\beta^1, \bar{\alpha}^{N,2}, \ldots, \bar{\alpha}^{N,N}) \geq J + \bar{A},$$

which provides us with the appropriate tool to choose A and avoid having to consider $(\beta_t^1)_{0 \leq t \leq T}$ whose expected square integral is too large. $\square$

A simple inspection of the last part of the above proof shows that a stronger result actually holds when $\mathbb{E}\int_0^T |\beta_t^1|^2 dt \leq A$. Indeed, the estimates (4.8), (4.17), and (4.20) can be used as in (4.14) to deduce (up to a modification of c_A)

(4.26) $$\bar{J}^{N,i}(\beta^1, \bar{\alpha}^{N,2}, \ldots, \bar{\alpha}^{N,N}) \geq J - c_A N^{-1/(d+4)}, \quad 2 \leq i \leq N.$$

COROLLARY 4.3. *Under assumptions* (A.1)–(A.7), *not only does*

$$\left((\bar{\alpha}_t^{N,i} = \hat{\alpha}(t, X_t^i, \mu_t, u(t, X_t^i)))_{1 \leq i \leq N}\right)_{0 \leq t \leq T}$$

form an approximate Nash equilibrium of the N-player game (4.5)–(4.6) *but*
   (i) *there exists an integer N_0 such that, for any $N \geq N_0$ and $\bar{A} > 0$, there exists a constant $A > 0$ such that, for any player $i \in \{1, \ldots, N\}$ and any admissible strategy $\beta^i = (\beta_t^i)_{0 \leq t \leq T}$,*
(4.27)

$$\mathbb{E}\int_0^T |\beta_t^i|^2 dt \geq A \quad \Longrightarrow \quad \bar{J}^{N,i}(\bar{\alpha}^{N,1}, \ldots, \bar{\alpha}^{N,i-1}, \beta^i, \bar{\alpha}^{N,i+1}, \ldots, \bar{\alpha}^{N,N}) \geq J + \bar{A}.$$

(ii) *Moreover, for any $A > 0$, there exists a sequence of positive real numbers $(\epsilon_N)_{N \geq 1}$ converging toward 0, such that for any admissible strategy $\beta^1 = (\beta^1_t)_{0 \leq t \leq T}$ for the first player,*

$$(4.28) \qquad \mathbb{E} \int_0^T |\beta^1_t|^2 dt \leq A \quad \Longrightarrow \quad \min_{1 \leq i \leq N} \bar{J}^{N,i}(\beta^1, \bar{\alpha}^{N,2}, \ldots, \bar{\alpha}^{N,N}) \geq J - \varepsilon_N.$$

5. Appendix: Proof of Lemma 3.10. We focus on the approximation of the running cost f (the case of the terminal cost g is similar) and we ignore the dependence of f upon t to simplify the notation. For any $n \geq 1$, we define f_n as the truncated Legendre transform:

$$(5.1) \qquad f_n(x, \mu, \alpha) = \sup_{|y| \leq n} \inf_{z \in \mathbb{R}^d} \big[\langle y, x - z \rangle + f(z, \mu, \alpha)\big]$$

for $(x, \alpha) \in \mathbb{R}^d \times \mathbb{R}^k$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$. By standard properties of the Legendre transform of convex functions,

$$(5.2) \qquad f_n(x, \mu, \alpha) \leq \sup_{y \in \mathbb{R}^d} \inf_{z \in \mathbb{R}^d} \big[\langle y, x - z \rangle + f(z, \mu, \alpha)\big] = f(x, \mu, \alpha).$$

Moreover, by strict convexity of f in x,

$$
\begin{aligned}
f_n(x, \mu, \alpha) &\geq \inf_{z \in \mathbb{R}^d} \big[f(z, \mu, \alpha)\big] \geq \inf_{z \in \mathbb{R}^d} \big[\gamma|z|^2 + \langle \partial_x f(0, \mu, \alpha), z \rangle \big] + f(0, \mu, \alpha) \\
(5.3) \qquad &\geq -\frac{1}{4\gamma}|\partial_x f(0, \mu, \alpha)|^2 + f(0, \mu, \alpha),
\end{aligned}
$$

so that f_n has finite real values. Clearly, it is also n-Lipschitz continuous in x.

Step 1. We first check that the sequence $(f_n)_{n \geq 1}$ converges towards f, uniformly on bounded subsets of $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^k$. So for any given $R > 0$, we restrict ourselves to $|x| \leq R$ and $|\alpha| \leq R$, and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, such that $M_2(\mu) \leq R$. By (A.5), there exists a constant $c > 0$, independent of R, such that

$$(5.4) \quad \sup_{z \in \mathbb{R}^d} \big[\langle y, z \rangle - f(z, \mu, \alpha)\big] \geq \sup_{z \in \mathbb{R}^d} \big[\langle y, z \rangle - c|z|^2\big] - c(1 + R^2) = \frac{|y|^2}{4c} - c(1 + R^2).$$

Therefore,

$$(5.5) \qquad \inf_{z \in \mathbb{R}^d} \big[\langle y, x - z \rangle + f(z, \mu, \alpha)\big] \leq R|y| - \frac{|y|^2}{4c} + c(1 + R^2).$$

By (5.3) and (A.5), $f_n(t, x, \mu, \alpha) \geq -c(1 + R^2)$, c depending possibly on γ, so that optimization in the variable y can be done over points $y^\star$ satisfying

$$(5.6) \qquad -c(1 + R^2) \leq R|y^\star| - \frac{|y^\star|^2}{4c} + c(1 + R^2), \quad \text{that is} \quad |y^\star| \leq c(1 + R),$$

In particular, for n large enough (depending on R),

$$(5.7) \qquad f_n(x, \mu, \alpha) = \sup_{y \in \mathbb{R}^d} \inf_{z \in \mathbb{R}^d} \big[\langle y, x - z \rangle + f(z, \mu, \alpha)\big] = f(x, \mu, \alpha).$$

So on bounded subsets of $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^k$, f_n and f coincide for n large enough. In particular, for n large enough, $f_n(0, \delta_0, 0)$, $\partial_x f_n(0, \delta_0, 0)$, and $\partial_\alpha f_n(0, \delta_0, 0)$ exist,

coincide with $f(0, \delta_0, 0)$, $\partial_x f(0, \delta_0, 0)$, and $\partial_\alpha f(0, \delta_0, 0)$, respectively, and are bounded by c_L as in (A.5). Moreover, still for $|x| \leq R$, $|\alpha| \leq R$, and $M_2(\mu) \leq R$, we see from (5.2) and (5.6) that optimization in z can be reduced to $z^\star$ satisfying

$$\langle y^\star, x - z^\star \rangle + f(z^\star, \mu, \alpha) \leq f(x, \mu, \alpha) \leq c(1 + R^2),$$

the second inequality following from (A.5). By strict convexity of f in x, we obtain

$$-c(1 + R)|z^\star| + \gamma |z^\star|^2 + \langle \partial_x f(0, \mu, \alpha), z^\star \rangle + f(0, \mu, \alpha) \leq c(1 + R^2),$$

so that, by (A.5), $\gamma |z^\star|^2 - c(1 + R)|z^\star| \leq c(1 + R^2)$, that is

$$(5.8) \qquad\qquad\qquad\qquad |z^\star| \leq c(1 + R).$$

Step 2. We now investigate the convexity property of $f_n(\cdot, \mu, \cdot)$ for a given $\mu \in \mathcal{P}_2(\mathbb{R}^d)$. For any $h \in \mathbb{R}$, $x, e, y, z_1, z_2 \in \mathbb{R}^d$, and $\alpha, \beta \in \mathbb{R}^k$, with $|y| \leq n$ and $|e|, |\beta| = 1$, we deduce from the convexity of $f(\cdot, \mu, \cdot)$ that

$$2 \inf_{z \in \mathbb{R}^d} \left[\langle y, x - z \rangle + f(z, \mu, \alpha) \right]$$

$$\leq \left\langle y, (x + he - z_1) + (x - he - z_2) \right\rangle + 2f\left(\frac{z_1 + z_2}{2}, \mu, \frac{(\alpha + h\beta) + (\alpha - h\beta)}{2} \right)$$

$$\leq \langle y, x + he - z_1 \rangle + f(z_1, \mu, \alpha + h\beta) + \langle y, x - he - z_2 \rangle + f(z_2, \mu, \alpha - h\beta) - 2\lambda h^2.$$

Taking infimum with respect to z_1, z_2 and supremum with respect to y, we obtain

$$(5.9) \qquad f_n(x, \mu, \alpha) \leq \frac{1}{2} f_n(x + he, \mu, \alpha + h\beta) + \frac{1}{2} f_n(x - he, \mu, \alpha - h\beta) - \lambda h^2.$$

In particular, the function $\mathbb{R}^d \times \mathbb{R}^k \ni (x, \alpha) \hookrightarrow f_n(x, \mu, \alpha) - \lambda |\alpha|^2$ is convex. We prove later on that it is also continuously differentiable so that (2.8) holds.

In a similar way, we can investigate the semi-concavity property of $f_n(\cdot, \mu, \cdot)$. For any $h \in \mathbb{R}$, $x, e, y_1, y_2 \in \mathbb{R}^d$, $\alpha, \beta \in \mathbb{R}^k$, with $|y_1|, |y_2| \leq n$ and $|e|, |\beta| = 1$,

$$\inf_{z \in \mathbb{R}^d} \left[\langle y_1, x + he - z \rangle + f(z, \mu, \alpha + h\beta) \right] + \inf_{z \in \mathbb{R}^d} \left[\langle y_2, x - he - z \rangle + f(z, \mu, \alpha - h\beta) \right]$$

$$= \inf_{z \in \mathbb{R}^d} \left[\langle y_1, x - z \rangle + f(z + he, \mu, \alpha + h\beta) \right] + \inf_{z \in \mathbb{R}^d} \left[\langle y_2, x - z \rangle + f(z - he, \mu, \alpha - h\beta) \right].$$

By expanding $f(\cdot, \mu, \cdot)$ up to the second order, we see that

$$\inf_{z \in \mathbb{R}^d} \left[\langle y_1, x + he - z \rangle + f(z, \mu, \alpha + h\beta) \right] + \inf_{z \in \mathbb{R}^d} \left[\langle y_2, x - he - z \rangle + f(z, \mu, \alpha - h\beta) \right]$$

$$\leq \inf_{z \in \mathbb{R}^d} \left[\langle y_1 + y_2, x - z \rangle + 2f(z, \mu, \alpha) \right] + c|h|^2$$

for some constant c. Taking the supremum over y_1, y_2, we deduce that

$$f_n(x + he, \mu, \alpha + h\beta) + f_n(x - he, \mu, \alpha - h\beta) - 2f_n(x, \mu, \alpha) \leq c|h|^2.$$

So for any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, the function $\mathbb{R}^d \times \mathbb{R}^k \ni (x, \alpha) \hookrightarrow f_n(x, \mu, \alpha) - c[|x|^2 + |\alpha|^2]$ is concave and $f_n(\cdot, \mu, \cdot)$ is $\mathcal{C}^{1,1}$, the Lipschitz constant of the derivatives being uniform in $n \geq 1$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$. Moreover, by definition, the function $f_n(\cdot, \mu, \cdot)$ is n-Lipschitz continuous in the variable x, that is $\partial_x f_n$ is bounded, as required.

Step 3. We now investigate (A.5). Given $\delta > 0$, $R > 0$, and $n \geq 1$, we consider $x \in \mathbb{R}^d$, $\alpha \in \mathbb{R}^k$, $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$ such that

(5.10) $$\max\big(|x|, |\alpha|, M_2(\mu), M_2(\mu')\big) \leq R, \quad W_2(\mu, \mu') \leq \delta.$$

By (A.5) and (5.8), we can find a constant c' (possibly depending on γ) such that

(5.11) $$\begin{aligned} f_n(x, \mu', \alpha) &= \sup_{|y| \leq n} \inf_{|z| \leq c(1+R)} \big[\langle y, x - z\rangle + f(z, \mu', \alpha)\big] \\ &\leq \sup_{|y| \leq n} \inf_{z \leq c(1+R)} \big[\langle y, x - z\rangle + f(z, \mu, \alpha) + c_L(1 + R + |z|)\delta\big] \\ &= \sup_{|y| \leq n} \inf_{z \in \mathbb{R}^d} \big[\langle y, x - z\rangle + f(z, \mu, \alpha)\big] + c'(1 + R)\delta. \end{aligned}$$

This proves local Lipschitz-continuity in the measure argument as in (A.5).

In order to prove local Lipschitz-continuity in the variables x and α, we use the $\mathcal{C}^{1,1}$-property. Indeed, for x, μ, and α as in (5.10), we know that

(5.12) $$\big|\partial_x f_n(x, \mu, \alpha)\big| + \big|\partial_\alpha f_n(x, \mu, \alpha)\big| \leq \big|\partial_x f_n(0, \mu, 0)\big| + \big|\partial_\alpha f_n(0, \mu, 0)\big| + cR.$$

By (5.7), for any integer $p \geq 1$, there exists an integer n_p, such that, for any $n \geq n_p$, $\partial_x f_n(0, \mu, 0)$ and $\partial_\alpha f_n(0, \mu, 0)$ are, respectively, equal to $\partial_x f(0, \mu, 0)$ and $\partial_\alpha f(0, \mu, 0)$ for $M_2(\mu) \leq p$. In particular, for $n \geq n_p$,

(5.13) $$\big|\partial_x f_n(0, \mu, 0)\big| + \big|\partial_\alpha f_n(0, \mu, 0)\big| \leq c\big(1 + M_2(\mu)\big) \quad \text{whenever} \quad M_2(\mu) \leq p,$$

so that (5.12) implies (A.5) whenever $n \geq n_p$ and $M_2(\mu) \leq p$. We get rid of these restrictions by modifying the definition of f_n. Given a probability measure $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ and an integer $p \geq 1$, we define $\Phi_p(\mu)$ as the push-forward of μ by the mapping $\mathbb{R}^d \ni x \hookrightarrow \big[\max\big(M_2(\mu), p\big)\big]^{-1} px$ so that $\Phi_p(\mu) \in \mathcal{P}_2(\mathbb{R}^d)$ and $M_2(\Phi_p(\mu)) \leq \min(p, M_2(\mu))$. Indeed, if X has μ as distribution, then the random variable $X_p = pX/\max(M_2(\mu), p)$ has $\Phi_p(\mu)$ as distribution. It is easy to check that Φ_p is Lipschitz continuous for the 2-Wasserstein distance, uniformly in $n \geq 1$. We then consider the approximating sequence

$$\hat{f}_p : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^k \ni (x, \mu, \alpha) \hookrightarrow f_{n_p}\big(x, \Phi_p(\mu), \alpha\big), \quad p \geq 1,$$

instead of $(f_n)_{n \geq 1}$ itself. Clearly, on any bounded subset, $\hat{f}_p$ still coincides with f for p large enough. Moreover, the conclusion of the second step is preserved. In particular, the conclusion of the second step together with (5.11), (5.12), and (5.13) say that (A.5) holds (for a possible new choice of c_L). From now on, we get rid of the symbol "hat" in $(\hat{f}_p)_{p \geq 1}$ and keep the notation $(f_n)_{n \geq 1}$ for $(\hat{f}_p)_{p \geq 1}$.

Step 4. It only remains to check that f_n satisfies the bound (A.6) and the sign condition (A.7). Since $|\partial_\alpha f(x, \mu, 0)| \leq c_L$, the Lipschitz property of $\partial_\alpha f$ implies that there exists a constant $c \geq 0$ such that $|\partial_\alpha f(x, \mu, \alpha)| \leq c \; \forall (x, \mu, \alpha) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^k$ with $|\alpha| \leq 1$. In particular, for any $n \geq 1$, it is plain to see that $f_n(x, \mu, \alpha) \leq f_n(x, \mu, 0) + c|\alpha|$, for any $(x, \mu, \alpha) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^k$ with $|\alpha| \leq 1$, so that $|\partial_\alpha f_n(x, \mu, 0)| \leq c$. This proves (A.6).

Finally, we can modify the definition of f_n once more to satisfy (A.7). Indeed, for any $R > 0$, there exists an integer n_R, such that, for any $n \geq n_R$, $f_n(x, \mu, \alpha)$ and $f(x, \mu, \alpha)$ coincide for $(x, \mu, \alpha) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^k$ with $|x|, |\alpha|, M_2(\mu) \leq R$ so that $\langle x, \partial_x f_n(0, \delta_x, 0)\rangle \geq -c_L(1 + |x|)$ for $|x| \leq R$ and $n \geq n_R$. Next we choose a smooth

function $\psi : \mathbb{R}^d \hookrightarrow \mathbb{R}^d$, satisfying $|\psi(x)| \leq 1$ for any $x \in \mathbb{R}^d$, $\psi(x) = x$ for $|x| \leq 1/2$, and $\psi(x) = x/|x|$ for $|x| \geq 1$, and we set $\hat{f}_p(x, \mu, \alpha) = f_{n_p}(x, \Psi_p(\mu), \alpha)$ for any integer $p \geq 1$ and $(x, \mu, \alpha) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^k$, where $\Psi_p(\mu)$ is the push-forward of μ by the mapping $\mathbb{R}^d \ni x \hookrightarrow x - \overline{\mu} + p\psi(p^{-1}\overline{\mu})$. Recall that $\overline{\mu}$ stands for the mean of μ. In other words, if X has distribution μ, then $\hat{X}_p = X - \mathbb{E}(X) + p\psi(p^{-1}\mathbb{E}(X))$ has distribution $\Psi_p(\mu)$.

Ψ_p is Lipschitz continuous with respect to W_2, uniformly in $p \geq 1$. Moreover, for any $R > 0$ and $p \geq 2R$, $M_2(\mu) \leq R$ implies $|\int_{\mathbb{R}^d} x' d\mu(x')| \leq R$ so that $p^{-1}|\int_{\mathbb{R}^d} x' d\mu(x')| \leq 1/2$, that is $\Psi_p(\mu) = \mu$ and, for $|x|, |\alpha| \leq R$, $\hat{f}_p(x, \mu, \alpha) = f_{n_p}(x, \mu, \alpha) = f(x, \mu, \alpha)$. Therefore, the sequence $(\hat{f}_p)_{p \geq 1}$ is an approximating sequence for f which satisfies the same regularity properties as $(f_n)_{n \geq 1}$. In addition,

$$\langle x, \partial_x \hat{f}_p(0, \delta_x, 0) \rangle = \langle x, \partial_x f_{n_p}(0, \delta_{p\psi(p^{-1}x)}, 0) \rangle = \langle x, \partial_x f(0, \delta_{p\psi(p^{-1}x)}, 0) \rangle$$

for $x \in \mathbb{R}^d$. Finally, we choose $\psi(x) = [\rho(|x|)/|x|]x$ (with $\psi(0) = 0$), where ρ is a smooth nondecreasing function from $[0, +\infty)$ into $[0, 1]$ such that $\rho(x) = x$ on $[0, 1/2]$ and $\rho(x) = 1$ on $[1, +\infty)$. If $x \neq 0$, then the above right-hand side is equal to

$$\langle x, \partial_x f(0, \delta_{p\psi(p^{-1}x)}, 0) \rangle = \frac{|p^{-1}x|}{\rho(|p^{-1}x|)} \langle p\psi(p^{-1}x), \partial_x f(0, \delta_{p\psi(p^{-1}x)}, 0) \rangle$$

$$\geq -c_L \frac{|p^{-1}x|}{\rho(|p^{-1}x|)} \left(1 + |p\psi(p^{-1}x)| \right).$$

For $|x| \leq p/2$, we have $\rho(p^{-1}|x|) = |p^{-1}x|$, so that the right-hand side coincides with $-c_L(1 + |x|)$. For $|x| \geq p/2$, we have $\rho(p^{-1}|x|) \geq 1/2$ so that

$$-\frac{|p^{-1}x|}{\rho(|p^{-1}x|)} \left(1 + |p\psi(p^{-1}x)| \right) \geq -2p^{-1}|x| \left(1 + |p\psi(p^{-1}x)| \right) \geq -2p^{-1}|x| \left(1 + p \right) \geq -4|x|.$$

This proves that (A.7) holds with a new constant. $\quad\square$

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